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WHAT TO DO ABOUT ELECTION INVERSIONS UNDER PROPORTIONAL REPRESENTATION?

Dan S. Felsenthal and Nicholas R. Miller

An election inversion occurs when a party, or coalition of parties, wins more seats in the legislature than another party, or coalition of parties, despite the fact that the other party, or coalition of parties, won a larger number of votes cast in an election. It is well known that such inversions can and do occur under the kind of 'majoritarian' electoral systems commonly used in English-speaking countries. It has recently been observed that such inversions not only can and do occur also under party-list proportional representation (PR) systems, but also that they are unavoidable due to the 'whole number problem'. This finding raises the question of whether and how the problem of election inversions under PR systems might be solved or mitigated. We consider a number of options and point out their relative advantages and disadvantages.

Introduction

It is well known that *election inversions*, in which one party wins a majority of seats in the legislature (or, in the case of US presidential elections, of Electoral College votes) despite the fact that another party won a larger proportion of popular votes, can and do occur under the 'majoritarian' electoral systems commonly used in English-speaking countries. Examples include the 2000 US presidential election in which Republican George W. Bush won a majority of Electoral College votes although Democrat Albert Gore Jr won more popular votes, the 1979 Canadian general election in which the Conservative Party won a majority of parliamentary seats although Liberal candidates collectively won considerably more popular votes, and the 1951 British general election in which the post-war Labour government was turned out of power by the Conservative Party although Labour candidates collectively won more popular votes. When they occur, these inversions attract some comment and their possible occurrence is commonly regarded as an important defect in 'majoritarian' systems and as a reason to favour proportional systems.

However, two recent studies have shown that a similar phenomenon can and (arguably) has occurred under list systems of proportional representation (PR). Since such systems typically produce parliaments in which no party wins a majority of either popular votes or seats, an election inversion may be defined in this context as a situation in which a coalition of parties collectively supported by a majority of voters fails to win a majority of seats while the complementary coalition supported by only a minority of voters wins a majority of seats (and might form a government).

In the first study, Kurrild-Klitgaard (2013) identified two examples of election inversions under the Danish PR system. Kurrild-Klitgaard's first case exemplified what he calls the 'federal effect' resulting from the fact that Greenland and the Faroe Islands (like small states in the U.S.

Electoral College) are allocated seats in excess of what their populations warrant. Kurrild-Klitgaard's second case exemplified what he calls the 'threshold effect' resulting from the fact that parties that receive less than 2% of the total vote are denied seats in parliament.

Thus, Kurrild-Klitgaard's examples resulted from *avoidable* imperfections in the Danish (and many other) PR systems deliberately introduced to serve goals other than proportionality, for example, providing enhanced representation for sparsely populated areas and deterring proliferation of many tiny parties. In a follow-up study, Miller (2015) carried Kurrild-Klitgaard's analysis a step further by showing that neither the 'federal effect' nor the 'threshold effect', nor any other avoidable imperfections in PR systems, are necessary for election inversions. Miller demonstrated by both hypothetical and empirical examples that, provided that there are three or more parties, election inversions can and do occur even under the purest types of PR—namely, those that (i) use a single national constituency (and thereby preclude Kurrild-Klitgaard's 'federal effect'), (ii) impose no explicit threshold for winning seats (and thereby reduce Kurrild-Klitgaard's 'threshold effect' to a minimum), and (iii) employ a highly proportional electoral formula, for example, either the (unmodified) Sainte-Laguë (a.k.a. Webster's) or Largest Remainder–Hare (LR–H) (a.k.a. Hamilton's) formulas, in preference to the D'Hondt (a.k.a. Jefferson's) formula (that favours larger parties at the expense of smaller ones). This is because even the purest PR systems have *unavoidable* imperfections that result from the *whole number problem*—that is, from the fact that the number of seats to be apportioned among parties is, in universal practice, much smaller than the number of votes cast in the election and that parties must be awarded seats in terms of relatively small whole numbers while party vote shares are essentially continuous quantities. As a result, perfect proportionality can essentially never be achieved.¹

Miller (2015) demonstrated this theoretical possibility with the two closely related examples displayed in Table 1. To implement the Largest Remainder formula, quotas are calculated for each party. In the first example (the first three lines), the quotas rounded down sum

TABLE 1
Hypothetical election inversions under pure PR (LR–H or Sainte-Laguë) with 35 seats

Party	% Party Votes	% Coalition Votes	Party Quota	Party Seats	Coalition Seats
A	49.7	49.7	17.395	18	18
B	29.4	50.3	10.290	10	17
C	20.9		7.315	7	
A	50.3	50.3	17.605	17	17
B	30.4	49.7	10.640	11	18
C	19.3		6.755	7	

to 34; party *A* has the largest remainder and gets the extra seat for a majority of 18 out of 35, despite having a smaller vote share than the coalition of *B* and *C*. In the second example (the bottom three lines), the quotas rounded down sum to 33; parties *B* and *C* have the two largest remainders and get the two extra seats for a collective majority of 18 out of 35, despite having a smaller vote share than party *A*. The Sainte-Laguë formula produces the same seat allocations and thus the same inversion examples.

