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UNIFORM CONVERGENCE OF INTERLACED EULER METHOD FOR STIFF STOCHASTIC DIFFERENTIAL EQUATIONS*

IOANA CIPCIGAN† AND MURUHAN RATHINAM†

This manuscript is dedicated to the memory of the first author Ioana Cipcigan who passed away before its acceptance for publication.

Abstract. In contrast to stiff deterministic systems of ordinary differential equations, in general, the implicit Euler method for stiff stochastic differential equations is not effective. This paper introduces a new numerical method for stiff differential equations which consists of interlacing large implicit Euler time steps with a sequence of small explicit Euler time steps. We emphasize that uniform convergence with respect to the time scale separation parameter ε is a desirable property of a stiff solver. We prove that the means and variances of this interlaced method converge uniformly in ε for a suitably chosen test problem. We also illustrate the effectiveness of this method via some numerical examples.

Key words. stochastic differential equations, stiffness, uniform convergence, implicit methods, Euler methods, absolute stability

AMS subject classifications. 65L20, 65C30, 60H10

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1. Introduction.

In deterministic as well as stochastic dynamic models, stiff systems, i.e., systems with vastly different time scales where the fast scales are stable, are very common. It is well known that the implicit Euler method is well suited for stiff deterministic equations (modeled by ODEs) while the explicit Euler method is not. In particular, once the fast transients are over, the implicit Euler allows for the choice of time steps comparable to the slowest time scale of the system. In stochastic systems modeled by stochastic differential equations (SDEs), the picture is more complex. While the implicit Euler has better stability properties over the explicit Euler, it underestimates the stationary variance. See, for instance, [2], [3], [9], [10]. In general, one may not expect any method to work successfully by taking time steps of the order of the slowest time scale.

Let us first consider the following system of ODEs:

$$\frac{dX(t)}{dt} = \frac{1}{\varepsilon}\, a(X(t),\, Y(t)),$$
$$\frac{dY(t)}{dt} = f(X(t),\, Y(t)),$$

where ε represents the ratio between the time scales of the system and we assume that a is such that $X(t)$ is stable. It is well known that when $\varepsilon \ll 1$, the implicit Euler method is more effective than the explicit Euler method. Note that ε is a measure of stiffness. It is worth noting that the implicit Euler works well regardless of ε. When ε is not so small, the explicit Euler method may be more efficient because of the computational savings, however, the implicit Euler method with the same step size is expected to be equally

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†Mathematics and Statistics, University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250 (muruhan@umbc.edu).

accurate. Thus, one may expect an effective stiff solver, such as the implicit Euler, to *converge uniformly* in ε as the step size approaches zero. For the deterministic part of the test linear system considered in this paper, this uniform convergence holds. However, this is not the case for stochastic systems and the effectiveness of the implicit Euler is compromised. Some relevant numerical results may be found in [3], [9].

In this paper, we propose an efficient numerical method for stiff systems of SDEs driven by Brownian motion. More exactly, we are interested in systems of the form

$$
\begin{aligned}
dX(t) &= \frac{1}{\varepsilon}\, a(X(t), Y(t))\,dt + \frac{1}{\sqrt{\varepsilon}}\, b(X(t), Y(t))\,dB_1(t),\\
dY(t) &= f(X(t), Y(t))\,dt + g(X(t), Y(t))\,dB_2(t),
\end{aligned}
$$

where B_1 and B_2 are independent Brownian motions. Our goal is to devise a numerical scheme which works effectively for a range of ε values, not necessarily very small. Thus, in general, we only assume that $0 < \varepsilon < 1$ and attempt to devise a method which is convergent uniformly in ε. It is worth mentioning at this point that this is what differentiates our approach from the averaging type methods, such as the projective integration [5], which are primarily concerned with the $\varepsilon \to 0$ behavior. See also [4], [6].

We present a composite time-stepping strategy for solving stiff systems of SDEs, by interlacing a large implicit Euler time step ($O(1)$) with a sequence of m small explicit Euler time steps ($O(1/\varepsilon)$). The alternation of large and small steps resembles the projective integration methods studied in [5]. However, unlike in the projective integration, we do not compute the averages (over time or ensemble) of the fast dynamics. We merely alternate the time steps. Furthermore, our large time steps are implicit and the small time steps are explicit. The motivation for our method comes from the asymptotic moment analysis, where numerical methods with constant step size τ are applied to a suitably chosen test SDE. In this analysis, in addition to stability, we compute and compare the asymptotic moments (usually the first two) of the method with those of the true solution. This analysis shows that the variance computed by the explicit Euler method is larger than the variance of the stationary distribution, while the implicit Euler method underestimates it.

We further present the convergence analysis of this new method as applied to a test problem. This analysis shows the existence of a range of values for the number of small explicit time steps, m, such that the method converges uniformly in the time scale separation parameter ε. The outline of the paper is as follows. In section 2, we summarize some results of the asymptotic moment analysis applied to the Euler methods, motivate the interlaced method, and comment on the comparison with the trapezoidal method. In section 3, we provide the error analysis for the first two moments of the interlaced method when applied to a suitably chosen test system. In section 4, we present the main result of the paper, the uniform convergence of the interlaced method. Section 5 presents some numerical examples which demonstrate the effectiveness of the method. Finally, in section 6, we make some concluding remarks and discuss some extensions of the method as future work.

2. Interlaced Euler method.

In this section, we introduce the interlaced Euler method. The method consists of interlacing one large implicit time step of size k with a sequence of m small explicit time steps of size τ. For stiff stochastic systems, the implicit Euler underestimates the stationary variance and the explicit Euler overestimates it [2], [3], [10]. By interlacing, we seek to obtain a composite method which gives an asymptotic numerical variance close to the exact stationary variance. This provides us the criterion for determining m, the number of explicit time steps. We shall choose a test problem that enables the analysis of finding a formula for m.

2.1. Motivation. The standard test problem for absolute stability analysis in deterministic systems is $\dot{x} = -\lambda x$. In stochastic systems, it is important to choose a test problem which has nonzero asymptotic variance [2], [10]. This leads us to choose the following test SDE:

$$dX(t) = -\lambda X(t)dt + \lambda \bar{x}dt + \mu X(t)dB(t), \tag{2.1}$$

and we are interested in the stationary mean and variance of $X(t)$, provided that the first two moments are stable. The differential equations for the moments [3], [8] show that in order to have finite asymptotic first two moments, the following two stability conditions must be satisfied:

$$\lambda > 0, \qquad \mu^2 < 2\lambda. \tag{2.2}$$

It can be shown that

$$E[X(\infty)] = \lim_{t\to\infty} E[X(t)] = \bar{x},$$
$$Var(X(\infty)) = \lim_{t\to\infty} Var(X(t)) = \frac{\mu^2 \bar{x}^2}{2\lambda - \mu^2}.$$

We consider the Euler methods with a fixed time step, applied to this SDE. The Euler methods belong to the family of stochastic theta methods, which we briefly review here (see, for instance, [7], [8]). Consider the general SDE

$$dX(t) = a(X(t))dt + b(X(t))dB(t). \tag{2.3}$$

The stochastic theta method applied to (2.3) is

$$\hat{X}_{n+1} = \hat{X}_n + (1-\theta)a(\hat{X}_n)h + \theta a(\hat{X}_{n+1})h + b(\hat{X}_n)dB_n,$$

where h is a fixed time step, $\hat{X}_n$ is the numerical approximation at time $t_n = nh$, and dB_n are independently and identically distributed Gaussian random variables with mean 0 and variance h. Taking $\theta = 0$, we obtain the explicit Euler method, $\theta = 1$ gives the implicit Euler method (also known as semi-implicit Euler, as referred to in [8]), and $\theta = 1/2$ gives the trapezoidal method.

For explicit Euler with fixed time step τ, applied to our test problem, it is easy to see [3], [7], [9] that the stability conditions for the mean and variance is

$$\tau < \frac{2\lambda - \mu^2}{\lambda^2}, \tag{2.4}$$

provided (2.2) holds. When stability holds it follows that

$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x},$$
$$Var(\hat{X}_\infty) = \frac{1}{1 - \frac{\lambda^2\tau}{2\lambda-\mu^2}} Var(X(\infty)) > Var(X(\infty)).$$

Therefore, the asymptotic mean of the numerical solution obtained by the explicit Euler method is the same as the asymptotic mean of the exact solution but the stationary variance is overestimated.

For the case of the implicit Euler method with fixed time step k applied to our test problem, the moments are unconditionally stable, provided (2.2) holds. Moreover, the asymptotic mean and variance are

$$E[\hat{X}_\infty] = E[X(\infty)] = \bar{x},$$
$$Var(\hat{X}_\infty) = \frac{1}{1 + \frac{\lambda^2 k}{2\lambda - \mu^2}} Var(X(\infty)) < Var(X(\infty)).$$

Here we see that the asymptotic mean of the numerical solution obtained by the implicit Euler method is the same as the asymptotic mean of the exact solution but the stationary variance is underestimated. These two observations give us the basic idea for the interlaced Euler method which consists of interlacing one implicit time step of size k with m explicit time steps of size τ satisfying the stability condition (2.4). It can be easily shown that this method is stable for all k and m and it computes the asymptotic mean correctly. Moreover, the asymptotic variance $Var(\hat{X}_\infty)$ is a function of τ, k, and m. We provide below the formula for the ratio $VQ(m, \tau, k) = Var(\hat{X}_\infty)/Var(\hat{X}(\infty))$, which we shall refer to as *variance quotient*

$$VQ(m,\tau,k) = \frac{2\lambda - \mu^2}{2\lambda - \mu^2 - \lambda^2\tau}\left\{1 - \frac{(\lambda^2k^2 + \lambda^2k\tau)[(1-\lambda\tau)^2 + \mu^2\tau]^m}{(1+\lambda k)^2 - (1+\mu^2 k)[(1-\lambda\tau)^2 + \mu^2\tau]^m}\right\}. \tag{2.5}$$

It is possible to choose m such that this ratio is close to 1. With such a choice, we expect the method not only to be stable but also to compute the first two asymptotic moments correctly.

2.2. Test problem. So far we have discussed the absolute stability and the asymptotic moments for a scalar test problem. Our ultimate goal is to obtain a method which applies to a stiff system of SDEs. We choose the following system of stochastic differential equations with two different time scales:

$$dX(t) = -\frac{\lambda_0}{\varepsilon}X(t)dt + \frac{\lambda_0}{\varepsilon}\bar{x}dt + \frac{\mu_0}{\sqrt{\varepsilon}}X(t)dB_1(t),$$
$$dY(t) = -\lambda_0 Y(t)dt + \beta X(t)dB_2(t), \tag{2.6}$$

where the fast equation resembles our previous scalar test SDE and B_1, B_2 are two independent Brownian motions. Here $\varepsilon > 0$ represents the time scale separation parameter. When $\varepsilon \ll 1$, the system is stiff. It is instructive to consider the $\varepsilon \to 0$ limit behavior obtained via the singular perturbation theory for SDEs, which gives the following reduced equation for Y [5], [6]:

$$d\tilde{Y}(t) = -\lambda_0 \tilde{Y}(t)dt + \beta\sqrt{E[(X^*)^2]}\,dB(t),$$

where B is another Brownian motion. In the above equation, X^* denotes the random variable with the same asymptotic distribution as $X(t)$. It is clear that the slow variable $Y(t)$ evolves depending on the second moment of the fast variable. Thus, one would expect the implicit Euler method applied to this system with step size of the order of $1/\lambda_0 \gg \varepsilon/\lambda_0$ (the time scale of the slow dynamics) to underestimate the second moment of the fast variable and hence lead to inaccurate computation of the variance of the slow variable.

It is relevant at this point to discuss certain parameters of importance.

- T_f represents the relaxation time of the fast dynamics. In (2.6), $T_f = \varepsilon/\lambda_0$.
- T_s represents the time to resolve the slow dynamics. In (2.6), $T_s = 1/\lambda_0$.
- T represents the time interval of simulation.

It is convenient to introduce the following nondimensional parameters:

- ε represents the time scale separation, $\varepsilon = T_f/T_s$. We assume that $\varepsilon < 1$ but not necessarily $\varepsilon \ll 1$.
- $\eta = \mu_0^2/2\lambda_0$ represents the ratio between drift and diffusion. Note that $\eta < 1$ is required from (2.2).
- α_1 represents the accuracy parameter for the implicit time step. Thus, $k = \alpha_1 T_s = \alpha_1/\lambda_0$.
- α_2 represents the accuracy parameter for the explicit time step. Thus, $\tau = \alpha_2 T_f = \alpha_2 \varepsilon/\lambda_0$. Note that $\alpha_2 < 2(1-\eta)$ is required from (2.4).
- ρ represents the interval of simulation in terms of the slow time scale. Thus, $T = \rho T_s$.

The absolute stability and asymptotic moment analysis of the previous subsection corresponds to the $\varepsilon \to 0$ limit behavior of the fast subsystem. Indeed, making the change of variables $t' = t/\varepsilon,\ T' = T/\varepsilon$, we obtain the system

$$\begin{aligned} dX(t) &= -\lambda_0 X(t)dt + \lambda_0 \bar{x} dt + \mu_0 X(t) d\tilde{B}_1(t), \\ dY(t) &= -\lambda_0 \varepsilon Y(t)dt + \beta\sqrt{\varepsilon} X(t) d\tilde{B}_2(t), \end{aligned}$$

(where $\tilde{B}_1$ and $\tilde{B}_2$ are two different independent Brownian motions) and taking the limit $\varepsilon \to 0$, we obtain $T' \to \infty$ and $Y =$ constant. If we consider a fixed time step $h' = h/\varepsilon$ with $h \to 0$, we see that the analysis of the dynamics of the initial system reduces to the asymptotic analysis of the fast variable. Thus, it was instructive to study the asymptotic behavior of the fast component.

