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Option Pricing with a Pentanomial Lattice Model that Incorporates Skewness and Kurtosis

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ABSTRACT *This paper analyzes a pentanomial lattice model for option pricing that incorporates skewness and kurtosis of the underlying asset. The lattice is constructed using a moment matching procedure, and explicit positivity conditions for branch probabilities are provided in terms of skewness and kurtosis. We also explore the limiting distribution of this lattice, which is compound Poisson, and give a Fourier transform based formula that can be used to more efficiently price European call and put options. An example illustrates some of the features of this model in capturing volatility smiles and smirks.*

KEY WORDS: Lattice, volatility smile, option pricing

Introduction

Lattices for option pricing were first introduced in 1979 in the pioneering work of Cox *et al.* (1979) and Rendleman and Bartter (1979). In particular, Cox *et al.* used a binomial lattice to model geometric Brownian motion and an exponential Poisson process. An attractive property of their model is that the binomial lattice for geometric Brownian motion is consistent with the standard Black–Scholes formula for European options in that no mismatch occurs if a lattice is used to price an American option where early exercise is not optimal. Since that initial work, lattices have become a standard tool in option pricing, especially for valuing American options.

Due to the simplicity and versatility of lattice models, a number of extensions to the basic model have been proposed. Lattices have been constructed for more than a single underlying asset, and for more complicated models of a single underlying asset (Rubinstein, 1998; Yamada and Primbs, 2001; Amin, 1993; Boyle, 1988). In particular, lattice models have also been used extensively to extract implied volatility surfaces as in the well known works of Rubinstein (1994), Derman and Kani (1994a, b), Dupire (1994). Additionally, lattice models have been proposed as a method of

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capturing skewness and kurtosis in the underlying asset. Specifically, Rubinstein (1994) proposed a lattice model that incorporates skewness and kurtosis by using an Edgeworth expansion. While such models have been proposed, they are far fewer in number than models that have been proposed for European option pricing under skewness and kurtosis. In that area, researchers have proposed exponential Levy process models (Carr and Madan, 1999; Chan, 1999), Edgeworth and Gram–Charlier expansions (Corrado and Su, 1996, 1997; Jarrow and Rudd, 1988), and others (Barndorff-Nielsen, 1998).

This paper looks at the issue of incorporating skewness and kurtosis directly in a lattice model using a moment-matching procedure, and explores the pricing consequences of such a model. By capturing the first four moments, this lattice model is able to match skewness and kurtosis, and hence can produce volatility smiles and smirks. For a range of skewness and kurtosis values, we provide explicit positivity conditions for branch probabilities. Hence, we give a thorough characterization of this lattice model and its features and limitations. Additionally, we analyse the possible limits in continuous time of this model, which lead to probability distributions that can be used to price European options in a manner consistent with the assumptions of the lattice model. It turns out that Fourier transform techniques (Carr and Madan, 1999) can be used to more efficiently price European options using the limiting distributions, and those pricing formulas are derived as well. Therefore, beginning from a simple lattice model, we develop a consistent and complete approach to pricing both American (using the lattice) and European (using Fourier transform techniques) options. Important aspects of this model are highlighted along the way.

The paper proceeds as follows. In the following section we develop the basic pentanomial lattice model. Conditions for positivity of probabilities are provided. We compute the limiting distribution in the third section. The fourth section uses Fourier transform pricing in conjunction with the limiting distribution to present a pricing formula for European calls and puts. The fifth section provides a numerical example and illustrates some of the features of the model.

The Lattice Model

Since an exponential Levy process model has the form

$$S_t = S_0 e^{X_t}$$

by constructing a lattice to match X_t, exponentiation provides a lattice model for S_t. Even before considering the problem of creating a lattice for the process X_t, we begin with the more generic setup of determining a discrete random variable that matches a given set of moments.

A Discrete Moment-matching Random Variable

We begin with the generic set-up of matching moments of a random variable X with a discrete random variable Z. Hence, consider a random variable X. Let m_j denote its jth raw moment, μ_j its jth central moment, and c_j its jth cumulant. We will construct a discrete random variable Z that matches the moments of the random variable X.

Let Z denote the discrete random variable which takes the values

$$Z = m_1 + (2l - L - 1)\alpha, \quad l = 1, \ldots, L \quad \text{with probabilities } p_l$$

where α is a parameter and m_1 is the mean of X. Note that the distance between outcomes is 2α. Hence, Z is a discrete random variable that may take on L values. Note that this choice of values for Z is such that sums of Z correspond to a recombining lattice.

Equations for the moments of Z. We would like to require that the moments of Z match those of X. Using the central moments of X, this would require the following equations to hold:

$$\sum_{l=1}^{L} ((2l - L - 1)\alpha)^j p_l = \mu_j \tag{1}$$

where the μ_j are the jth central moments of X, and $\mu_1 = 0$. (The condition that $\mu_1 = 0$ ensures that the mean of Z is m_1.)

Matching four moments. To proceed further, let us consider matching four moments, and choose $L = 5$. (Hence, this will ultimately correspond to a pentanomial lattice (i.e. five branches).) We choose to focus on four moments because in financial problems it is common to consider skewness and kurtosis of underlying asset return distributions, which requires knowledge of the first four moments. With four moments, one could conceivably use a quadranomial lattice (i.e. four branches), however the recombination condition along with the requirement of non-negative probabilities is quite limiting in terms of the range of skewness and kurtosis that can be captured. A pentanomial lattice allows for a much wider range of behaviour without significant additional complication, and hence is our focus.

