Dynamical systems viewpoint of ODEs

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A very general class of systems of ODEs in \( n \) dependent real variable \( x = (x_1, \ldots, x_n) \) may be written in the form

\[
F(x^{(r)}, x^{(r-1)}, \ldots, x^{(1)}, x, t) = 0,
\]

where \( t \) is the independent (real) variable and \( x^{(i)} \) denotes the \( i \)th derivative of \( x \) with respect to \( t \), and \( F = 0 \) describes \( n \) scalar equations (since one has \( n \) “unknowns” \( x \)) so that \( F : \mathbb{R}^{rn+n+1} \to \mathbb{R}^n \).

When \( F \) does not depend on \( t \) the ODE system is said to be autonomous or if \( t \) is considered to be time then time independent. If \( F \) does depend on \( t \) then one uses the terminology non-autonomous or time dependent.

A large class of systems of ODEs (higher order and non-autonomous) can be rewritten (after introduction of extra variables if necessary) in the autonomous first order form

\[
\dot{x} = f(x),
\]

where \( \dot{x} \) denotes the derivative of \( x \) with respect to \( t \), the independent variable which is regarded as time (this interpretation is not strictly necessary from a mathematical point of view) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) may be regarded as a vector field on \( \mathbb{R}^n \). Thus \( x \) constitutes the vector valued dependent variable.

In order to see the generality of the above form, we consider the following examples:

1. \( \dot{y} = f(\dot{y}, y) \) where \( y \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \).

2. \( \dot{y} = f(y, t) \) where \( y \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \).

For the first autonomous example set \( x_i = y_i \) and \( x_{n+i} = \dot{y}_i \) for \( i = 1, \ldots, n \). Then one obtains

\[
\dot{x}_i = F_i(x) = F_i(x_1, \ldots, x_{2n}), \quad i = 1, \ldots, 2n,
\]
where

\[ F_i(x) = x_{n+i}, \quad i = 1, \ldots, n, \]

and

\[ F_i(x) = f_i(x_{n+1}, \ldots, x_{2n}, x_1, \ldots, x_n), \quad i = n + 1, \ldots, 2n. \]

For the second non-autonomous example set \( x_i = y_i \) for \( i = 1, \ldots, n \) and set \( x_{n+1} = t \). Then we obtain

\[ \dot{x}_i = F_i(x), \quad i = 1, \ldots, n + 1, \]

where

\[ F_i(x_1, \ldots, x_{n+1}) = f_i(x_1, \ldots, x_n, x_{n+1}), \quad i = 1, \ldots, n, \]

and

\[ F_{n+1}(x_1, \ldots, x_{n+1}) = 1. \]

A solution or trajectory of the system (1) is then a function \( x : (a, b) \to \mathbb{R}^n \) from an interval \((a, b) \subset \mathbb{R}\) (regarded as a time interval) into \( \mathbb{R}^n \) such that \( x \) is differentiable for each \( t \in (a, b) \) and the derivative (denoted by \( \dot{x}(t) \)) is equal to \( f(x(t)) \). When \( f \) is sufficiently nice we can be assured of the existence of solutions. However there will typically be infinitely many solutions unless an initial condition (IC) is specified. The IC typically takes the form of a specification of the value of solution \( x \) at time \( t = 0 \). Thus the pair of equations

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0, \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( x_0 \in \mathbb{R}^n \) respectively describe the vector field and the IC is referred to as an Initial Value Problem (IVP). We note that in the autonomous form specifying the initial condition at time 0 loses no generality.

It may be a source of confusion when one uses \( x \) for the solution and hence a function or a parametrized curve on \( \mathbb{R}^n \) and particular points on \( \mathbb{R}^n \) which may also be denoted by \( x \) or \( x_0 \) etc. With mathematical maturity we hope this confusion goes away!

When the vector field is sufficiently nice, we are assured the existence of a unique solution for the IVP which is defined on an interval. In this situation it useful to denote the unique solution corresponding to IC \( x_0 \) by \( \phi(t, x_0) \). In other words, \( \phi(t, x) \) is the solution at time \( t \) corresponding to the IC \( x \). Some times one uses \( \phi_i(x) \). The mapping \( \phi \) is referred to as the flow of the vector field.