However, we are also interested in the situations when ε might not be much smaller than 1. Let us consider the deterministic situation corresponding to $\mu_0 = 0, \beta = 0$ which is given by

$$\begin{aligned} X'(t) &= -\frac{\lambda_0}{\varepsilon} X(t) + \frac{\lambda_0}{\varepsilon}\bar{x}, \\ Y'(t) &= -\lambda_0 Y(t). \end{aligned} \tag{2.7}$$

Then the implicit Euler solution with time step $k = \alpha_1/\lambda_0$ is

$$\begin{aligned} \hat{X}_n &= (X(0) - \bar{x})M^n + \bar{x}, \\ \hat{Y}_n &= Y(0)P^n, \end{aligned}$$

where $n = T/k = \lambda_0 T/\alpha_1$ and

$$\begin{aligned} M &= \frac{1}{1 + \frac{\lambda_0}{\varepsilon}k} = \frac{\varepsilon}{\alpha_1 + \varepsilon}, \\ P &= \frac{1}{1 + \lambda_0 k} = \frac{1}{1 + \alpha_1}. \end{aligned}$$

The exact solution of (2.7) is given by

$$\begin{aligned} X(T) &= (X(0) - \bar{x})e^{-\frac{\lambda_0}{\varepsilon}T} + \bar{x}, \\ Y(T) &= Y(0)e^{-\lambda_0 T}. \end{aligned}$$

It can be shown that as $\alpha_1 \to 0$, M^n converges uniformly in ε to $e^{-(\lambda_0/\varepsilon)T}$. It is therefore clear that the implicit Euler for ODEs converges uniformly in ε. Thus, the implicit Euler method applied to the system (2.7) performs effectively, regardless of the size of ε. On the other hand, it may be shown that explicit Euler method as well as the trapezoidal method applied to this system, with step size $k = \alpha_1/\lambda_0$, do not converge uniformly in ε. The lack of uniform convergence manifests in a subtle way in the trapezoidal case, which we discuss in section 2.3.

The investigation of the deterministic case (2.7) leads us to look for a method that is uniformly convergent with respect to ε, at least in the first two moments for the stochastic case. If such a method is available, one would expect it to work for very stiff systems as well as moderately stiff systems. The asymptotic moment analysis in section 2.1 suggests that the implicit Euler method with step size $k = \alpha_1/\lambda_0$ does not converge uniformly for the second moment. Our conjecture is that no method with step size $k = \alpha_1/\lambda_0$ can converge uniformly for the second moment. Revisiting the interlaced method introduced in the previous subsection, we ask the following question: "Is there a range of values for m (possibly depending on α_1, α_2, ε, and η) such that the method is uniformly convergent (in ε) for the first two moments?"

The choice of (the optimal) m for which the variance quotient $VQ(m) = 1$ given by the asymptotic moment analysis is shown below in terms of the nondimensional parameters

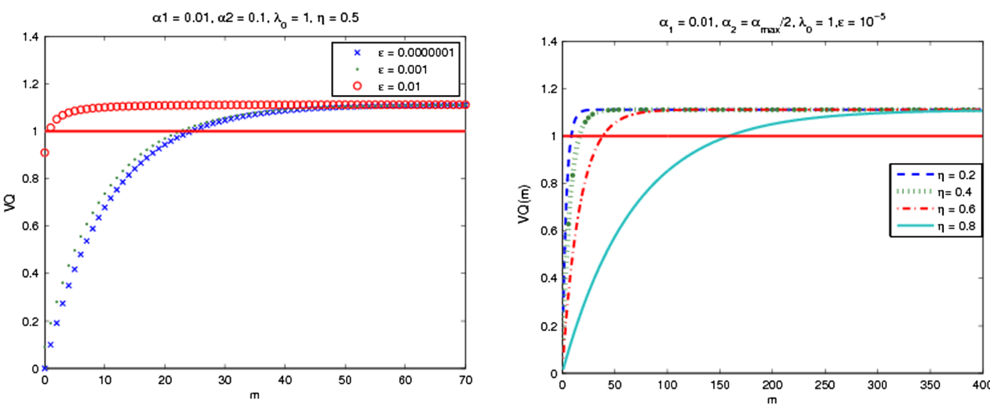

FIG. 2.1. *$VQ(m)$ plotted against m. The left figure shows the dependence on ϵ, and the right figure shows the dependence on η.*

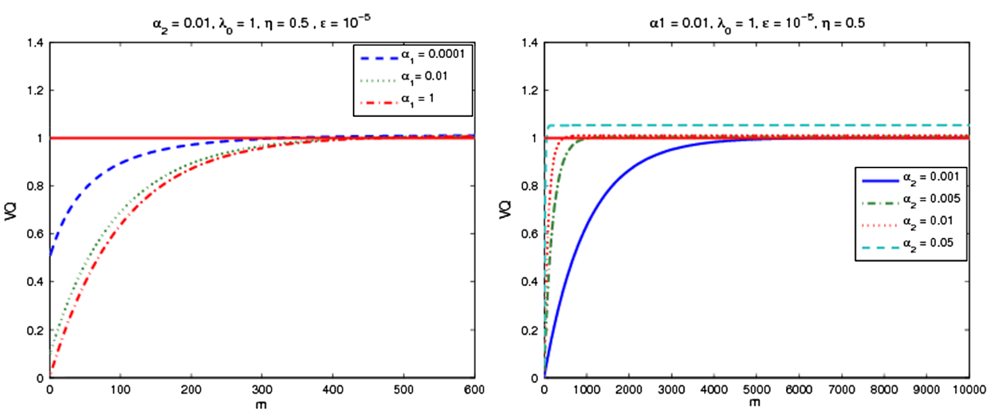

FIG. 2.2. *$VQ(m)$ plotted against m. The left figure shows the dependence on α_1, and the right figure shows the dependence on α_2.*

$$m = \frac{\ln\left(\dfrac{(\epsilon^2 + 2\alpha_1\epsilon + \alpha_1^2)\alpha_2}{\alpha_2\epsilon^2 + 2\alpha_1\alpha_2\epsilon + 2\alpha_1^2(1-\eta)}\right)}{\ln\left(1 + \alpha_2^2 - 2\alpha_2(1-\eta)\right)}. \tag{2.8}$$

We show the dependence of the optimal m on the problem parameters η and ϵ, as well as the numerical method parameters, α_1 and α_2, in Figures 2.1 and 2.2. Here we plot VQ as a function of m and the constant function 1. The intersection of the graphs gives the optimal value of m. We observe that the optimal m is a decreasing function of ϵ and α_2 and an increasing function of η and α_1.

The asymptotic moment analysis also suggests that the interlaced method with the choice of m given by(2.8) might be uniformly convergent. Indeed, in this paper, we prove the existence of a range of values for m, which includes the one given by (2.8), such that uniform convergence of the first two moments holds.

2.3. Comparison with the trapezoidal method. It must be noted that the asymptotic moment analysis shows that the trapezoidal method captures the first two moments exactly. However, the trapezoidal method is not uniformly convergent, even for the first moment. Note that the differential equations for the first moments are the same as the ODE system (2.7). For the trapezoidal method applied with time step $k = \alpha_1 T_s$, the numerical amplification factor for mean of the fast variable is

$$M_t = \frac{2 - \frac{\alpha_1}{\varepsilon}}{2 + \frac{\alpha_1}{\varepsilon}}.$$

It is not difficult to see that $M_t^n = M_t^{\rho/\alpha_1}$ does not converge uniformly to $e^{-\lambda_0 T/\varepsilon} = e^{-\rho/\varepsilon}$ as $\alpha_1 \to 0$. The lack of uniform convergence specifically presents a problem when ρ and ε are both small. In this case, since ε is very small, the trapezoidal method reaches the steady state much later than the true solution, and if ρ is small, the time interval does not allow the trapezoidal method to catch up with the true solution. In addition, this phenomenon becomes more pronounced when the eigenvalues are complex with large imaginary parts [3]. The work in [3] shows numerical examples where the interlaced method is superior to the trapezoidal method. In section 5, we also present some comparisons of the interlaced method with the trapezoidal method.

3. Error analysis.

In this section, we derive linear inequalities for the global errors of the first two (nonmixed) moments for the test problem (2.6). In our derivation, we use the ordinary differential equations for the first two moments of the exact solution and the difference equations for the corresponding moments of the numerical solution. Thus, our error analysis is similar to the ODE case (see [1], for instance).

In section 2.2, the interlaced time step h is given by $h = k + m\tau$ with the implicit time step $k = \alpha_1/\lambda_0$ and the explicit time step $\tau = \varepsilon\alpha_2/\lambda_0$. For simplicity, we choose $\alpha_1 = \alpha$ and $\alpha_2 = F\alpha$, where F is a constant. The stability condition (2.4) becomes $F\alpha < 2(1-\eta)$, and the composite time step is

$$h = k + m\tau = \frac{\alpha + Fm\alpha\epsilon}{\lambda_0}.$$

Note that in this setup, m is given by the following formula:

$$m = \frac{\ln\left(\dfrac{F\varepsilon^2 + 2F\alpha\varepsilon + 2(1-\eta)\alpha}{F\varepsilon^2 + 2F\alpha\varepsilon + F\alpha^2}\right)}{-\ln\left(1 - 2F(1-\eta)\alpha + F^2\alpha^2\right)}. \tag{3.1}$$

Therefore, the convergence analysis will be done with respect to α. More specifically, we will prove the uniform convergence with respect to ε as $\alpha \to 0$.

We apply the interlaced method to system (2.6). The time interval of simulation is $[0, T]$, and it is discretized as follows: $0 = t_0 < t_1 < \cdots < t_N = T$, with $t_n = nh$. We denote by $\hat{X}_n$ and $\hat{Y}_n$ the numerical approximations of the fast and slow variables, respectively, at time t_n. Throughout this paper, we will use the following notations:

$$g(t) = E[X(t)], \quad \hat{g}_n = E[\hat{X}_n], \quad h(t) = E[Y(t)], \quad \hat{h}_n = E[\hat{Y}_n],$$
$$u(t) = E[X^2(t)], \quad \hat{u}_n = E[\hat{X}_n^2], \quad v(t) = Var(Y(t)), \quad \hat{v}_n = Var(\hat{Y}_n).$$

The numerical solution inside the composite time step plays an important role in the analysis. We denote by $\hat{X}_n^{1/2,l}$ the numerical solution corresponding to the fast component at time $t_n + k + l\tau$, that is, inside the $(n+1)$st composite time step. Then for any $l = 0, \ldots, m-1$ let

$$\hat{g}_n^{1/2,l} = E[\hat{X}_n^{1/2,l}], \qquad \hat{u}_n^{1/2,l} = E[(\hat{X}_n^{1/2,l})^2].$$

Note that throughout this paper we will also use the notations $\lambda = \lambda_0/\varepsilon$ and $\mu = \mu_0/\sqrt{\varepsilon}$.

The differential equations for the first two moments of a general vector linear SDE system are derived in [8]. Here we present the corresponding equations for our test problem.

LEMMA 3.1. *The ordinary differential equations for the first two moments of the exact solution of system* (2.6) *are*

$$g'(t) = -\frac{\lambda_0}{\varepsilon} g(t) + \frac{\lambda_0}{\varepsilon}\bar{x},$$
$$u'(t) = -\frac{2\lambda_0 - \mu_0^2}{\varepsilon} u(t) + 2\frac{\lambda_0}{\varepsilon}\bar{x}g(t),$$
$$h'(t) = -\lambda_0 h(t),$$
$$v'(t) = -2\lambda_0 v(t) + b^2 u(t).$$

The difference equations for the first two moments of the numerical solution obtained by the implicit/explicit Euler with constant time step can be easily derived using the properties of the Brownian increments. In this section, we will only state the corresponding lemmas and will omit their proofs.

Before we proceed with the analysis, we make two important remarks.

Remark 1. Our error analysis in this section and the proof of uniform convergence in section 4 are still valid if the two Brownian motions B_1 and B_2 in (2.6) are the same $(B_2 = B_1)$ instead of being two independent Brownian motions. This is because our analysis is confined to the nonmixed first two moments.

Remark 2. Throughout the rest of this paper we shall derive several estimates for errors which involve certain constants which we shall usually label $C, C_1, C_2 \ldots$, etc. These constants are independent of α and ϵ. We shall not attempt to obtain sharp estimates of these constants. Given this fact, we shall adopt the naming convention that a constant denoted by C, C_1, etc., has scope only within the statement of a theorem and/or its proof environment. Two constants denoted by C appearing in different theorems or lemmas are not meant to be the same.

3.1. Mean of the fast variable. Our first task is to derive the global error for the mean of the fast variable. Using the difference equation for the numerical solution and the differential equation for the first moment of the exact solution, we obtain the linear inequality which characterizes the error propagation for the mean.

LEMMA 3.2. *The mean of the fast variable obtained by the implicit Euler method with time step k is given by*

$$E[\hat{X}_{n+1}] = ME[\hat{X}_n] + N,$$

where

$$M = M(\alpha, \varepsilon) = \frac{1}{1+\lambda k} = \frac{\varepsilon}{\alpha + \varepsilon}, \qquad N = N(\alpha, \varepsilon) = \bar{x}\frac{\lambda k}{1+\lambda k} = \bar{x}\frac{\alpha}{\alpha+\varepsilon}.$$

LEMMA 3.3. *The mean of the fast variable obtained by the explicit Euler method with time step τ is given by*

$$E[\hat{X}_{n+1}] = AE[\hat{X}_n] + B,$$

where

$$A = A(\alpha) = 1 - \lambda\tau = 1 - F\alpha, \qquad B = B(\alpha) = \bar{x}\lambda\tau = \bar{x}F\alpha.$$

The following result is immediate from the previous two lemmas.

