HW 6 : MATH 301/02, Spring 2016, due in class on March 24th Thu

Please come prepared to discuss ALL these problems in class which will require Sections 3.1 and 3.2. You may use theorems from these sections unless explicitly asked otherwise. You are required to turn in the starred problems in class on March 26th Thu.

Q1 * Prove the limits of the following sequences. You may assume lim(1/n) = 0.

(a) \( \frac{(-1)^n + n^2}{n^3 + \sin(n)} \).
(b) \( \frac{4n}{n + \sin(n^2)} \).
(c) \( \sqrt{n + 1} - \sqrt{n} \).

Q2 Suppose that \((x_n)\) is a bounded sequence and that \((y_n)\) converges to 0. Prove that \((x_ny_n)\) converges to 0.

Q3 Suppose that \((x_n)\) converges to \(x > 0\). Prove that \((x_n)\) has a tail sequence with all terms that are positive.

Q4 * A convergent sequence \((x_n)\) has the property that \(x_n > 0\) for \(n\) even and \(x_n < 0\) for \(n\) odd. Prove that \(\lim(x_n) = 0\).

Q5 Prove directly from the \(\epsilon-K\) definition that the sequence \((-1)^n\) does not converge.

Q6 Suppose that \((x_n)\) is an integer valued sequence, meaning that \(x_n \in \mathbb{Z}\) for all \(n \in \mathbb{N}\). Moreover, suppose that \((x_n)\) converges to \(x\). Prove that there exists \(K \in \mathbb{N}\) such that \(x_n = x\) for all \(n \geq K\), i.e. \((x_n)\) is eventually constant.

Q7 Suppose that two sequences \((x_n)\) and \((y_n)\) both converge to \(L \in \mathbb{R}\) and that we define the interlaced sequence \((z_n)\) as follows: for all \(n \in \mathbb{N}\), \(z_{2n-1} = x_n\) and \(z_{2n} = y_n\). Prove that \((z_n)\) also converges to \(L\).

Q8 Suppose that \(\lim(x_n) = x\) and \(\lim(y_n) = y\) where \(y > x\). Define the sequence \((z_n)\) by \(z_n = \max\{x_n, y_n\}\) for each \(n \in \mathbb{N}\). Prove that \((z_n)\) converges to \(y\).

Q9 * A sequence \((x_n)\) has infinitely many terms which are equal to \(a \in \mathbb{R}\). Suppose that \((x_n)\) converges. Prove that the limit of \((x_n)\) is \(a\).