Two theoretical points require elaboration. First, all PR formulas are (weakly) ‘monotonic’ with respect to individual parties—that is, if party *A* has a larger vote share than party *B*, every formula assigns party *A* at least as many parliamentary seats as party *B*.² Thus if there are just two parties, election inversions cannot occur under PR, in contrast to the situation under ‘majoritarian’ electoral systems.³ But if there are three or more parties, no formula can avoid the possibility that there exists a partition of the parties into sets such that set *A* collectively wins a larger vote share than set *B* but set *B* is collectively assigned more parliamentary seats.

Second, one easy way to construct hypothetical examples of inversion possibilities is to allow the number of parties that are so small that they win no seats (even in the absence of an explicit seat threshold) to proliferate to the extent that they collectively win a significant proportion of the total popular vote. It may then be easy to find a coalition of parties that controls a majority of the parliamentary seats but that collectively wins less than a majority of the total vote. However, it remains likely that such a coalition collectively wins a majority of the (smaller) vote cast for the (seat-winning) parties actually represented in parliament. Thus it is important to note that Miller’s theoretical claim holds even if there are no small non-seat-winning parties (as is demonstrated by the examples in Table 1).

Two further points should be noted with respect to identifying empirical examples of election inversions under PR. First, when we examine empirical data to find examples of election inversions, we may want, for the reason noted just above, to calculate vote shares on the basis of the total vote for seat-winning parties only, rather than the total vote for all parties.

Second, in so far as ‘majoritarian’ electoral systems, in accordance with Duverger’s Law, produce two-party systems (as is certainly true in the United States but is not true at present in the United Kingdom or Canada), there is only one possible partition of parties into complementary sets, so an inversion either occurs or does not. But PR produces multiparty systems, so there are many possible partitions of parties into complementary sets (even if we exclude non-seat-winning parties). Certainly most of these partitions will not entail inversions but one or more may do so. Of course, such a partition may produce, on the side controlling a majority of seats without the support of a majority of votes, a highly implausible coalition of ideologically or otherwise incompatible parties that would never form a government. Moreover, even if such a partition does produce a plausible governing coalition, it may not be the one that actually forms.

Miller’s (2015) empirical examination of recent election results in Israel and the Netherlands (plus re-examination of Kurrild-Klitgaard’s Danish case) turns up many examples of inversions (under both the actual apportionment of seats and under the Largest Remainder and/or Sainte-Laguë formulas without a seat threshold), some of which hold up when vote shares pertain to seat-winning parties only, and several of which entail plausible governing coalitions.

Examples: The 2013 Elections in Israel and Germany

Table 2 presents the results of the 2013 election in Israel, which is noted for having a relatively pure PR system, particularly by having a single nationwide constituency combined with a relatively low seat threshold.⁴ However, it uses the less proportional D’Hondt formula.