LEMMA 3.4. *The mean of the fast variable obtained by the interlaced method is given by*

$$\hat{g}_{n+1} = A^m M \hat{g}_n + A^m N + B\sum_{i=0}^{m-1} A^i. \tag{3.2}$$

Proof. The proof is straightforward using Lemmas 3.2, 3.3, and B.1. □

The next step is to write the mean of the fast variable in the same format as the mean of the numerical solution. We first write the exact mean using the implicit/explicit Euler format by means of the Taylor expansion, and then we obtain the exact solution written in the same format as the interlaced method solution.

Remark 3. Since the ultimate goal is to prove the uniform convergence with respect to ε, for Taylor expansion it is useful to work with the integral remainder rather than the more common Lagrange remainder.

LEMMA 3.5. *The mean of the fast variable written in the explicit Euler format is*

$$g(t+\tau) = Ag(t) + B + TruncMeanFastE(t, \tau),$$

where $TruncMeanFastE(t,\tau)$ satisfies

$$|TruncMeanFastE(t,\tau)| \leq C\alpha^2 e^{-\frac{\lambda_0}{\varepsilon}t} \quad \forall\ t \geq 0 \tag{3.3}$$

with constant C independent of ε.

Proof. Using the ODE for the mean of the fast component, we obtain

$$\begin{aligned} g(t+\tau) &= (1-\lambda\tau)g(t) + \bar{x}\lambda\tau + TruncMeanFastE(t,\tau) \\ &= Ag(t) + B + TruncMeanFastE(t,\tau) \end{aligned}$$

with $TruncMeanFastE(t,\tau) = \int_t^{t+\tau} g'(s)(t+\tau-s)ds$. From Lemma B.6, we know that there exists a constant C, independent of ε, such that $|g''(t)| \le C(\lambda_0^2/\varepsilon^2)$ $e^{-(\lambda_0/\varepsilon)t}$ for all $t \ge 0$. Some tedious manipulations yield

$$|TruncMeanFastE(t,\tau)| \le \int_t^{t+\tau} |g''(s)|(t+\tau-s)ds \le C\alpha^2 e^{-\frac{\lambda_0}{\varepsilon}t},$$

which completes the proof. ∎

LEMMA 3.6. *The mean of the fast variable written in the implicit Euler format is*

$$g(t_n+k) = Mg(t_n) + N + TruncMeanFastI(t_n,k),$$

where $TruncMeanFastI(t_n,k)$ *satisfies*

$$|TruncMeanFastI(t_n,k)| \le C(e^{-\frac{\lambda_0 h}{\varepsilon}})^n \frac{e^{\frac{\alpha}{\varepsilon}} - \frac{\alpha}{\varepsilon} - 1}{e^{\frac{\alpha}{\varepsilon}}} \tag{3.4}$$

with constant C *independent of* ε*.*

Proof. We have $g(t_n) = g(t_n+k) - kg'(t_n+k) + TruncMeanFastI(t_n,k)$ with

$$TruncMeanFastI(t_n,k) = \int_{t_n}^{t_n+k} g''(s)(t_n-s)ds,$$

and using the ODE for the mean, we get

$$\begin{aligned} g(t_n+k) &= \frac{1}{1+\lambda k}g(t_n) + \bar{x}\frac{\lambda k}{1+\lambda k} + \frac{1}{1+\lambda k}TruncMeanFastI(t_n,k) \\ &= Mg(t_n) + N + MTruncMeanFastI(t_n,k). \end{aligned}$$

This completes the first part of the proof.

To prove (3.4), we use the bound for $g''(t)$ from Lemma B.6, and computing the resulting integral, we obtain (3.4). ∎

COROLLARY 3.7. *The truncation error of the fast mean in the implicit format satisfies*

$$\begin{aligned} &|TruncMeanFastI(t_n,k)| \le C_1\frac{\alpha}{\varepsilon}e^{-Fm\alpha} \quad \forall\ n \le 1, \\ &\sum_{i=0}^{n-1} |TruncMeanFastI(t_i,k)| \le \frac{\alpha}{\varepsilon}, \\ &\sum_{i=0}^{n-1} |TruncMeanFastI(t_i,k)| \le 1. \end{aligned}$$

Further, we write the exact mean in the same format as the mean of numerical solution given by the interlaced method.

LEMMA 3.8. *The mean of the fast variable satisfies*

$$g(t_n + k + m\tau) = A^m M g(t_n) + A^m N + B \sum_{l=0}^{m-1} A^l + A^m M\, TruncMeanFastI(t_n, k) + \sum_{l=0}^{m-1} A^{m-1-l}\, TruncMeanFastE(t_n + k + l\tau, \tau). \tag{3.5}$$

Proof. Using Lemmas B.1 and 3.5, we obtain $g(t + m\tau)$, and then we use Lemma 3.6 with $t = t_n + k$ to obtain (3.5). ◻

Remark 4. Note that $TruncMeanFastE(t_n + k + l\tau, \tau)$ represents the truncation error from the explicit Euler expansion at step $t_n + k + l\tau$ and from (3.3) we obtain

$$|TruncMeanFastE\,(t_n + k + l\tau, \tau)| \leq C\alpha^2 e^{-\frac{\lambda_0}{\varepsilon}(t_n+k+l\tau)},$$

which gives

$$|TruncMeanFastE(t_n + k + l\tau, \tau)| \leq C\alpha^2 e^{-\frac{\alpha}{\varepsilon}} (e^{-\frac{\lambda_0 h}{\varepsilon}})^n (e^{-F\alpha})^l. \tag{3.6}$$

Combining the results from Lemma 3.8 with Lemma 3.4, we obtain our main result of this subsection.

THEOREM 3.9. *The error for the mean of the fast variable, $e_n = \hat{g}_n - g(t_n)$, satisfies the linear inequality*

$$|e_{n+1}| \leq A^m M |e_n| + A^m M |TruncMeanFastI(t_n, k)| + \sum_{l=0}^{m-1} A^{m-1-l} |TruncMeanFastE(t_n + k + l\tau, \tau)|. \tag{3.7}$$

Proof. Subtracting (3.2) from (3.5) we obtain the result. ◻

3.2. Mean of the slow variable. The linear inequality for the global error for the mean of the slow variable can be obtained through a similar derivation as for the fast variable. We present here only the main result for the mean of the slow variable.

THEOREM 3.10. *The error for the mean of the slow variable, $s_n = \hat{h}_n - h(t_n)$, satisfies the linear inequality*

$$|s_{n+1}| \leq P^m Q |s_n| + P^m Q |TruncMeanSlowI(t_n, k)| + \sum_{l=0}^{m-1} P^{m-1-l} |TruncMeanSlowE(t_n + k + l\tau, \tau)|, \tag{3.8}$$

where

$$P = \frac{1}{\alpha+1}, \qquad Q = 1 - F\alpha\varepsilon,$$
$$|TruncMeanSlowI(t_n, k)| \leq C_1\alpha^2,$$
$$|TruncMeanSlowE(t_n + k + l\tau, \tau)| \leq C_2\alpha^2\varepsilon$$

with constants C_1, C_2 independent of ε.

3.3. Second moment of the fast variable. In this section, we derive the propagation error for the second moment of the fast variable using a technique similar to the one in section 3.1.

LEMMA 3.11. *The second moment of the fast variable obtained by the implicit Euler method with time step k is given by*

$$E[\hat{X}_{n+1}^2] = I_2 E[\hat{X}_n^2] + I_1 E[\hat{X}_n] + RI,$$

where the functions I_2, I_1, and RI *satisfy*

$$I_2 = I_2(\alpha, \varepsilon) = \frac{1+\mu^2 k}{(1+\lambda k)^2} = \frac{\varepsilon^2 + 2\eta\alpha\varepsilon}{(\alpha+\varepsilon)^2} \leq C\frac{\varepsilon}{\alpha+\varepsilon},$$
$$I_1 = I_1(\alpha, \varepsilon) = 2\bar{x}\frac{\lambda k}{(1+\lambda k)^2} = 2\bar{x}\frac{\alpha\varepsilon}{(\alpha+\varepsilon)^2} \leq 2\bar{x}\frac{\alpha}{\alpha+\varepsilon},$$
$$RI = RI(\alpha, \varepsilon) = \bar{x}^2\left(\frac{\lambda k}{1+\lambda k}\right)^2 = \bar{x}^2\frac{\alpha^2}{(\alpha+\varepsilon)^2}$$

with the constant C independent of ε.

LEMMA 3.12. *The second moment of the fast variable obtained by the explicit Euler method with time step τ is given by*

$$E[\hat{X}_{n+1}^2] = E_2 E[\hat{X}_n^2] + E_1 E[\hat{X}_n] + RE,$$

where the functions E_2, E_1, and RE *satisfy*

$$E_2 = E_2(\alpha) = (1-\lambda\tau)^2 + \mu^2\tau = 1 - F\alpha[2(1-\eta) - F\alpha],$$
$$E_1 = E_1(\alpha) = 2\bar{x}\lambda\tau(1-\lambda\tau) = 2\bar{x}F\alpha(1-F\alpha) \leq C\alpha,$$
$$RE = RE(\alpha) = \bar{x}^2\lambda^2\tau^2 = \bar{x}^2F^2\alpha^2$$

with the constant C independent of ε.

Remark 5. Note that the stability condition $F\alpha < 2(1-\eta)$ implies $E_2 < 1$.

LEMMA 3.13. *The second moment of the fast variable obtained by the interlaced method is given by*

$$\hat{u}_{n+1} = E_2^m I_2 \hat{u}_n + E_2^m I_1 \hat{g}_n + E_1 \sum_{l=0}^{m-1} E_2^{m-1-l}\hat{g}_n^{1/2,l} + E_2^m RI + RE \sum_{l=0}^{m-1} E_2^l. \tag{3.9}$$

Proof. The proof follows from Lemmas 3.11, 3.12, and B.1. ∎

Now that we have obtained the formula for the second moment of the fast variable, we can prove some useful properties of m, which will play an important role in our uniform convergence proof and which are summarized in the following corollary.

COROLLARY 3.14. *The optimal m given by formula* (3.1) *satisfies the following inequalities:*

$$E_2^m\left(\frac{\alpha}{\alpha+\varepsilon}\right)^2 \le C_1\alpha(1-E_2^m), \tag{3.10}$$

$$\sqrt{E_2}^m\frac{\alpha}{\alpha+\varepsilon} \le C_2\sqrt{\alpha}, \tag{3.11}$$

$$A^m\frac{\alpha}{\alpha+\varepsilon} \le C_3\sqrt{\alpha} \tag{3.12}$$

with constants C_1, C_2, and C_3 independent of ε.

Proof. Recall that m is chosen such that the asymptotic variance obtained by the interlaced method to be equal to the true asymptotic variance. This is $\hat{u}_\infty = u(\infty)$, and from Lemma 3.1 we have $u(\infty) = \bar{x}^2/1-\eta$. Taking the limit $n\to\infty$ in (3.9) we obtain

$$\hat{u}_\infty = E_2^mI_2\hat{u}_\infty + E_2^mI_1\hat{g}_\infty + E_1\sum_{l=0}^{m-1}E_2^{m-1-l}\hat{g}_\infty^{1/2,l} + E_2^mRI + RE\sum_{l=0}^{m-1}E_2^l.$$

It can be easily shown that $\hat{g}_\infty = \hat{g}_\infty^{1/2,l} = \bar{x}$ for all $l = 0, \ldots, m-1$, and using the formulas for E_1, E_2, RI, RE and simplifying the resulting equation we get

$$\frac{\alpha}{\alpha+\varepsilon} = \frac{F\alpha}{\sqrt{F\alpha(2(1-\eta)-F\alpha)}}\sqrt{\frac{1-E_2^m(\alpha)}{E_2^m(\alpha)}}. \tag{3.13}$$

We can relax this condition by replacing "=" by "≤." This implies

$$E_2^m\left(\frac{\alpha}{\alpha+\varepsilon}\right)^2 \le C_1\alpha(1-E_2^m)$$

with $C_1 = F/1-\eta$, which proves the first inequality. Taking the square root in the above inequality, we obtain (3.11), with $C_2 = \sqrt{C_1}$. Note also that for any $\alpha \le (1-2\eta)/F$ we have $1-F\alpha \le \sqrt{1-F\alpha[2(1-\eta)-F\alpha]}$. This gives $A \le \sqrt{E_2}$ which yields (3.12). ∎

The next step in our derivation is to write the exact solution in the same format.

LEMMA 3.15. *The second moment of the fast variable written in the implicit Euler format is given by*

$$u(t_n+k) = I_2u(t_n) + I_1g(t_n) + RestI + TruncSecondMomentFastI(t_n,k),$$

where the functions RestI *and* TruncSecondMomentFastI *satisfy*

$$|RestI| \le C_1\left(\frac{\alpha}{\alpha+\varepsilon}\right)^2, \tag{3.14}$$

$$\sum_{i=0}^{n-1}|TruncSecondMomentFastI(t_i,k)| \le C_2\frac{\alpha}{\alpha+\varepsilon} \tag{3.15}$$

with the constants C_1, C_2 independent of ε.

Proof. Using the ODE for the second moment of the fast variable we obtain

$$u(t_n+k) = u(t_n) - 2(1-\eta)\lambda k u(t_n+k) + 2\bar{x}\lambda k g(t_n+k) + TrU(t_n,k) \tag{3.16}$$

with $TrU(t_n,k) = \int_{t_n}^{t_n+k} u'(t)(t-t_n)dt$. Using the bound $|u''(t)| \leq C_1(1/\varepsilon^2)e^{-(C_2/\epsilon)t}$ for all $t \geq 0$ from Lemma B.6, with constants C_1, C_2 independent of ε, an easy manipulation yields

$$|TrU(t_n,k)| \leq C_1 e^{-\frac{C_2 t_n}{\varepsilon}}\left(1 - e^{-\frac{C_2 k}{\varepsilon}} - \frac{C_2 k}{\varepsilon}\, e^{-\frac{C_2 k}{\varepsilon}}\right).$$

Further, we obtain

$$\sum_{i=0}^{n-1} |TrU(t_i,k)| \leq C_1 \frac{1 - e^{-\frac{C_2 k}{\varepsilon}} - \frac{C_2 k}{\varepsilon}\, e^{-\frac{C_2 k}{\varepsilon}}}{1 - e^{-\frac{C_2 k}{\varepsilon}}} \leq C_3 \frac{\alpha}{\varepsilon}, \tag{3.17}$$

where we have used $\sum_{i=0}^{n-1}(e^{-C_2 h/\varepsilon})^i < 1/(1-e^{-C_2 k/\varepsilon})$ and Lemma B.3.

Moreover, we have $g(t_n+k) = g(t_n) + TrG(t_n,k)$ with $TrG(t_n,k) = (\bar{x} - X_0)e^{-\lambda t_n}(1-e^{-\lambda k})$ satisfying

$$\sum_{i=0}^{n-1} |TrG(t_i,k)| \leq C(1-e^{-\lambda k})\sum_{i=0}^{n-1} e^{-\lambda t_i} \leq C \tag{3.18}$$

with $C = |X_0 - \bar{x}|$. The solution is therefore given by

$$\begin{aligned}
u(t_n+k) &= u(t_n) - 2(1-\eta)\lambda k u(t_n+k) + 2\bar{x}\lambda k[g(t_n) + TrG(t_n,k)] + TrU(t_n,k)\\
&= \frac{1+2\eta\lambda k}{1+2(1-\eta)\lambda k}\, u(t_n) + 2\bar{x}\frac{\lambda k}{(1+\lambda k)^2}\, g(t_n)\\
&\quad + \left(\frac{1}{1+2(1-\eta)\lambda k} - \frac{1+2\eta\lambda k}{(1+\lambda k)^2}\right) u(t_n)\\
&\quad + 2\bar{x}\lambda k\left(\frac{1}{1+2(1-\eta)\lambda k} - \frac{1}{(1+\lambda k)^2}\right) g(t_n)\\
&\quad + 2\bar{x}\frac{\lambda k}{1+2(1-\eta)\lambda k}\, TrG(T_n,k) + \frac{1}{1+2(1-\eta)\lambda k}\, TrU(t_n,k)\\
&= u(t_n) + I_1 g(t_n) + RestI + TruncSecondMomentFast(t_n,k),
\end{aligned}$$

where

$$\begin{aligned}
RestI = &\left(\frac{1}{1+2(1-\eta)\lambda k} - \frac{1+2\eta\lambda k}{(1+\lambda k)^2}\right) u(t_n)\\
&+ 2\bar{x}\lambda k\left(\frac{1}{1+2(1-\eta)\lambda k} - \frac{1}{(1+\lambda k)^2}\right) g(t_n).
\end{aligned}$$

Note that from Lemma B.6 we know that $|u(t)|$ and $|g(t)|$ are uniformly bounded for any $t \geq 0$, and some easy manipulations show that the coefficients of $u(t)$ and $g(t_n)$ in the above expression are also uniformly bounded. Using this in the above equation, we obtain (3.14).

To prove (3.15), note that

$$TruncSecondMomentFastI(t_n,k) = \frac{2\bar{x}\lambda k\, TrG(t_n,k)}{1+2(1-\eta)\lambda k} + \frac{TrU(t_n,k)}{1+2(1-\eta)\lambda k}.$$

Using $1/(1+2(1-\eta)\lambda k) \leq \max\{1, 2(1-\eta)\}(1/1+\lambda k)$ and the bounds from (3.17) and (3.18), we obtain (3.15). ∎

LEMMA 3.16. *The second moment of the fast variable written in the explicit Euler format is given by*

$$u(t+\tau) = E_2 u(t) + E_1 g(t) + RestE,$$

where RestE *satisfies* $|RestE| \leq C\alpha^2$, *with constant* C *independent of* ε.

Proof. Using the ODE for $u(t)$ and matching the coefficients with those given by explicit Euler solution, we obtain

$$\begin{aligned} u(t_n+\tau) &= [(1-\lambda\tau)^2 + \mu^2\tau]u(t_n) + 2\bar{x}\lambda\tau(1-\lambda\tau)g(t_n) + \frac{\tau^2}{2}u'(\xi_n) \\ &\quad + [1-(2\lambda-\mu^2)\tau - (1-\lambda\tau)^2 - \mu^2\tau]u(t_n) + 2\bar{x}\lambda\tau[1-(1-\lambda\tau)]g(t_n) \\ &= E_2 u(t_n) + E_1 g(t_n) + RestE, \end{aligned}$$

where

$$\begin{aligned} RestE &= \frac{\tau^2}{2}u''(\xi_n) + [1-(2\lambda-\mu^2)\tau - (1-\lambda\tau)^2 - \mu^2\tau]u(t_n) \\ &\quad + 2\bar{x}\lambda\tau[1-(1-\lambda\tau)]g(t_n) \\ &= \frac{\tau^2}{2}u''(\xi_n) - \lambda^2\tau^2 u(t_n) + 2\bar{x}\lambda^2\tau^2 g(t_n), \end{aligned}$$

with $\xi_n \in (t_n, t_n+\tau)$. Using the bounds for u'', g, and u from Lemma B.6, we obtain

$$|RestE| \leq C\lambda^2\tau^2 = CF^2\alpha^2,$$

with the constant C independent of ε. ∎

LEMMA 3.17. *The second moment of the fast variable satisfies*

$$\begin{aligned} u(t_n+k+m\tau) &= E_2^m I_2 u(t_n) + E_2^m(\alpha) I_1 g(t_n) \\ &\quad + E_2^m RestI + E_2^m TruncSecondMomentFastI(t_n, k) \\ &\quad + E_1 \sum_{l=0}^{m-1} E_2^{m-1-l} g(t_n+k+l\tau) \\ &\quad + RestE \sum_{l=0}^{m-1} E_2^l. \end{aligned} \tag{3.19}$$

Proof. The proof is straightforward, first using Lemmas B.1 and 3.16 to obtain $u(t+m\tau)$ and then using Lemma 3.15 with $t = t_n + k$. ∎

LEMMA 3.18. *The error of the second moment of the fast variable,* $f_n = \hat{u}_n - u(t_n)$, *satisfies the linear inequality*

$$|f_{n+1}| \le E_2^m I_2 |f_n| + C_1(1 - E_2^m)|e_n| + C_2\alpha(1 - E_2^m) + C_3 E_2^m |TruncSecondMomentFastI(t_n, k)| + C_4\alpha \sum_{l=0}^{m-1} E_2^{m-1-l} |e_n^{1/2,l}|, \tag{3.20}$$

with constants C_1, C_2, C_3, and C_4 independent of ε.

Proof. Subtracting (3.9) from (3.19), we obtain

$$|f_{n+1}| \le E_2^m I_2 |f_n| + E_2^m I_1 |e_n| + E_2^m(|RestI| + |RI|) + E2^m |TruncSecondMomentFastI(t_n, k)| + E_1 \sum_{l=0}^{m-1} E_2^{m-1-l} |e_n^{1/2,l}| + (|RestE| + |RE|) \sum_{l=0}^{m-1} E_2^l.$$

Further, we have

$$E_2^m I_1 = C_1 E_2^m \frac{\alpha\varepsilon}{(\alpha+\varepsilon)^2} < C_1 E_2^m \frac{\alpha^2}{(\alpha+\varepsilon)^2}\frac{1}{\alpha} < C_2\alpha(1 - E_2^m)\frac{1}{\alpha} = C_2(1 - E_2^m),$$

and using the bounds for $RestI$, RI, $RestE$, RE, and the bound for $E_2^m(\alpha^2/(\alpha+\varepsilon)^2)$ from inequality (3.10), we obtain the result. ∎

3.4. Variance of the slow variable. Finally, in this section, we derive a linear inequality for the global error of the slow variable.

LEMMA 3.19. *The variance of the slow variable obtained by the implicit Euler method is given by*

$$Var(\hat{Y}_{n+1}) = I_y Var(\hat{Y}_n) + I_x E[\hat{X}_n^2],$$

where the functions I_y and I_x are given by

$$I_y = I_y(\alpha) = \frac{1}{(1+\lambda_0 k)^2} = \frac{1}{(1+\alpha)^2},$$
$$I_x = I_x(\alpha) = \frac{b^2 k}{(1+\lambda_0 k)^2} = \frac{b^2}{\lambda_0}\frac{\alpha}{(1+\alpha)^2}.$$

LEMMA 3.20. *The variance of the slow variable obtained by the explicit Euler method is given by*

$$Var(\hat{Y}_{n+1}) = E_y Var(\hat{Y}_n) + E_x E[\hat{X}_n^2],$$

where the functions E_y and E_x are given by

$$E_y = E_y(\alpha, \varepsilon) = (1 - \lambda_0\tau)^2 = (1 - F\alpha\varepsilon)^2, \quad E_x = E_x(\alpha, \varepsilon) = b^2\tau = \frac{b^2 F}{\lambda_0}\alpha\varepsilon.$$

LEMMA 3.21. *The variance of the slow variable obtained by the interlaced method is given by*

$$\hat{v}_{n+1} = E_y^m I_y \hat{v}_n + E_y^m I_x \hat{u}_n + E_x \sum_{l=0}^{m-1} E_y^{m-1-l} \hat{u}_n^{1/2,l}. \tag{3.21}$$

Proof. The result is a straightforward application of Lemmas 3.19, 3.20, and B.1. ∎

Next, using the ODE for the second moment, we want to write the exact solution in the same format.

LEMMA 3.22. *The variance of the slow variable written in the implicit Euler format is*

$$v(t_n + k) = I_y v(t_n) + I_x u(t_n) + RestSlowI + TruncVarSlowI(t_n, k),$$

where the functions RestSlowI *and* TruncVarSlowI *satisfy*

$$|RestSlowI| \leq C_1 \alpha^2, \tag{3.22}$$

$$\sum_{i=0}^{n-1} |TruncSlowVarI(t_i, k))| \leq C_2 \alpha, \tag{3.23}$$

with constants C_1, C_2 independent of ε.

Proof. To write $v(t+k)$ in the same format as the corresponding numerical solution, we first Taylor expand $v(t+k)$ using the explicit format and then we match the coefficients with the coefficients of the variance of the numerical solution given by the implicit Euler method.

$$\begin{aligned} v(t_n + k) &= v(t_n) + kv'(t_n) + TruncVarSlowI(t_n, k) \\ &= (1 - 2\lambda_0 k) v(t_n) + b^2 k u(t_n) + TruncVarSlowI(t_n, k) \\ &= \frac{1}{(1+\lambda_0 k)^2} v(t_n) + \frac{b^2 k}{(1+\lambda_0 k)^2} u(t_n) + TruncVarSlowI(t_n, k) \\ &\quad + \left(1 - 2\lambda_0 k - \frac{1}{(1+\lambda_0 k)^2}\right) v(t_n) + b^2 k \left[1 - \frac{1}{(1+\lambda_0 k)^2}\right] u(t_n) \\ &= I_y v(t_n) + I_x u(t_n) + RestSlowI + TruncVarSlowI(t_n, k), \end{aligned}$$

with $TruncVarSlowI(t_n, k) = \int_{t_n}^{t_n+k} v'(t_n)(t_n + k - t)\,dt$ and

$$RestSlowI = \left(1 - 2\lambda_0 k - \frac{1}{(1+\lambda_0 k)^2}\right) v(t_n) + b^2 k \left(1 - \frac{1}{(1+\lambda_0 k)^2}\right) u(t_n).$$

Using the fact that $v(t)$ and $|u(t)$ are uniformly bounded for all $t \geq 0$ and simplifying the coefficients of $u(t_n), v(t_n)$, we obtain $RestSlowI \leq C\alpha^2$, with C independent of ε. From Lemma B.6, we have $|v''(t)| < C_1 + C_2[(C/\varepsilon)e^{-(C/\varepsilon)t}]$ for all $t \geq 0$ which gives

$$\begin{aligned} |TruncVarSlowI(t_n, k)| &\leq \int_{t_n}^{t_n+k} |v''(t)|(t_n + k - t)\,dt \\ &\leq C_1 \frac{k^2}{2} + \frac{C_2}{C}(Ck - \varepsilon + \varepsilon e^{-C\frac{k}{\varepsilon}}) e^{-C\frac{t_n}{\varepsilon}}. \end{aligned}$$

Therefore,

$$\sum_{i=0}^{n-1} |TruncVarSlowI(t_i,k)| < C_1 \sum_{i=0}^{n-1} \alpha^2 + \frac{C_2}{C}(Ck - \varepsilon + \varepsilon e^{-C\frac{k}{\varepsilon}}) \sum_{i=0}^{n-1} (e^{-\frac{Ch}{\varepsilon}})^i$$
$$< C_3\alpha + \frac{C_2}{C} \frac{Ck - \varepsilon + \varepsilon e^{-C\frac{k}{\varepsilon}}}{1 - e^{-\frac{Ck}{\varepsilon}}}.$$

Note that in the above inequality we have used $n\alpha < nh \leq T$ and $e^{-Ch/\varepsilon} < e^{-Ck/\varepsilon}$. Finally, Lemma B.4 implies

$$\sum_{i=0}^{n-1} |TruncVarSlowI(t_i,k)| < C_3\alpha + \frac{C_2}{C} Ck < C_4\alpha$$

and this completes the proof. ∎

LEMMA 3.23. *The variance of the slow component written in the explicit Euler format is*

$$v(t+\tau) = E_y v(t) + E_x u(t) + RestSlowE,$$

where the function RestSlowE *satisfies*

$$|RestSlowE| \leq C\alpha^2\varepsilon,$$

with constant C *independent of* ε.