TABLE 2
Results of the 2013 Israeli election

List no.	List name	Votes	Vote %	Seats (actual)	% votes (seat-winning only)	Seats (pure PR) ^a	% votes (seat-winning only)
1	Likud Yisrael Beitenu	885,163	23.34	31	25.12	28	23.57
2	Yesh Atid	543,458	14.33	19	15.42	17	14.47
3	Yisrael Labor Party	432,118	11.39	15	12.26	14	11.51
4	Habayit Hayehudi	345,985	9.12	12	9.82	11	9.21
5	Shas	331,868	8.75	11	9.42	11	8.84
6	United Tora Judaism	195,892	5.16	7	5.56	6	5.22
7	Hatenua	189,167	4.99	6	5.37	6	5.04
8	Meretz	172,403	4.55	6	4.89	6	4.59
9	United Arab List	138,450	3.65	4	3.93	4	3.69
10	Hadash	113,439	2.99	4	3.22	4	3.02
11	National Dem. Assembly	97,030	2.56	3	2.75	3	2.58
12	Kadima	78,974	2.08	2	2.24	3	2.10
13	Otzma Leyisrael	66,775	1.76	0	–	2	1.78
14	Am Shalem	45,690	1.20	0	–	1	1.22
15	Green Leaf— Liberal List	43,734	1.15	0	–	1	1.16
16	Eretz Hadasha	28,080	0.74	0	–	1	.75
17	Koach Lehaspia	28,049	0.74	0	–	1	.75
18	Hayisraelim	18,939	0.50	0	–	1	.50
19	The Green and Young	8117	0.21	0	–	0	–
20	Dor Bonei Haaretz	5975	0.16	0	–	0	–
21	Chaim Bekavod	3640	0.10	0	–	0	–
22	Da-am— Workers Party	3546	0.09	0	–	0	–
23	We Are Brothers	2899	0.08	0	–	0	–
24	Tzedek Hevrat	2877	0.08	0	–	0	–
25	Kulanu Haverim	2176	0.06	0	–	0	–
26	The Pirates	2076	0.05	0	–	0	–
27	The Economics Party	1972	0.05	0	–	0	–
28	Mitkademet Liberalit	1352	0.04	0	–	0	–
29	Light	1027	0.03	0	–	0	–
30	Brit Olam	761	0.02	0	–	0	–
31	Hatikva Leshinui	649	0.02	0	–	0	–
32	Moreshet Avot	461	0.01	0	–	0	–
	Total	3,792,742	100.00	120	100.00	120	100.00

^aHypothetical pure PR seat allocation (assuming no explicit threshold and using largest remainder or Sainte-Laguë formula); data taken from Miller (2015, Table 4).

Source: http://www.bechirof.gov.il/elections19/eng/list/results_eng.aspx.

Thus, Table 2 also presents the seat allocation that would result if Israel used the more proportional Sainte-Laguë or Largest Remainder formula (which produce identical allocations in this case) and with no explicit seat threshold.

TABLE 3
Results of the 2013 German election

Party List	Votes	Vote %	Seats	Coalition		(*)
				Vote %	Seats	Seats
Christian Dem. Union (CDU)	14,921,877	34.13	255	46.31	311	328
Christian Soc. Union (CSU)	3,243,569	7.42	56			
Free Dem. Party (FDP)	2,083,533	4.76	0			
Social Dem. Party (SPD)	11,252,215	25.73	193	42.77	320	242
The Greens	3,694,057	8.45	63			
The Left	3,755,699	8.59	64			61
Alternative for Germany	2,056,985	4.70	0	4.70	0	0
Pirate Party	959,177	2.19	0	2.19	0	0
Others	1,759,744	4.02	0	4.02	0	0
Total	43,726,856	100.00	631	100.00	631	631

^aWith CDU/CSU + FDP 'enhanced *apparentement*'.

Source: http://www.bundeswahlleiter.de/en/bundestagswahlen/BTW_BUND_13/ergebnisse/bundesergebnisse/index.html

A patient reader equipped with a calculator can verify that a number of inversion possibilities appear in this table. With respect to the actual seat allocation, coalitions of lists 1 + 2 + 3 and 1 + 2 + 4 (neither of which is highly implausible) control 65 and 62 seats, respectively (61 being an absolute majority) based on 49.06% and 46.79% of the total vote, respectively; however, even the latter figure represents more than 50% of the vote cast for the 12 seat-winning parties. But the coalition of lists 1 + 2 + 5 (also not entirely implausible) controls 61 seats on the basis of 46.42% of the total vote and 49.96% of the vote for seat-winning parties. More strikingly, the coalition of lists 1 + 3 + 4 + 5 also controls 61 seats on the basis 46.38% of the total vote and 49.91% of the vote for seat-winning parties, and this coalition came close to forming a government. Ultimately, the more broad-based governing coalition of lists 1 + 2 + 4 + 7 (with 68 seats and supported by 51.78% of the total vote) formed.⁵ When we 'purify' the electoral formula to produce the second seat allocation, none of the coalitions previously identified remains in control of 61 seats but at least four new inversion possibilities arise, all of which survive as inversion possibilities when vote support is restricted to seat-winning parties (which now collectively receive 99.01% of the vote); however, their political plausibility may be questionable.⁶