Proof. Using the ODE for $v(t)$, we get

$$\begin{aligned} v(t+\tau) &= (1 - 2\lambda_0\tau)v(t) + b^2\tau u(t) + \frac{\tau^2}{2} v''(\xi) \\ &= (1-\lambda_0\tau)^2 v(t) + b^2\tau u(t) + \tau^2 \left(\frac{v''(\xi)}{2} - \lambda_0^2 u(t) \right) \\ &= E_y v(t) + E_x u(t) + RestSlowE \end{aligned}$$

with $RestSlowE = \tau^2(\frac{v''(\xi)}{2} - \lambda_0^2 u(t))$ and $\xi \in (t, t+\tau)$. From Lemma B.6, there exists a constant C_1 independent of ε such that $|\frac{v''(\xi)}{2} - \lambda_0^2 u(t)| < \frac{C_1}{\varepsilon}$ for all $t \geq 0$ and $\xi \in (t, t+\tau)$. Hence

$$|RestSlowE| < \left(\frac{F^2}{\lambda_0^2} \alpha^2 \varepsilon^2 \right) \left(\frac{C_1}{\varepsilon} \right) < C\alpha^2\varepsilon,$$

which proves the lemma. ∎

LEMMA 3.24. *The variance of the slow component satisfies*

$$\begin{aligned} v(t_n + k + m\tau) &= E_y^m I_y(\alpha) v(t_n) + E_y^m I_x(\alpha) u(t_n) E_y^m RestSlowI \\ &\quad + E_y^m TruncVarSlow(t_n,k) + E_x \sum_{l=0}^{m-1} E_y^{m-1-l} u(t_n + k + l\tau) \\ &\quad + RestSlowE \sum_{l=0}^{m-1} E_y^l. \end{aligned} \tag{3.24}$$

Proof. The result follows immediately from Lemmas 3.22, 3.23, and B.1. ∎

LEMMA 3.25. *The global error of the variance of the slow component*

$$es_n = Var(\hat{Y}_n) - Var(Y(t_n))$$

satisfies the linear inequality

$$|es_{n+1}| \leq E_y^m I_y |es_n| + E_y^m I_x |f_n| + E_y^m |RestSlowI| + E_y^m |TruncSlowI(t_n, k)| + E_x \sum_{l=0}^{m-1} E_y^{m-1-l} |f_n^{1/2,l}| + |RestSlowE| \sum_{l=0}^{m-1} E_y^l. \tag{3.25}$$

Proof. Subtracting (3.24) from (3.21), we obtain the result. ∎

Remark 6. As observed in the previous derivations, there exists a discrepancy between the difference equations for the second moments obtained from the numerical solutions given by the implicit/explicit Euler methods applied to SDEs, and the difference equations obtained by applying the implicit/explicit Euler methods to the differential equations for the second moments. This explains the presence of the terms *RI*, *RE*, *RestI*, *RestE*, *RestSlowI*, and *RestSlowE* which we shall refer to as displacement terms.

TABLE 3.1
Amplification factors.

Equation: fast mean Type: implicit	$M = \frac{\varepsilon}{\alpha+\varepsilon}$
Equation: fast mean Type: explicit	$A = 1 - F\alpha$
Equation: fast second moment Type: implicit Term: fast second moment	$I_2 = \frac{\varepsilon^2+2\eta\alpha\varepsilon}{(\alpha+\varepsilon)^2} \leq C_1 \frac{\varepsilon}{\alpha+\varepsilon}$
Equation: fast second moment Type: implicit Term: fast mean	$I_1 = 2\bar{x}\frac{\alpha\varepsilon}{(\alpha+\varepsilon)^2} \leq C_2 \frac{\alpha}{\alpha+\varepsilon}$
Equation: fast second moment Type: explicit Term: fast second moment	$E_2 = 1 - F\alpha[2(1-\eta) - F\alpha]$
Equation: fast second moment Type: explicit Term: fast mean	$E_1 = 2\bar{x}F\alpha(1 - F\alpha) \leq C_3\alpha$
Equation: slow variance Type: implicit Term: slow variance	$I_y = \frac{1}{(1+\alpha)^2}$
Equation: slow variance Type: implicit Term: fast second moment	$I_x = \frac{b^2}{\lambda_0}\frac{\alpha}{(1+\alpha)^2}$
Equation: slow variance Type: explicit Term: slow variance	$E_y = (1 - F\alpha\varepsilon)^2$
Equation: slow variance Type: explicit Term: fast second moment	$E_x = \frac{b^2 F}{\lambda_0}\alpha\varepsilon$

TABLE 3.2
Displacement terms.

Equation: fast second moment Type: implicit	$\lvert RestI\rvert + \lvert RI\rvert \leq C_1 \frac{\alpha^2}{(\alpha+\varepsilon)^2}$
Equation: fast second moment Type: explicit	$\lvert RestE\rvert + \lvert RE\rvert \leq C_2\alpha^2$
Equation: slow variance Type: implicit	$\lvert RestSlowI\rvert \leq C_3\alpha^2$
Equation: slow variance Type: explicit	$\lvert RestSlowE\rvert \leq C_4\alpha^2\varepsilon$

TABLE 3.3
Truncation errors.

$\lvert TruncMeanFastI(t_n, k)\rvert \leq C_1 \frac{e^{\frac{\alpha}{\varepsilon}} - \frac{\alpha}{\varepsilon} - 1}{e^{\frac{\alpha}{\varepsilon}}} (e^{-Fm\alpha})^n (e^{-\frac{\alpha}{\varepsilon}})^n$
$\sum_{i=0}^{n-1} \lvert TruncMeanFastI(t_i, k)\rvert \leq \frac{\alpha}{\varepsilon}$
$\sum_{i=0}^{n-1} \lvert TruncMeanFastI(t_i, k)\rvert \leq 1$
$\lvert TruncMeanFastE(t_n + k + l\tau, \tau)\rvert \leq C_2\alpha^2 e^{-\frac{\alpha}{\varepsilon}} (e^{-\frac{\lambda_0 h}{\varepsilon}})^n (e^{-F\alpha})^l$
$\sum_{i=0}^{n-1} \lvert TruncSecondMomentFastIFastI(t_i, k)\rvert \leq C_3 \frac{\alpha}{\alpha+\varepsilon}$
$\sum_{i=0}^{n-1} \lvert TruncVarSlow(t_i, k)\rvert \leq C_4\alpha$

The truncation errors for the moments, the amplification factors, and the displacement terms play an important role in the uniform convergence proof. Tables 3.1, 3.2, and 3.3 summarize these properties.

4. Uniform convergence. In this section, we prove the uniform convergence with respect to ε for the composite time step and for the first two moments of the fast and slow variables. Specifically, we show that each of the errors corresponding to these four moments satisfies

$$\lim_{\alpha\to 0} \sup_{\varepsilon>0} Error(\alpha, \epsilon) = 0.$$

To prove this, we derive uniform bounds in ε for each error.

THEOREM 4.1. *The composite time step h converges uniformly in ε to 0 as $\alpha \to 0$.*

Proof. We have

$$h = k + m\tau = \frac{\alpha}{\lambda_0} + m\frac{F\alpha\varepsilon}{\lambda_0} = \frac{1}{\lambda_0}(\alpha + Fm\alpha\varepsilon).$$

Recall that (3.1) gives the following equation for m:

$$m = \frac{\ln\left(\dfrac{F\varepsilon^2 + 2F\alpha\varepsilon + 2(1-\eta)\alpha}{F\varepsilon^2 + 2F\alpha\varepsilon + F\alpha^2}\right)}{-\ln\left(1 - 2F(1-\eta)\alpha + F^2\alpha^2\right)}.$$

Assuming $F\alpha \le 1-\eta$, Lemma B.2 implies

$$\frac{1}{-\ln\left(1 - F\alpha[2(1-\eta) - F\alpha]\right)} < \frac{1}{F(1-\eta)\alpha}.$$

Further, using $\ln(1+x) \le 2\sqrt{x}$ for all $x \ge 0$, we obtain

$$\ln\left(1 + \frac{2(1-\eta)\alpha - F\alpha^2}{F\varepsilon^2 + 2F\alpha\varepsilon + F\alpha^2}\right) < 2\sqrt{\alpha}\,\frac{\sqrt{2(1-\eta) - F\alpha}}{\sqrt{F\varepsilon^2 + 2F\alpha\varepsilon + F\alpha^2}} < C_1\frac{\sqrt{\alpha}}{\varepsilon},$$

where $C_1 = 2(\sqrt{2(1-\eta)}/\sqrt{F})$. Combining these two results, we obtain

$$m \le \frac{C_1\sqrt{\alpha}}{F(1-\eta)\alpha\varepsilon}.$$

Therefore, $m\alpha\varepsilon < C_2\sqrt{\alpha}$, with C_2 independent of ε. This yields $h = (1/\lambda_0)(\alpha + Fm\alpha\varepsilon) < (1/\lambda_0)(\alpha + FC_2\sqrt{\alpha})$ for all $\varepsilon > 0$.

Hence there exists a constant C such that $h < C\sqrt{\alpha}$ for all $\varepsilon > 0$ (and all α small enough) which implies that h converges uniformly to 0 as $\alpha \to 0$. ∎

Remark 7. Note that $m\alpha\varepsilon \le C\sqrt{\alpha}$ is a sufficient condition for the uniform convergence of the composite time step.

Table 4.1 summarizes the properties of m that we have obtained so far which will be used in our uniform convergence proofs.

THEOREM 4.2. *The global error for the mean of the fast variable is uniformly bounded in* ε, *for any* $n \ge 0$*; that is, there exists a constant* C *independent of* ε *such that*

$$|e_n| \le C\sqrt{\alpha} \quad \forall\ n \ge 0.$$

Proof. From (3.7), we have $|e_{n+1}| \le a|e_n| + c_n + d_n$, where

$$\begin{aligned} a &= A^m M = (1 - F\alpha)^m \frac{1}{1 + \frac{\alpha}{\varepsilon}}, \\ c_n &= A^m M |TruncMeanFastI(t_n, k)|, \\ d_n &= \sum_{l=0}^{m-1} A^{m-1-l} |TruncMeanFastE(t_n + k + l\tau, \tau)|. \end{aligned}$$

TABLE 4.1
Properties of m.

$E_2^m(\frac{\alpha}{\alpha+\varepsilon})^2 \le C_1\alpha(1 - E_2^m)$
$E_2^m \frac{\alpha}{\alpha+\varepsilon} \le C_2\sqrt{\alpha}$
$\sqrt{E_2}^m \frac{\alpha}{\alpha+\varepsilon} \le C_3\sqrt{\alpha}$
$A^m \frac{\alpha}{\alpha+\varepsilon} \le C_4\sqrt{\alpha}$
$m\alpha\varepsilon \le C_5\sqrt{\alpha}$

Using $e_0 = 0$, we get $|e_n| \le \sum_{i=0}^{n-1} a^{n-1-i} c_i + \sum_{i=0}^{n-1} a^{n-1-i} d_i$. First, we will find some useful bounds for a and d_n. We have

$$a = (1 - F\alpha)^m \frac{1}{1 + \frac{\alpha}{\varepsilon}} \le \frac{1}{1 + Fm\alpha + \frac{\alpha}{\varepsilon}} \le \frac{1}{1+x},$$

where $x = \lambda_0 h/\varepsilon = Fm\alpha + \alpha/\varepsilon$. Some easy manipulations give

$$\begin{aligned} d_n &= \sum_{l=0}^{m-1} A^{m-1-l} |TruncMeanFastE(t_n + k + l\tau, \tau)| \\ &\le \sum_{l=0}^{m-1} (1 - F\alpha)^{m-1-l} C\alpha^2 e^{-\frac{\alpha}{\varepsilon}} (e^{-\frac{\lambda_0 h}{\varepsilon}})^n (e^{-F\alpha})^l \\ &\le C_1 \alpha x e^{-x} (e^{-x})^n, \end{aligned}$$

where we have used $1 - F\alpha < e^{-F\alpha}$. Let us denote $S_1 = \sum_{i=0}^{n-1} a^{n-1-i} c_i$ and $S_2 = \sum_{i=0}^{n-1} a^{n-1-i} d_i$. We have

$$\begin{aligned} S_1 &= \sum_{i=0}^{n-1} a^{n-1-i} c_i \le \sum_{i=0}^{n-1} c_i = A^m M \sum_{i=0}^{n-1} |TruncMeanFastI(t_i, k)| \le A^m \frac{\alpha}{\alpha + \varepsilon}, \\ S_2 &= \sum_{i=0}^{n-1} a^{n-1-i} d_i \le C_1 \alpha x e^{-x} \sum_{i=0}^{n-1} \frac{(e^{-x})^i}{(1+x)^{n-1-i}} \le C_2 \alpha. \end{aligned}$$

This implies $|e_n| \le C_1\alpha + C_2 A^m(\alpha/(\alpha+\varepsilon))$ for all $n \ge 0$. Further, using $A^m(\alpha/(\alpha+\varepsilon)) \le C\sqrt{\alpha}$, we obtain

$$|e_n| \le C\sqrt{\alpha} \quad \forall\ n \ge 0,$$

which completes the proof. Let us note that we were able to obtain the uniform convergence for any $n \ge 0$ due to the property (3.12) of m. ☐

The second moment of the fast variable also depends on the mean of the fast variable inside the composite time step. Here we derive the global mean for the corresponding error.

THEOREM 4.3. *The error of the mean of the fast variable inside the composite time step,* $e_n^{1/2,l} = \hat{g}_n^{1/2,l} - g(t_n + k + l\tau)$, $l = 0, \ldots, m-1$, *satisfies the inequality*

$$|e_n^{1/2,l}| \le C_1\sqrt{\alpha} + A^l(\varepsilon/(\alpha+\varepsilon)) |TruncMeanFastI(t_n, k)| \quad \forall\ n \ge 0. \tag{4.1}$$

Proof. Taking $m = l$ in (3.5) and (3.2) and subtracting, we obtain

$$\begin{aligned} |e_n^{1/2,l}| \le\ & A^l M |e_n| + A^l M |TruncMeanFastI(t_n, k)| \\ &+ \sum_{j=0}^{l-1} A^{l-1-j} |TruncMeanFastE(t_n + k + j\tau, \tau)| \end{aligned}$$

for any $l = 0, \ldots, m-1$. Let us denote

$$S_l = \sum_{j=0}^{l-1} A^{l-1-j} |TruncMeanFastE(t_n + k + j\tau, \tau)|.$$

Using

$$A = 1 - F\alpha < e^{-F\alpha}, \qquad Fl\alpha e^{-Fl\alpha} \leq 1,$$
$$|TruncMeanFastE(t_n + k + j\tau, \tau)| < C\alpha^2 e^{\frac{\alpha}{\varepsilon}} (e^{-\frac{\lambda_0 h}{\varepsilon}})^n (e^{-F\alpha})^j,$$

we obtain $S_l \leq C_1 \alpha$. Further, $|e_n| \leq C\sqrt{\alpha}$ for all $n \geq 0$ implies

$$|e_n^{1/2,l}| \leq C\sqrt{\alpha} + A^l M\, TruncMeanFastI(t_n, k),$$

which combined with $M = \varepsilon/(\alpha + \varepsilon)$ yields (4.1). ∎

The proof of uniform convergence for the mean of the slow variable is similar to the convergence proof for the mean of the fast variable; here we present only the main result.

THEOREM 4.4. *The global error for the mean of the slow variable, $s_n = \hat{h}_n - h(t_n)$, is uniformly bounded in ε; that is, there exists a constant C independent of ε such that*

$$|s_n| \leq C\sqrt{\alpha} \quad \forall\ n \geq 0.$$

THEOREM 4.5. *The global error of the second moment of the fast component is uniformly bounded; that is, there exists a constant C independent of ε such that*

$$|f_n| \leq C\sqrt{\alpha} \quad \forall\ n \geq 2. \tag{4.2}$$

Proof. Using the bounds for e_n and $e_n^{1/2,l}$, which are satisfied for any $n \geq 0$, and applying Theorem 3.18, we get

$$\begin{aligned}|f_{n+1}| \leq\ & E_2^m I_2 |f_n| + C_1\sqrt{\alpha}(1 - E_2^m) + C_2\alpha(1 - E_2^m) \\ & + C_3 E_2^m |TruncSecondMomentFastI(t_n, k)| \\ & + \alpha\left(C_4\sqrt{\alpha}\sum_{l=0}^{m-1} E_2^l + C_5 \frac{\varepsilon}{\alpha + \varepsilon}\sum_{l=0}^{m-1} A^l E_2^{m-1-l} |TruncMeanFastI(t_n, k)|\right).\end{aligned}$$

Recall that $E_2 = 1 - F\alpha[2(1-\eta) - F\alpha]$ and by assuming $F\alpha \leq 1 - \eta$, we obtain $1/(1 - E_2) < C(1/\alpha)$, with $C = 1/(F(1-\eta))$. This implies $\sum_{l=0}^{m-1} E_2^m \leq C((1-E_2^m)/\alpha)$. Moreover, $A \leq \sqrt{E_2}$, which implies $\sum_{l=0}^{m-1} A^l E_2^{m-1-l} \leq Cm\sqrt{E_2^m}$, with constant C independent of ε. Using these two results and combining the like terms, we obtain

$$\begin{aligned}|f_{n+1}| \leq\ & E_2^m I_2 |f_n| + C_1\sqrt{\alpha}(1 - E_2^m) + C_2 E_2^m |TruncSecondMomentFastI(t_n, k)| \\ & + C_3\alpha\frac{\varepsilon}{\alpha + \varepsilon} m\sqrt{E_2^m} |TruncMeanFastI(t_n, k)|.\end{aligned}$$

Further, $f_0 = 0$ yields

$$
\begin{aligned}
|f_n| \le\, & C_1\sqrt{\alpha}(1-E_2^m)\sum_{i=0}^{n-1}(E_2^m I_2)^i \\
& + C_2 E_2^m \sum_{i=0}^{n-1}(E_2^m I_2)^{n-1-i}|TruncSecondMomentFastI(t_i,k)| \\
& + C_3 m\frac{\alpha\varepsilon}{\alpha+\varepsilon}\sqrt{E_2^m}\sum_{i=0}^{n-1}(E_2^m I_2)^{n-1-i}|TruncMeanFastI(t_i,k)|.
\end{aligned}
$$

Using $\sum_{i=0}^{n-1} E_2^m I_2 \le \sum_{i=0}^{n-1} E_2^m \le 1/(1-E_2^m)$ and $(E_2^m I2)^{n-1-i} \le 1$, we obtain

$$
\begin{aligned}
|f_n| \le\, & C_1\sqrt{\alpha} + C_2 E_2^m \sum_{i=0}^{n-1}|TruncSecondMomentFastI(t_i,k)| \\
& + C_3 m\frac{\alpha\varepsilon}{\alpha+\varepsilon}\sqrt{E_2^m}(E_2^m I_2)^{n-1}|TruncMeanFastI(t_0,k)| \\
& + C_4 m\frac{\alpha\varepsilon}{\alpha+\varepsilon}\sqrt{E_2^m}\sum_{i=1}^{n-1}|TruncMeanFastI(t_i,k)|.
\end{aligned}
$$

Using the inequalities

$$
\begin{aligned}
& (E_2^m I_2)^{n-1} < E_2^m I_2 < C E_2^m \frac{\varepsilon}{\alpha+\varepsilon} \quad \forall\ n \ge 2, \\
& |TruncMeanFastI(t_i,k)| \le e^{-Fm\alpha}\frac{e^{\frac{\alpha}{\varepsilon}}-\frac{\alpha}{\varepsilon}-1}{e^{\frac{\alpha}{\varepsilon}}}(e^{-\frac{\alpha}{\varepsilon}})^i \quad \forall\ i \ge 1, \\
& |TruncMeanFastI(t_0,k)| \le \frac{\alpha^2}{\varepsilon^2},
\end{aligned}
$$

the properties of m listed in Table 4.1, and those of the truncation errors listed in Table 3.3, we obtain

$$
|f_n| \le C_1\sqrt{\alpha} + C_2(m\alpha E_2^m)\sqrt{E_2^m}\frac{\alpha^2}{(\alpha+\varepsilon)^2} + C_3(m\alpha e^{-Fm\alpha})\sqrt{E_2^m}\frac{\alpha}{\alpha+\varepsilon}
$$

for all $n \ge 2$. It can be easily shown that $m\alpha E_2^m < C_1$ and $m\alpha e^{-Fm\alpha} < C_2$, with constants C_1, C_2 independent of ε, which combined with $\sqrt{E_2^m}(\alpha/(\alpha+\varepsilon)) \le C_3\sqrt{\alpha}$ yields (4.2). ∎

Remark 8. Note that the term $\sqrt{E_2^m}(\alpha/(\alpha+\varepsilon))$ plays an important role in the uniform convergence. Specifically, $\alpha/(\alpha+\varepsilon)$ does not converge uniformly to 0, but when multiplied by $\sqrt{E_2}^m$ we obtain the uniform convergence. This also explains why implicit Euler method (which corresponds to $m = 0$) does not converge uniformly in ε as $\alpha \to 0$.

The variance of the numerical solution for the slow component depends on the numerical solution for the fast component inside the composite time step. The following theorem characterizes the second moment of the fast component inside the composite time step.

THEOREM 4.6. *The error of the second moment of the fast component inside the composite time step satisfies*

$$|f_n^{1/2,l}| \leq C_1\sqrt{\alpha} + C_2\sqrt{E_2^l}(\alpha/(\alpha+\varepsilon)) + C_3 E_2^l|TruncSecondMomentFastI(t_n,k)| \tag{4.3}$$

for all $l = 0, \ldots, m-1$ *and* $n \geq 2$.

Proof. Taking $m = l$ in (3.19) and (3.9) and using the bounds for E_1 and *RestE*, *RE*, we obtain

$$\begin{aligned}|f_n^{1/2,l}| \leq{}& E_2^l I_2|f_n| + E_2^l I_1|e_n| + E_2^l\Big(\alpha/(\alpha+\varepsilon)\Big)^2 \\ &+ E_2^l(\alpha)|TruncSecondMomentFastI(t_n,k)| \\ &+ C_1\alpha\sum_{j=0}^{l-1} E_2^{l-1-j}|e_n^{1/2,j}| + C_2\alpha^2\sum_{j=0}^{l-1} E_2^j.\end{aligned}$$

Let us denote $S_l = \sum_{j=0}^{l-1} E_2^{l-1-j}|e_n^{1/2,j}|$. Using the bound for $e_n^{1/2,l}$ from Theorem 4.3 and $A < \sqrt{E_2}$, we obtain

$$S_l \leq C_1\frac{\sqrt{\alpha}}{\alpha} + C_2\frac{\varepsilon}{\alpha+\varepsilon}|TruncMeanFastI(t_n,k)|\Big(l\sqrt{E_2^l}\Big).$$

Next, using $|e_n| \leq C_1\sqrt{\alpha}$ and $|f_n| \leq C_2\sqrt{\alpha}$ and combining the like terms, we obtain

$$\begin{aligned}|f_n^{1/2,l}| \leq{}& C_1\sqrt{\alpha} + C_2 E_2^l\Big(\alpha/(\alpha+\varepsilon)\Big)^2 \\ &+ C_3 E_2^l|TruncSecondMomentFastI(t_n,k)| \\ &+ C_4\alpha\frac{\varepsilon}{\alpha+\varepsilon}|TruncMeanFastI(t_n,k)|\Big(l\sqrt{E_2^l}\Big)\end{aligned}$$

for all $n \geq 2$.

From Corollary 3.7, we have

$$|TruncMeanFastI(t_n,k)| \leq C\frac{\alpha}{\varepsilon}e^{-Fm\alpha} \quad \forall\ n \geq 1.$$

Using $l\alpha e^{-Fm\alpha} \leq 1/F$, we obtain

$$\alpha\frac{\varepsilon}{\alpha+\varepsilon}|TruncMeanFastI(t_n,k)|l\sqrt{E_2^l} \leq \sqrt{E_2^l}\frac{\alpha}{\alpha+\varepsilon},$$

and (4.3) follows. ∎

THEOREM 4.7. *The global error for the variance of the slow component is uniformly bounded; that is, there exists a constant* C *independent of* ε *such that*

$$|es_n| < C\frac{1}{\ln\left(\frac{2(1-\eta)}{F\alpha}\right)} \quad \forall\ n \geq 2. \tag{4.4}$$

Proof. Using the bounds for E_x, *RestSlowI*, *RestSlowE*, and I_x in Theorem 3.25, we get

$$|es_{n+1}| \leq E_y^m I_y |es_n| + C_1 E_y^m \alpha |f_n| + C_2 \alpha^2 E_y^m + E_y^m |TruncVarSlowI(t_n, k)| + C_3 \alpha \varepsilon \sum_{l=0}^{m-1} E_y^{m-1-l} |f_n^{1/2,l}| + C_4 \alpha^2 \varepsilon \sum_{l=0}^{m-1} E_y^l.$$

Using $es_0 = 0$, we obtain

$$\begin{aligned} |es_n| \leq{}& C_1 E_y^m \alpha \sum_{i=0}^{n-1} (E_y^m I_y)^{n-1-i} |f_i| + C_2 \alpha^2 E_y^m \sum_{i=0}^{n-1} (E_y^m I_y)^i \\ &+ E_y^m \sum_{i=0}^{n-1} (E_y^m I_y)^{n-1-i} |TruncVarSlowI(t_i, k)| \\ &+ C_3 \alpha \varepsilon \sum_{i=0}^{n-1} \left((E_y^m I_y)^{n-1-i} \sum_{l=0}^{m-1} E_y^{m-1-l} |f_i^{1/2,l}| \right) \\ &+ C_4 \alpha^2 \varepsilon ((1 - E_y^m)/(1 - E_y)) \sum_{i=0}^{n-1} (E_y^m I_y)^i. \end{aligned} \tag{4.5}$$

Let us denote $S = \sum_{l=0}^{m-1} E_y^{m-1-l} |f_i^{1/2,l}|$. Using the bound for $f_i^{1/2,l}$ from Theorem 4.6, we obtain

$$S \leq C_1 \sqrt{\alpha} \sum_{l=0}^{m-1} E_y^l + C_2 \frac{\alpha}{\alpha + \varepsilon} \sum_{l=0}^{m-1} E_y^{m-1-l} E_2^l + C_3 |TruncSecondMomentFastI(t_i, k)| \sum_{l=0}^{m-1} E_y^{m-1-l} E_2^l$$

for all $i \geq 2$.