Table 3 presents the results of the 2013 German election.⁷ Germany uses the more proportional Sainte-Laguë formula but in combination with a high 5% seat threshold. While almost as many party lists received votes in Germany as in Israel, the vote was distinctly more concentrated among the leading parties, only five of which passed the demanding seat threshold. The most notable feature of the election results is that the Free Democratic Party (FDP) for the first time fell below the 5% seat threshold and therefore failed to win any seats in parliament. The Christian Democratic Union (CDU)/Christian Social Union (CSU) (effectively a single party) had been in coalition with the FDP going into the election, which was generally perceived (like other recent German elections) as a contest between this centre-right coalition and the rival prospective centre-left Social Democratic Party (SPD) and Green coalition. As given in Table 3, the outgoing governing coalition in fact outpolled its

rival by 46.31% to 42.77% even when the latter is augmented by the Left Party, while the latter won a majority of 320 seats, thereby producing an election inversion based on 'threshold effects'. Since the Left Party (led by the last leader of the old East German ruling party) was not regarded as an acceptable coalition partner, the election produced protracted and difficult negotiations between the CDU/CSU and the SPD that ultimately produced a 'grand coalition' of the two largest parties.

What to Do about Election Inversions?

Given the theoretical and empirical possibility of election inversions under PR, two questions arise. First, is the problem sufficiently severe to warrant institutional reforms to eliminate or mitigate the possibility of inversions? Second, what might these reforms be? In this section, we begin by addressing the first question and conclude that doing nothing might be deemed an acceptable option. However, it is still worth considering the menu of possible reforms. We first consider a simple but radical institutional reform that completely eliminates the possibility of inversions (with respect to seat-winning parties), but it could arguably be seen as in some fundamental ways undermining the proper character of parliamentary government. We then examine various less radical options, one of which guarantees that a coalition supported by a majority of the votes cast for seat-winning parties can form a government supported by a parliamentary majority.

Doing Nothing

Under 'majoritarian' electoral systems, election inversions can occur between parties and can in theory be extreme. As Miller (2012: 112–4) and others have observed, given uniform districts (i.e., districts with equal numbers of voters electing the same number of representatives or electors) and a strict two-party election, it is theoretically possible for a party which is supported by barely more than 25% of the electorate to win a majority of seats (or electoral votes). As Miller (2012: 96–7) further points out, a counterfactual two-party version of the 1860 US presidential election in which Lincoln's opponents combine into a unified ticket provides a semi-realistic version of an extreme inversion in which Lincoln, with less than 40% of the popular vote, still wins a comfortable electoral vote majority. But in realistic practice, election inversions under 'majoritarian' systems are typically close-run things. In the 2000 US presidential election, Bush beat Gore by four electoral votes (271 vs. 267) while Gore beat Bush by a margin of about 0.52% in the popular vote (50,996,062 vs. 50,465,169). In the 1951 UK general election, the Conservative Party (including its National Liberal allies) won 26 more seats than Labour (321 vs. 295 seats), though Labour candidates collectively received a 0.83% greater share of the total votes than the Conservatives and their allies (13,948,385 vs. 13,717,850 votes). Inversions are also quite rare, occurring in a few per cent of all elections and about 10–15% of all two-party elections in which the winning party wins no more than 51% of the vote.⁸

It would be fair to say that when they occur under 'majoritarian' systems, election inversions cause some comments and complaints, but they have never caused a crisis of legitimacy for the incoming government or administration (though a string of inversions in the late 1970s and early 1980s helped push New Zealand to replace its traditional First-Past-the-Post system with a German-style mixed member PR system in 1996).

Miller's (2015) search, for examples, of election inversions in Israel, the Netherlands and Denmark suggests that in countries with about 10 or more seat-winning parties, including several that win only a few seats, most elections produce *potential* election inversions, but few of these entail coalitions that are politically viable, and even fewer entail coalitions that actually formed governments. He identifies only three coalition governments supported by less than half of the total vote: in Israel following the 1981 elections, in Denmark following the 1990 elections (Kurrild-Klittgaard's [2013] most persuasive example) and in the Netherlands following the 2010 elections, but in none of these cases was the governing coalition supported by less than a majority of the votes cast for parties actually represented in parliament.⁹

Moreover, election inversions under PR are necessarily very close-run things, in which a coalition with very slightly less than half the total vote (and perhaps also less than half the vote for seat-winning parties) wins very slightly more than half the seats in parliament. Perhaps for this reason they have attracted little or no political attention. Indeed, we are not aware of any scholarly recognition of the phenomenon prior to the paper by Kurrild-Klittgaard (2013). Given these considerations, maintaining existing PR systems and tolerating the possibility of election inversions may be a quite viable option.

We should, however, consider reform options, the first of which provides a simple and complete but radical solution to the problem.