Further, let us denote $S_m = \sum_{l=0}^{m-1} E_y^{m-1-l} E_2^l$. The sequence $\{E_y^{m-1-l}\}_{l \geq 0}$ is an increasing sequence, and the sequence $\{E_2^l\}_{l \geq 0}$ is a decreasing sequence. Chebyshev's inequality implies

$$\begin{aligned} S_m &= \sum_{l=0}^{m-1} E_y^{m-1-l} E_2^l \\ &\leq \frac{1}{m} \left(\sum_{l=0}^{m-1} E_y^{m-1-l} \right) \left(\sum_{l=0}^{m-1} E_2^l \right) = \frac{1}{m} \frac{1 - E_y^m}{1 - E_y} \frac{1 - E_2^m}{1 - E_2}. \end{aligned}$$

Recall that $E_y = (1 - F\alpha\varepsilon)^2$. An easy calculation shows that there exists a constant C independent of ε such that $1/(1 - E_y) \leq C(1/\alpha\varepsilon)$. This implies

$$S \leq C_1 \sqrt{\alpha} \frac{1 - E_y^m}{\alpha\varepsilon} + C_2 \frac{\alpha}{\alpha + \varepsilon} \frac{1}{m} \frac{1 - E_y^m}{\alpha\varepsilon} \frac{1 - E_2^m}{1 - E_2} + C_3 m |TruncSecondMomentFastI(t_i, k)|.$$

We use the above bound in (4.5), as well as the following inequalities:

$$\sum_{i=0}^{n-1}(E_y^m I_y)^i \leq \sum_{i=0}^{n-1} I_y^i \leq \sum_{i=0}^{n-1}\Big(1/(1+\alpha)\Big)^i \leq 2/\alpha,$$
$$\sum_{i=0}^{n-1}|TruncVarSlowI(t_i,k)| \leq C\alpha,$$
$$\sum_{i=0}^{n-1}|TruncSecondMomentFastI(t_i,k)| \leq C\alpha/(\alpha+\varepsilon),$$
$$m\alpha\varepsilon \leq C\sqrt{\alpha}, \quad |f_1| < 1, \quad |f_i| < C\sqrt{\alpha} \quad \forall\, i \geq 2,$$
$$|f_0^{1/2,l}| < 1, \quad |f_1^{1/2,l}| < 1,$$
$$|f_i^{1/2,l}| < C_1\sqrt{\alpha} + C_2E_2^l(\alpha/(\alpha+\varepsilon)) + C_3E_2^l|TruncSecondMomentFastI(t_i,k)| \ \ \forall\, i \geq 2$$

to prove that

$$|es_n| \leq C_1\sqrt{\alpha} + C_2\frac{1}{m}\frac{\alpha}{\alpha+\varepsilon}\frac{1-E_2^m}{1-E_2}.$$

Let us denote $T(\alpha,\varepsilon) = \frac{1}{m}\frac{\alpha}{\alpha+\varepsilon}\frac{1-E_2^m}{1-E_2}$. Some tedious manipulations in (3.13) yield

$$\frac{1-E_2^m}{1-E_2} = \frac{1}{F^2(\alpha+\varepsilon)^2 + F\alpha[2(1-\eta)-F\alpha]}$$

which combined with (3.1) gives

$$T(\alpha,\varepsilon) = \frac{\alpha}{\alpha+\varepsilon}\frac{-\ln(1-F\alpha[2(1-\eta)-F\alpha])}{\ln\left(\frac{F^2(\alpha+\varepsilon)^2+F\alpha[2(1-\eta)-F\alpha]}{F^2(\alpha+\varepsilon)^2}\right)}\frac{1}{F^2(\alpha+\varepsilon)^2+F\alpha[2(1-\eta)-F\alpha]}.$$

From Lemma B.5, we have that the function $T(\alpha,\varepsilon)$ is a decreasing function of ε, thus

$$T(\alpha,\varepsilon) \leq T(\alpha,0) = \frac{-\ln(1-F\alpha[2(1-\eta)-F\alpha])}{\ln\left(1+\frac{2(1-\eta)-F\alpha}{F\alpha}\right)}\frac{1}{2(1-\eta)F\alpha}$$
$$\leq \frac{-\ln(1-2(1-\eta)F\alpha)}{2(1-\eta)F\alpha}\frac{1}{\ln\left(\frac{2(1-\eta)}{F\alpha}\right)}.$$

Assuming $F\alpha < 1/2$, we have $2(1-\eta)F\alpha \leq 1-\eta$ which implies

$$\frac{-\ln(1-2(1-\eta)F\alpha)}{2(1-\eta)F\alpha} \leq \frac{-\ln\eta}{1-\eta}.$$

Hence $T(\alpha,\varepsilon) \leq C(1/\ln(2(1-\eta)/F\alpha))$ with $C = -\ln\eta/(1-\eta)$. Note also that $\sqrt{\alpha} \leq 1/\ln(1/\alpha) \leq 1/\ln(2(1-\eta)/F\alpha)$ provided $F > 2(1-\eta)$, and combining these two results, we obtain (4.4). ∎

Remark 9. Our convergence analysis shows that in order for the uniform condition to hold, we can relax the condition on m, by allowing a range of values as opposed to a single value. In fact, examining the proofs of uniform convergence, we observe that it is adequate that m satisfies the following three conditions:

$$m\alpha\varepsilon \leq C_1\sqrt{\alpha}, \tag{4.6}$$

$$E_2^m(\alpha)\frac{\alpha^2}{(\alpha+\varepsilon)^2} \leq C_2\alpha(1-E_2^m), \tag{4.7}$$

$$\frac{1}{m}\frac{\alpha}{\alpha+\varepsilon}\frac{1-E_2^m(\alpha)}{1-E_2} \leq C_3\frac{1}{\ln\left(\frac{2(1-\eta)}{F\alpha}\right)}, \tag{4.8}$$

where C_1, C_2, C_3 are some arbitrary constants independent of α and ϵ.

The first two inequalities are monotone in m and provide upper and lower bounds. The proofs provided so far demonstrate that there exist constants C_1, C_2, C_3 such that the optimal m given by formula (3.1) satisfies these three inequalities. On the other hand, enlarging these constants if necessary, one may obtain a nonempty interval of values (dependent on α and ε) for m that satisfies these three inequalities.

Since we do not have sharp estimates of these constants, the existence of this interval of values does not provide practical algorithms as such but rather provides some comfort that an approximate choice of m might be reasonable. This will be useful in circumstances when there are more than one scale separation; for instance, if the (multidimensional) fast subsystem has a scale separation by a factor of 10 or so within itself.

5. Numerical examples.

In this section, we consider several examples and illustrate via numerical experiments the efficiency of the interlaced method. We first apply the interlaced method to our test system (2.6). Further, we consider three other examples: a fully coupled 2D linear system, a linear system with a 3D fast subsystem, and a nonlinear system. The question we want to address is how to choose the optimal m in these situations. Our numerical examples suggest that we can use the choice of m given by (3.1).

5.1. Test problem.

First we apply the interlaced method to our test system (2.6) and we compare the results with the implicit Euler method. The setup of the problem is $\lambda_0 = 1$, $\mu_0 = 1$, $\varepsilon = 10^{-5}$, $\bar{x} = 100$, $\beta = 2$, and the time interval for simulations is [0,1]. The initial conditions are $X(0) = 300$, $Y(0) = 500$. The exact values of the variances are $Var(X(1)) = 10000$ and $Var(Y(1)) = 34594$.

For the numerical methods, we use $\alpha = 0.01$. This gives the implicit time step $k = 10^{-2}$. For the interlaced method, we take $F = 10$ which gives $m = 24$. Therefore, the interlaced time step is $h = 10^{-2} + 24 \cdot 10^{-6}$. The results are shown in Figures 5.1, 5.2, and 5.3, and Tables 5.1 and 5.2.

Figure 5.1 shows two sample paths for each variable along with the corresponding expected values. One sample path is obtained with benchmark explicit Euler with time step $\tau = 10^{-6}$ and the other one corresponds to the interlaced solution. The histograms of fast and slow variables at time $T = 1$ are shown in Figure 5.2. We compare the results obtained by the interlaced method and the implicit method, with the benchmark explicit Euler. We see that the implicit solver produces a distribution which is too narrow.

Figure 5.3 shows the time evolution of the fast and slow variances. The implicit method underestimates the variance of both components while the interlaced method gives the correct variances.

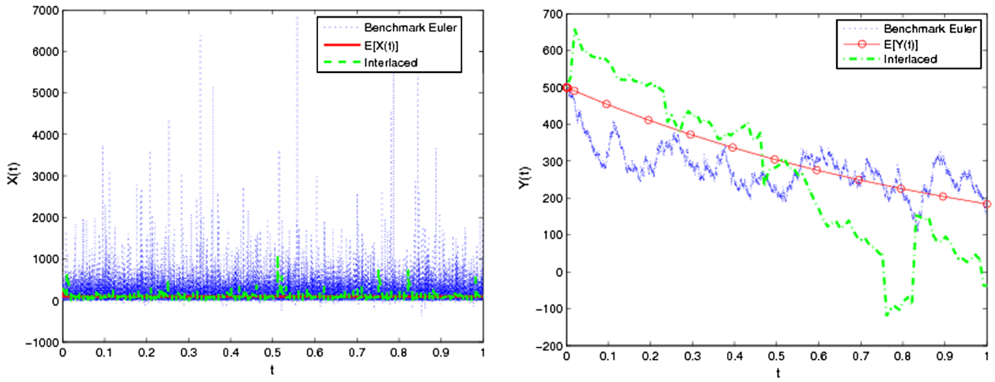

FIG. 5.1. *Sample paths for $X(t)$ and $Y(t)$ and the corresponding mean. The left figure shows a sample path for $X(t)$ obtained by benchmark explicit Euler, the interlaced method, and $E[X(t)]$. The right figure shows the corresponding sample paths for $Y(t)$ and $E[Y(t)]$.*

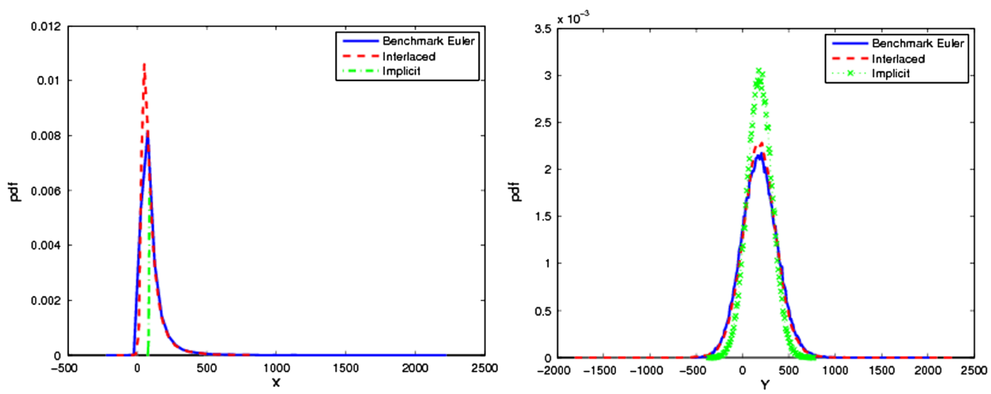

FIG. 5.2. *Histograms (100,000 samples) of $X(1)$ and $Y(1)$. The left figure shows the histogram of the fast variable obtained by benchmark explicit Euler, the interlaced method, and the implicit Euler. The right figure shows the histogram of the slow variable.*

Tables 5.1 and 5.2 show the values of the stationary variances of the fast and slow variables obtained with the interlaced method and implicit method with fixed time steps $k = 10^{-2}$, 10^{-5}, and 10^{-6}. The interlaced method with composite time step $h = 10^{-2} + 24 \cdot 10^{-6}$ gives the variance close to the true variance for both variables. The implicit method requires a much smaller time step, $k = 10^{-6}$, to compute the variances correctly. This makes the interlaced method almost 400 times faster than the implicit method for this example.

5.2. Fully coupled 2D system. Now we consider a fully coupled 2D system. The goal is to show that the optimal m given by (3.1) works in this case too. This is because for small ε, $Y(t)$ behaves like a constant in the equation of $X(t)$ and hence can be assimilated with $\bar{x}$. Since m does not depend on $\bar{x}$, we expect that for small values of ε the optimal m is independent of the slow variable.

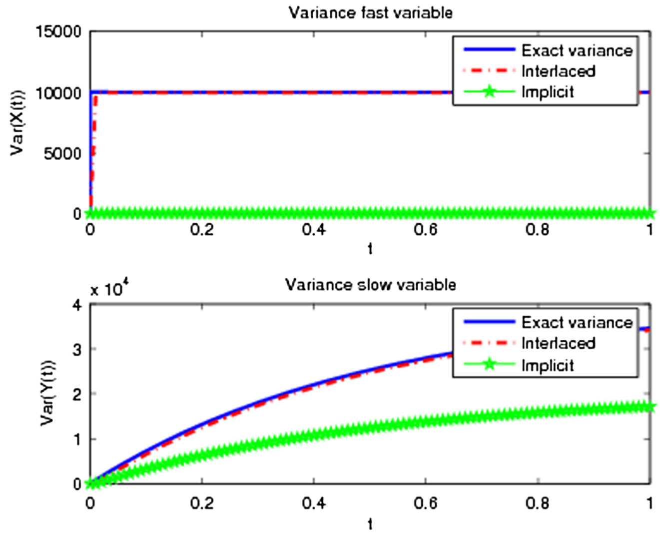

FIG. 5.3. *Variance of the exact solution and numerical solutions of system* (2.6). *The top figure shows the time evolution of the variance of the fast variable and the bottom figure shows the variance for the slow variable.*

TABLE 5.1
Variance of the fast variable of system (2.6).

Fast variable	Interlaced	Implicit	Implicit	Implicit
Time step	$10^{-2} + 24 \cdot 10^{-6}$	10^{-2}	10^{-5}	10^{-6}
$Var(\hat{X}(1))$	9958	10	5000	9091

TABLE 5.2
Variance of the slow variable of system (2.6).

Slow variable	Interlaced	Implicit	Implicit	Implicit
Time step	$10^{-2} + 24 \cdot 10^{-6}$	10^{-2}	10^{-5}	10^{-6}
$Var(\hat{Y}(1))$	34196	17148	25940	33014

We consider the following system:

$$dX(t) = \left(-\frac{1}{\epsilon}X(t) + \frac{0.1}{\epsilon}\,Y(t) + \frac{500}{\epsilon}\right)dt + \left(\frac{1}{\sqrt{\epsilon}}X(t) + \frac{0.01}{\sqrt{\epsilon}}\,Y(t)\right)dB(t),$$
$$dY(t) = (X(t) - Y(t) + 900)\,dt + (0.1X(t) + 0.001\,Y(t))\,dB(t). \tag{5.1}$$

We take $\varepsilon = 10^{-10}$ and the time simulation interval [0,0.1]. The initial conditions are $X(0) = 300$, $Y(0) = 500$, and the values for the exact variances are $Var(X(0.1)) = 319231$ and $Var(Y(0.1)) = 627$. We further apply the interlaced method with $\alpha = 0.0025$ and we compare the results with the trapezoidal method with different time steps. The results are shown in Tables 5.3 and 5.4. We can see that for this example,

Table 5.3
Interlaced method applied to system (5.1).

α	m	$Var(\hat{X}(0.1))$	$Var(\hat{Y}(0.1))$	CPU time
0.0025	149	319120	614	0.275s

Table 5.4
Trapezoidal method applied to system (5.1).

k	$Var(\hat{X}(0.1))$	$Var(\hat{Y}(0.1))$	CPU time
0.0025	2596	398	0.012s
0.000016	56681	416	0.532s
0.000008	166595	460	1.058s
0.000004	296996	548	2.117s
0.000002	318214	607	4.245s

the interlaced method is more efficient than the trapezoidal, which requires a much smaller time step in order to get the variances correct. This makes the interlaced method almost 20 times faster than the trapezoidal method.

5.3. Fast subsystem. Further, we investigate a linear system with a fast subsystem. When the fast subsystem is diagonal, we choose the optimal m corresponding to the fastest reaction. Here we consider the following system:

$$\begin{aligned} dX_1(t) &= \left(-\frac{10}{\epsilon}X_1(t) + \frac{100}{\epsilon}\right)dt + \frac{4}{\sqrt{\epsilon}}X_1(t)dB(t), \\ dX_2(t) &= \left(-\frac{9}{\epsilon}X_2(t) + \frac{500}{\epsilon}\right)dt + \frac{3}{\sqrt{\epsilon}}X_2(t)dB(t), \\ dX_3(t) &= \left(-\frac{7}{\epsilon}X_3(t) + \frac{300}{\epsilon}\right)dt + \frac{2}{\sqrt{\epsilon}}X_3(t)dB(t), \\ dX_4(t) &= -2X_4(t)dt - (2X_1(t) + 5X_2(t) + 7X_3(t))dB(t). \end{aligned} \tag{5.2}$$

We take $\varepsilon = 10^{-10}$ and the time interval for simulations [0,0.1]. The initial conditions are $X_1(0) = 300$, $X_2(0) = 500$, $X_3(0) = 100$, and $X_4(0) = 200$. The values for the exact variances at time t=0.1 are $Var(X_1(0.1))$=400, $Var(X_2(0.1))$=3086, $Var(X_3(0.1)) = 735$, and $Var(X_4(0.1)) = 49603$.

We apply the interlaced method with $\alpha = 0.0025$. This gives $m = 249$, and the results are shown in Table 5.5.

Next, we apply the trapezoidal method with the same time step as the interlaced method, $k = \alpha = 0.00025$. In this case, the trapezoidal method gives the incorrect variances as shown in Table 5.6. Our simulations show that a much smaller time step is needed for the trapezoidal method. For example, $k = \alpha/1000 = 2.5 \times 10^{-7}$ gives results similar to those obtained by the interlaced method, but only for the fast variables; the variance of the slow variable is still overestimated. Even for this choice of time step,

Table 5.5
Interlaced method applied to system (5.2).

α	m	$Var(\hat{X}_1(0.1))$	$Var(\hat{X}_2(0.1))$	$Var(\hat{X}_3(0.1))$	$Var(\hat{X}_4(0.1))$
0.00025	249	400	3153	743	51105

Table 5.6
Trapezoidal method applied to system (5.2).

α	$Var(\hat{X}_1(0.1))$	$Var(\hat{X}_2(0.1))$	$Var(\hat{X}_3(0.1))$	$Var(\hat{X}_4(0.1))$
0.00025	100	192	6	874283
0.000000025	539	3086	735	92592

Table 5.7
Variance of the fast/slow variables of system (5.3).

	Benchmark	Interlaced	Implicit
Variance fast variable: $X(0.1)$	10514	10360	42.27
Variance slow variable: $Y(0.1)$	84312	89473	49812

the interlaced method is three times faster than the trapezoidal, but as the results show, the trapezoidal methods need an even smaller time step.

5.4. Nonlinear system. Finally, we investigate nonlinear systems. We study systems which are nonlinear only in the slow variable. In this case, we can choose the optimal m to be determined by the fast variable only, that is, m given by(3.1). We consider the system

$$dX(t) = -\frac{\lambda_0}{\varepsilon} X(t)\,dt + \frac{\lambda_0}{\varepsilon}\bar{x}\,dt + \frac{\mu_0}{\sqrt{\varepsilon}} X(t)\,dB(t),$$

$$dY(t) = -\lambda_0 \sin(Y(t))\,dt + (bX(t) + c\cos(X(t)))\,dB(t), \tag{5.3}$$

and we compare the interlaced method with the benchmark explicit Euler. The setup for this problem is $\lambda_0 = 1$, $\mu_0 = 1$, $\bar{x} = 100$, $b = 2$, $c = 0.5$, and $\varepsilon = 10^{-5}$. The initial conditions are $X(0) = 500$, $Y(0) = 300$. For the interlaced method, we choose $\alpha = 0.01$ which gives $m = 24$. For the benchmark Euler, the time step is $\tau = 10^{-6}$. We run 100,000 Monte Carlo simulations, and the results are presented in Table 5.7. We can see that for this problem the interlaced method performs very well. Once again, we see that the variance obtained by the implicit Euler is underestimated for both fast and slow variables.

6. Concluding remarks. We have proposed a strategy called the "interlaced Euler method" for computing numerical solutions of stiff systems of SDEs by interlacing one large implicit time step with m small explicit time steps. We have proven the uniform convergence (of the mean and variance) of the interlaced method with respect to the time scale separation parameter ε for a singularly perturbed family of 2D linear systems. We have also shown via numerical experiments the efficiency of the interlaced Euler method for linear/nonlinear systems.

More importantly, the interlaced Euler method serves as a template and a proof of concept, rather than as a final product. We regard the key message to be the feasibility of the development of time-stepping methods for stiff SDEs whose performance is uniform in the time scale separation parameter ε. We have shown that such methods are possible when explicit time steps of the order of the fast time scale are interlaced with implicit time steps of the order of the slow time scale.

We have limited the convergence analysis to the mean and the variance to manage the complexity. Our proof of uniform convergence was "hand crafted" rather than obtained via a more general overarching approach. In the future, we hope that such an approach will be possible. We also anticipate the development of more sophisticated time-stepping methods that can deal with multiple time scales as well as adaptivity of the step sizes.

Appendix A. Derivation of the formula for the variance quotient. Equation (2.5) for the variance quotient is somewhat tedious to derive and was accomplished with the aid of Maple. We show some steps here. We first recall two basic facts from probability theory. If X and Y are two random variables, and if $E(Y|X)$ and $Var(Y|X)$, respectively, denote the conditional expectation and conditional variance of Y given X, then $E(Y)$ and $Var(Y)$ are given by

$$E(Y) = E(E(Y|X)) \tag{A.1}$$

and

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)). \tag{A.2}$$

Given the test equation

$$dX(t) = -\lambda X(t)dt + \lambda\bar{x}dt + \mu X(t)dB(t),$$

let us consider the application of one step of implicit Euler with step size k starting at state $\hat{X}_n$ to obtain an intermediate state $\hat{X}_n^{1/2,0}$. This is given by

$$\hat{X}_n^{1/2,0} = (1/(1+\lambda k))\hat{X}_n + (\mu/(1+\lambda k))\hat{X}_n(B(t_n+k) - B(t_n)) + \lambda\bar{x}(k/(1+\lambda k)).$$

Since $\hat{X}_n$ and $B(t_n+k) - B(t_n)$ are independent it follows that

$$E(\hat{X}_n^{1/2,0}|\hat{X}_n) = (1/(1+\lambda k))\hat{X}_n + \lambda\bar{x}(k/(1+\lambda k)),$$

and that

$$Var(\hat{X}_n^{1/2,0}|\hat{X}_n) = (\mu^2 k/(1+\lambda k)^2)(\hat{X}_n)^2.$$

By using (A.1) and (A.2) we obtain that

$$E(\hat{X}_n^{1/2,0}) = (1/(1+\lambda k))E(\hat{X}_n) + \lambda\bar{x}(k/(1+\lambda k)),$$

and that

$$Var(\hat{X}_n^{1/2,0}) = ((1+\mu^2 k)/(1+\lambda k)^2)\ Var(\hat{X}_n) + (\mu^2 k/(1+\lambda k)^2)(E(\hat{X}_n))^2.$$

Now consider the application of m steps of explicit Euler with step size τ starting at state $\hat{X}_n^{1/2,0}$. One obtains intermediate states $\hat{X}_n^{1/2,l}$ for $l = 1, \ldots, m$ with $\hat{X}_n^{1/2,m} = \hat{X}_{n+1}$. Following similar calculations, we may obtain the recurrences

$$E(\hat{X}_n^{1/2,l+1}) = (1 - \lambda\tau)E(\hat{X}_n^{1/2,l}) + \lambda\tau\bar{x}$$

and

$$Var(\hat{X}_n^{1/2,l+1}) = [(1 - \lambda\tau)^2 + \mu^2\tau]\,Var(\hat{X}_n^{1/2,l}) + \mu^2\tau(E(\hat{X}_n^{1/2,l}))^2.$$

Combining the above, we may obtain the recurrences for $E(\hat{X}_n)$ and $Var(\hat{X}_n)$ for the interlaced method. The recurrence for the mean is given by

$$E(\hat{X}_{n+1}) = MA^m E(\hat{X}_n) + NA^m + B\frac{1 - A^m}{1 - A},$$

where

$$M = \frac{1}{1 + \lambda k}, \quad N = \bar{x}\frac{\lambda k}{1 + \lambda k}, \quad A = 1 - \lambda\tau, \quad B = \lambda\tau\bar{x}.$$

By our condition on τ, $|A| < 1$ and $|M| < 1$ by assumption $\lambda > 0$. Thus, this recurrence is stable and the asymptotic value $E(\hat{X}_\infty) = \bar{x}$ is the exact asymptotic mean. The recurrence formula for $Var(\hat{X}_n)$ is considerably messy and was computed with the aid of Maple. We only show its form in partial detail:

$$Var(\hat{X}_{n+1}) = I_2 E_2^m\, Var(\hat{X}_n) + \tilde{A}(E(\hat{X}_n))^2 + \tilde{B}E(\hat{X}_n) + \tilde{C},$$

where

$$I_2 = \frac{1 + \mu^2 k}{(1 + \lambda k)^2}$$

and

$$E_2 = (1 - \lambda\tau)^2 + \mu^2\tau,$$

and $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ depend only on λ, μ, $\bar{x}$, τ, k. From the assumptions on the problem and on τ, it follows that $|E_2| < 1$ and $|I_2| < 1$ and hence stability of the recurrence is guaranteed. Additionally, the asymptotic variance of the method $Var(\hat{X}_\infty)$ must satisfy

$$Var(\hat{X}_\infty) = I_2 E_2^m\, Var(\hat{X}_\infty) + \tilde{A}\bar{x}^2 + \tilde{B}\,\bar{x} + \tilde{C}.$$

Finally, using the exact asymptotic variance of the problem $Var(X(\infty)) = \frac{\mu^2\bar{x}^2}{2\lambda - \mu^2}$, the variance quotient (2.5) may be obtained with the aid of Maple.

Finally, we observe that as $(\tau, k) \to (0, 0)$, it can be shown (again with some careful manipulations) from (2.5) that the variance quotient approaches 1. This is consistent with the fact that the Euler methods are convergent.

Appendix B. Some relevant lemmas. In this appendix, we collect some lemmas relevant for the analysis of uniform convergence. We are omitting the proof since they can be easily verified.

LEMMA B.1. *The solution of the linear difference equation*

$$x_{n+1} = Ax_n + By_n + Cz_n \quad \text{is} \quad x_n = A^n x_0 + B\sum_{i=0}^{n-1} A^{n-1-i} y_i + C\sum_{i=0}^{n-1} A^{n-1-i} z_i.$$

LEMMA B.2. *For any real α which satisfies $F\alpha < 1-\eta$, the following inequality holds:*

$$\frac{1}{-\ln\left(1 - F\alpha[2(1-\eta) - F\alpha]\right)} < \frac{1}{F(1-\eta)\alpha}.$$

LEMMA B.3. *For any real $x > 0$, we have $(1 - e^{-x} - xe^{-x})/(1 - e^{-x}) \leq x$.*

LEMMA B.4. *For any real $k > 0$, we have $(k - x + xe^{-k/x})/(1 - e^{-k/x}) \leq k \; \forall x > 0$.*

LEMMA B.5. *For any $\alpha > 0$ satisfying $F\alpha < 2(1-\eta)$, the function*

$$f(x) = \frac{-\alpha \, \ln\left(1 - F\alpha[2(1-\eta) - F\alpha]\right)}{(\alpha + x)(F^2(\alpha+x)^2 + F\alpha[2(1-\eta) - F\alpha])\ln\left(1 + \frac{F\alpha[2(1-\eta)-F\alpha]}{F^2(\alpha+x)^2}\right)}$$

is a decreasing function of x, for $x \geq 0$.

LEMMA B.6. *The first two moments of the fast and slow component satisfy the following inequalities:*

$$\begin{aligned}
&|g(t)| < C_1, \quad |h(t)| < C_2, \quad |u(t)| < C_3, \quad |v(t)| < C_4 \quad \forall\ t \geq 0 \\
&|u'(t)| < \frac{C_5}{\varepsilon}\, e^{-\frac{C_0}{\varepsilon}t}, \quad |u''(t)| < \frac{C_6}{\varepsilon^2}\, e^{-\frac{C_0}{\epsilon}t} \quad \forall\ t \geq 0 \\
&|g''(t)| < \frac{C_7}{\varepsilon^2}\, e^{-\frac{\lambda_0}{\varepsilon}t}, \quad |h''(t)| \leq C_8 \quad \forall\ t \geq 0 \\
&|v''(t)| < C_9 + \frac{C_{10}}{\varepsilon}\, e^{-\frac{C_0}{\varepsilon}t} \quad \forall\ t \geq 0,
\end{aligned}$$

where the constants C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10} are independent of ε.

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