The Voting Weight Solution

Corporate entities commonly assign to every shareholder a number of votes equal to the number of shares he or she holds. Since parliaments under PR arguably resemble an assembly of shareholders, every party in parliament might be assigned a number of votes that is exactly equal to the number of valid votes it received in an election. This idea is so simple and natural that it is striking that it has rarely been proposed. This proposal would achieve what Barthélemy et al. (2014: 114), call 'limit apportionment'. The same result can be obtained by assigning each party a quota of seats, where the sum of the quotas is equal to the number of seats in parliament but the quotas are fractions calculated out to many decimal places, rather than whole numbers. In either event, each party is assigned a *voting weight* precisely equal to the number of votes it received in the election, which weighted vote would be cast by the party in parliamentary divisions. Where a division is now carried by a majority of individual parliamentarians, it would instead be carried by a majority of the total voting weight (which would effectively make tie votes impossible).

However, this proposed solution raises a number of practical questions.

The threshold of representation. The first question is whether every party which participates in an election and receives at least one vote be represented in parliament. This would be theoretically possible and not unfair, since parties with virtually no support in the electorate would have virtually no voting weight in parliament. Even ordinary PR encourages the proliferation of parties, which may impede parliament's ability to function effectively and provides the rationale for seat thresholds. Representation of all parties with any electoral support would be highly impractical, since it would empower anyone to form his or her own party, vote for himself or herself, and gain a seat in parliament, presumably with certain attendant privileges (though essentially no parliamentary voting weight). Thus, it would probably be most practical to continue existing thresholds that parties must meet in

order to gain representation (and voting weight) in parliament. In this event, the voting weight solution would give any coalition of parties supported by a majority of the vote for seat-winning parties (even if not a majority of the vote for all parties) a majority of the parliamentary voting weight.

Representation of parties meeting the threshold. The exact number of seats (representatives) each represented party would receive in the legislature would no longer be important: every party meeting the threshold would be granted one or more seats which would not necessarily be related to the number of valid votes it received, and every party would be able to distribute its total weight among its representatives in any way it desires. It would be theoretically possible for a party with a greater weight to choose, for whatever reason, to be represented in the legislature by fewer representatives than a party with a smaller weight.¹⁰ However, the most direct way to implement this solution would be not to introduce any change in the total number of seats and the manner in which these seats are distributed among the parties.

Voting weight of individual members of parliament. In many parliamentary divisions (particularly investing governments or votes of confidence), parties vote as blocs, so how their voting weight is divided among individual members is inconsequential. But for occasions when no party discipline is imposed on members' votes, or when a party splits, a rule would be needed. The simplest would be to divide a party's weight equally among its representatives, though there are other (and more complicated) possibilities, for example, giving greater weight to party leaders.¹¹

Quorum requirements. In countries whose legislatures are elected under a PR system and in which a specific quorum is required in order to conduct business either in plenary sessions of the legislature and/or in its parliamentary committees (e.g., in Belgium, Denmark, Germany, Finland, the Netherlands, Norway), this quorum will have to be stated in terms of the required minimal sum of the weights of the attending members rather than in terms of their numbers.

Voting weight of a parliamentarian who defects from his party. Different countries have different rules regarding the possibility that one or more party members will defect from it without resigning from parliament. In the event such defections are permitted, the defecting member could retain his voting weight until the next election (the party from which the defection occurs losing weight accordingly).

Voting weight in committees. In the event a party assigns different weights to its representatives, it may choose to assign its 'heavier' members to the committees it considers more important. Thus the total weight of representatives assigned to a given committee (of fixed size) may vary and with it the quota required to pass a proposed resolution.

Voting weight is not equivalent to voting power. Students of voting power (e.g., Felsenthal and Machover 1998) may be concerned that making the *voting weight* of parties proportional to their electoral support is not the same as making their *voting power* proportional to their electoral support—and that the latter should be the objective. But in this respect, this proposed solution is no less fair than existing PR systems (and, if the number of parties

represented in parliament is fairly small, for example, no more than about six, relatively few configurations of voting power are possible in any case). And in so far as we are concerned with the voting power of individual representatives (whose numbers are relatively great, typically in the hundreds), the Penrose Limit Theorem suggests that voting power is approximately equal to voting weight.¹²

While the voting weight solution completely solves the election inversion problem, it may do so at considerable cost to the traditional character of parliamentary operations. Indeed, the etymology of the word 'parliament' pertains to speaking, not voting, and suggests deliberation in a context in which all members of a body of substantial size have equal standing. The voting weight solution is in some tension with this, at least moderately (if the total number of seats and the manner in which these seats are distributed among the parties were unchanged and if parties distributed voting weight equally among their members) and perhaps radically (if otherwise).¹³

Given these considerations, it is worthwhile to consider other less radical solutions to the election inversion problem.

Enhanced Apparentement

Given a single nationwide constituency (which precludes 'federal' or other 'apportionment effects'), election inversions cannot occur under any PR formula if there are only two parties. Thus if PR could be operated in conjunction with what was in effect a two-party system, election inversions would be precluded. A strongly enhanced version of the *apparentement* device used at various times in various countries might accomplish this.

Apparentement refers to a provision in some electoral systems that allows parties to enter into a pre-election alliance and for the alliance to be allocated seats based on the combined electoral support of its members. It is most commonly used in conjunction with the D'Hondt formula, which favours larger parties.¹⁴ Various forms of 'reinforced' PR can provide even stronger incentives for *apparentement* where it is permitted.¹⁵ Since it is equivalent to the fusion of parties for the purposes of apportioning seats, an *apparentement* between parties B and C in Table 1 would preclude both election inversions (though it would be contrary to their joint interests in the second example).

Typically, *apparentement* (i) requires that each individual party in the alliance (not just the alliance as a whole) must meet any seat threshold and (ii) does not require the parties to continue their alliance after the election. What we might call 'enhanced *apparentement*' would both drop the first stipulation and commit the members of the alliance to operate as a single entity (in effect, a single party) in post-election bargaining over government formation. Combined with the D'Hondt formula or otherwise 'reinforced' PR, enhanced *apparentement* could lead to essentially bipolar rivalry between centre-right and centre-left (or perhaps regional or communal) alliances that, while maintaining multiple parties with somewhat distinct identities, would preclude election inversions.

As mentioned, Germany has seen this kind of bipolar centre-right versus centre-left rivalry in the recent election, which produced reasonable results so long as both the FDP (as the prospective partner of the CDU/CSU) and the Greens (as the prospective partner of the SPD) met the seat threshold, but produced an anomalous results when the FDP fell below 5% in the most recent election. An 'enhanced *apparentement*' (at least on the centre-right side) would have avoided the anomaly, as is shown in the last column in Table 3.

Majoritarian Constraints on Seat Allocations

While it is uncommon for a single party to obtain an absolute majority of the national votes under any form of PR, it is nevertheless quite common for such countries with such systems (e.g., both Germany and the Netherlands) to include in their electoral law provisions that assure that a (single) party which obtains a majority of the votes is allotted a majority of the seats. Such a law would preclude the second example of an election inversion given in Table 1.

Pukelsheim (2014: 149–57) identifies three ways in which such a provision may be implemented. The first is ‘residual seat redirection’: one or more seats initially awarded to other parties are redirected to the majority-supported party sufficient to give it a bare majority of seats.¹⁶ The second is ‘seat augmentation’: the majority party is awarded one or more additional seats, sufficient to give it a bare majority of the augmented number of seats. The third is ‘majority–minority partition’: parties are partitioned into two sets, the majority party and all other parties, respectively; seats are then apportioned between the two sets (which guarantees the majority party a majority of seats, regardless of the apportionment formula) and finally, seats are apportioned among the parties in the second set.

Laws that prohibit an apportionment of a majority of seats to a (single) party which does not obtain the majority of the votes are less common, but in Israel a party that receives no more than half of the valid votes obtained by the parties that passed the threshold may not be allocated more than half of the seats.¹⁷ Such a law would preclude the first example of an election inversion given in Table 1.

Used in conjunction with the kind of ‘enhanced *apparentement*’ discussed in the previous subsection, such majoritarian constraints further mitigate the election inversion problem.

Coalition-wise Majoritarian Constraints on Seat Allocations

As Pukelsheim (2014: 153) notes, majoritarian constraints such as those discussed above that guarantee that a *single party* with majority vote support is awarded a majority of seats could be extended to do the same for a *coalition of parties* with majority vote support. The problem is that, while there can be at most one majority-supported party, in the absence of such a party there are always several (overlapping) coalitions of parties with majority vote support, and it may be that no single apportionment of seats among the parties can assure that every such coalition controls a majority of seats. However, only one *governing coalition* can form, and such an extension could prescribe that a formally proposed coalition government supported by a majority of the votes cast for seat-winning parties, if not initially allocated a majority of parliamentary seats, be awarded one or more (additional or redirected) seats such that the coalition also controls at least a bare majority of parliamentary seats.

Thus immediately following an election and as negotiations to form a government get under way, each party would be assigned a voting weight precisely proportional to its electoral support (as under the voting weight solution). A proposed governing coalition would be required to hold a majority of the total voting weight and would be then awarded additional or redirected seats sufficient for a bare majority.

In either event, rules would have to be established regarding which particular party (or parties) in the proposed governing coalition will be awarded the seats and, in the latter event, regarding which particular party (or parties) outside the coalition would lose seats. The most elegant way to do this would be to generalise the majority–minority partition option discussed

above: first partition the parliamentary parties into two sets, those in the proposed coalition and those not in it; then apportion seats between the two sets (automatically guaranteeing the first set a majority of seats); and then apportion seats to parties within each set. If desired, the total number of seats could be increased sufficiently to assure that no party would lose seats relative to the initial direct apportionment.

With respect to the first example in Table 1, if parties B and C (controlling 50.3% of the voting weight) were to propose a coalition government, seats would be first apportioned (using either the Sainte-Laguë or LR–H formulas, which produce identical allocations in all the following calculations) between the two sets A and B + C, and then apportioned between B and C. If the total number of seats is held at 35, the first apportionment allocates 17 seats to A and 18 seats to B + C; the second apportionment allocates 11 seats to B and 7 to C. If the total number of seats is augmented by 2, the first apportionment preserves A's initial allocation of 18 seats and gives B + C 19 seats; the second apportionment then gives 11 seats to B and 8 seats to C.¹⁸

It probably would be more acceptable to augment the number of seats as necessary in order to avoid denying seats to members of parliament who won seats based in the initial direct apportionment following an election. This would mean, however, that the size of parliament could not be fixed in advance and might depend on what governing coalition forms.¹⁹

Conclusion

If the number of available seats in a legislature is much smaller than the number of votes cast in the election, the whole number problem implies that no apportionment method can guarantee that the number of seats assigned to parties in the legislature is exactly proportional to the number of votes each party received in the election. When there are more than two parties, this impossibility may occasionally lead to potential—or even actual—election inversions.

This fact raises the question of whether and how the problem of election inversions under PR systems might be solved or mitigated. We have considered a number of options.

The very fact that election inversions under PR have heretofore attracted little or no attention suggests that it may be quite acceptable to continue to tolerate their possibility.

At the other extreme, we have outlined a simple but radical solution that completely eliminates the possibility of election inversions while strictly maintaining the idea of PR. Instead of apportioning each party a number of seats which unavoidably is almost always only approximately proportional to the number of votes it received in an election, each party might be assigned a voting weight (to be used in all parliamentary divisions) exactly equal (or proportional) to the number of votes it received in an election. It is somewhat surprising that, as far as we know, this simple but radical solution has not previously been discussed in the literature. Though this solution raises practical problems and arguably is incompatible with the traditional notion of parliament as a deliberative body of equals, it surely warrants further discussion.

Between these extremes, we have identified several other options. 'Enhanced *apparentment*' and imposing majoritarian constraints in PR election laws can reduce but not preclude the possibility of election inversions. Generalising the idea of majoritarian constraints to cover proposed governing coalitions can eliminate election inversions but at the cost of either denying seats to members who initially won seats or creating a variable-sized legislature.

We do not offer any definitive conclusion to the question posed in the title of the article. Doing nothing about the problem of election inversions under PR is certainly the easiest and

most likely response to the problem and a reasonably acceptable one. The voting weight solution probably has more appeal as a subject for scholarly analysis than as a practical reform proposal. Imposing a coalition-wise majoritarian constraint effectively incorporates the other more modest proposals and can solve the problem without fundamentally affecting the character of parliamentary bodies.

NOTES

1. For an excellent discussion of alternative PR formulas for apportioning seats among parties, see Taagepera and Shugart (1989: 29–35). For a detailed history of similar methods used for apportioning seats in the U.S. House of Representatives among the various states, see Balinski and Young (1982). For a comprehensive survey of all PR systems currently used in European countries and in Israel, see Pukelsheim (2014).
2. However, party-wise inversions can occur if an apportionment formula is applied not just once but two or more times in succession—for example, if seats are first apportioned among regions or districts on the basis of population and then seats within each district are apportioned among parties on the basis of votes, or if seats are first apportioned among alliances and then among parties within each alliance (as when *apparentement* is permitted). For an example of the latter, see Pukelsheim (2014: 107).
3. Thus in the examples in Table 1, if parties B and C were fused into a single party, no inversions would occur. In the first example, adding the quotas of B and C gives the fused party the largest remainder and the extra seat. In the second example, adding the quotas of B and C gives the fused party 17 seats at the outset, but A now has the largest remainder and wins the extra seat. Sainte-Laguë again produces the same seat allocations and the same non-inversions.
4. The seat threshold in Israel, originally 1%, was increased to 1.5% in 1992, to 2% in 2004, and has been increased in 2014 to 3.25%.
5. Miller (2015), Table 3, identifies four more inversion possibilities; however, they all include Hadash (a mixed Jewish-Arab front of leftist-socialist organisations) and are thus politically implausible and, while all of them are supported by less than a majority of the total vote, three of them are supported by a majority of the vote for seat-winning parties.
6. These four coalitions are composed of parties 1 + 3 + 5 + 8 + 17 + 18; 1 + 3 + 5 + 8 + 16 + 18; 1 + 3 + 5 + 10 + 12 + 18; 3 + 4 + 5 + 7 + 8 + 10 + 11 + 12 + 16 + 17 + 18.
7. The votes shown are ‘second’ votes cast for party lists (as opposed to ‘first’ votes for individual candidates in local constituencies), which determine the overall allocation of seats among parties.
8. However, in random two-party elections (almost all of which are extremely close), inversions occur about 20% of the time (Feix et al. 2004; Felsenthal and Machover 1998, §3.3).
9. While Miller’s search for examples was not necessarily exhaustive, it is unlikely other examples exist in the data he examined.
10. In somewhat the same manner, under Article V of the U.S. Articles of Confederation in effect from 1781 to 1789, each state could send anywhere from two to seven delegates to Congress, but each state delegation cast a single vote whatever its size.
11. Some legislatures allow members on opposite sides of an upcoming division and who expect to be absent, to pair off and thereby not affect the outcome of the vote. Such pairing would be more difficult to execute if members may have different voting weights.

12. The Penrose Limit Theorem—implicit in Penrose (1952: 72) but for which he gave no rigorous proof—says that, in simple weighted voting games with a fixed quota (e.g., majority rule), share of voting power tends to approach share of voting weight as the number of voters increases. This has been proved in some special cases (Lindner and Machover 2004) and corroborated by simulation in a broader range of cases (Chang et al. 2006).
13. It may be worth noting that the voting weight solution to the problem of election inversions under PR cannot avoid inversions under ‘majoritarian’ electoral systems. For example, even if the U.S. Electoral College assigned to each state a (fractional) voting weight precisely proportional to the popular vote for President cast in the state (thereby eliminating ‘apportionment effects’ in the sense of Miller 2012), election inversions could still occur (because of ‘distribution effects’ in the sense of Miller 2012). However, if in addition, Presidential candidates in each state were awarded fractional shares of each state’s electoral vote precisely proportional to their popular votes in the state, election inversions would be impossible because this ‘double voting weight’ system would be mathematically equivalent to direct population election and thus superfluous.
14. For example, Israeli election law allows *apparentement* between pairs of parties, and in fact a number of smaller parties entered into such alliances in the 2013 elections. However, this did not affect the actual seat allocation presented in Table 2, which in this election was identical to that implied by D’Hondt without *apparentement*.
15. For example, the French electoral system in the 1950s had multi-member districts and awarded a party or *apparentement* supported by more than half of the votes in a district all the seats in the district; otherwise seats were allocated by D’Hondt (Lijphart 1994: 45–6). More recent provisions in Italian and Greek electoral law have given a significant seat bonus to the party that wins a plurality of the vote (Massetti 2006; Massicotte and Blais 1999).
16. The seats are redirected from parties with the weakest claim to the seats, according to the applicable apportionment formula. This method works best with the LR–H formula.
17. Cf. *The [Israeli] Knesset Election Law (consolidated version) 5729–1969, Article 81(d)(4)*.
18. In fact, the two-stage apportionment is superfluous in this case, as a direct apportionment of 37 seats among A, B, C produces the same allocation of seats. Indeed, given the very close-run nature of election inversions under PR, inversions typically disappear when the number of seats is slightly adjusted (up or down). (In the second example in Table 2, a direct apportionment of 37 seats also eliminates the inversion.)
19. Currently it is possible for the standard sizes of the legislatures in both Germany (with 598 members) and New Zealand (with 120 members) to grow as a result of allowing parties to keep their so-called ‘overhang’ seats and the need (in Germany since 2013) to compensate other parties for these ‘overhang’ seats by assigning to them additional so-called ‘balancing’ seats.

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