Solving (1) for p_l with $l = 1, \ldots, 5$ gives

$$Z_5 = \begin{cases} m_1 - 4\alpha & p_1 = \frac{\left(\mu_4 - 4\alpha^2\mu_2 - 4\alpha\mu_3\right)}{384\alpha^4} \\ m_1 - 2\alpha & p_2 = \frac{\left(-\mu_4 + 16\alpha^2\mu_2 + 2\alpha\mu_3\right)}{96\alpha^4} \\ m_1 & p_3 = 1 + \frac{\left(-20\alpha^2\mu_2 + \mu_4\right)}{64\alpha^4} \\ m_1 + 2\alpha & p_4 = \frac{-2\alpha\mu_3 - \mu_4 + 16\alpha^2\mu_2}{96\alpha^4} \\ m_1 + 4\alpha & p_5 = \frac{\left(\mu_4 - 4\alpha^2\mu_2 + 4\alpha\mu_3\right)}{384\alpha^4} \end{cases} \tag{2}$$

where α is still a free parameter. Note that because of symmetry we may take $\alpha \geqslant 0$ without loss of generality and that the choice $\alpha = 0$ leads to a degenerate lattice. Hence throughout the rest of this paper we shall assume that $\alpha > 0$. An important question is for what values of α are all the probabilities nonnegative. The following proposition answers this question.

Proposition 1. Provided $\mu_2\mu_4 \geq 3\mu_3^2$ and $25\mu_2^2 \geq 16\mu_4$ (or equivalently $\kappa \geqslant 3s^2 - 3$ and $\kappa \geq -\frac{23}{16}$, where $s = \frac{\mu_3}{\mu_2^{3/2}}$ denotes skewness and $\kappa = \frac{\mu_4}{\mu_2^2} - 3$ is excess kurtosis), there

exists a range of values of α given by

$$\frac{1}{16\mu_2}\left(\mu_3+\left(\mu_3^2+16\mu_2\mu_4\right)^{\frac{1}{2}}\right)\leq\alpha\leq\frac{1}{4\mu_2}\left(-2\mu_3+2\left(\mu_3^2+\mu_2\mu_4\right)^{\frac{1}{2}}\right) \tag{3}$$

that includes

$$\hat{\alpha}=\sqrt{\frac{\mu_4}{12\mu_2}}=\sigma\sqrt{\frac{3+\kappa}{12}} \tag{4}$$

for which all the probabilities p_l, $l=1, \ldots, 5$, are nonnegative.
The proof of this proposition is contained in Appendix A.

Proposition 1 provides a sufficient condition for the nonnegativity of the probabilities. Additionally, it can be shown that $\hat{a}$ will result in nonnegative probabilities for values of μ_2, μ_3, and μ_4 satisfying $\mu_2\mu_4\geq 3\mu_3^2$ and $3\mu_2^2\geq 2\mu_4$ (or equivalently $\kappa\geqslant 3s^2-3$ and $\kappa\geq-\frac{3}{2}$,), which is slightly more inclusive than the conditions in the proposition. However, (3) is not valid under these more inclusive conditions. For values of excess kurtosis below $-\frac{23}{16}$, the conditions for nonnegativity are more complex. We do not include these conditions as positive excess kurtosis ($\kappa>0$) is the typical situation of interest in finance, and is covered by Proposition 1.

The proposition determines a range of allowable skewness and kurtosis values that are compatible with a simple five branch lattice model. In fact, in the next section when a lattice model is considered explicitly, it will determine a valid range of skewness and kurtosis (or equivalently cumulants) that are consistent with a lattice model. The proposition also gives conditions on the spacing between outcomes of the random variable Z (given by the parameter α). This result determines allowable spacing in a lattice model as well. Finally, a simple formula for that spacing is given by $\hat{\alpha}$. Hence, this simple proposition provides the foundation for the creation of a lattice model, and also determines it features and limitations.

One may attempt to carry out a similar analysis as in Proposition 1 except for multinomial lattices with more than five branches. However, even with five branches, the algebra required to obtain the conditions in Proposition 1 is tedious, and with even more branches, it would require even further effort. When only skewness and kurtosis need to be captured, the pentanomial lattice is general enough to capture most parameter ranges of interest, and simple enough to allow for relatively clean characterizations, as in Proposition 1. Therefore, we find the pentanomial lattice a suitable compromise between complexity and practicality.

Creating a lattice model. Above, we have shown how to match the moments of a random variable X with a discrete random variable Z. To convert this to a lattice model, we first assume that X_t is a Levy process with moments. Then for any given time t, the results of the previous section indicate how to match the moments of X_t with a discrete random variable $Z(t)$. Since X_t is a Levy process, its cumulants scale linearly with time, and hence we may specify its cumulants at any time t by specifying its yearly cumulants. That is, let c_j be the jth cumulant of X_1, then the jth cumulant of X_t is $c_j t$.

Let τ be an increment in t that represents the step size of the lattice. To create a lattice model we model each increment X_τ with the discrete random variable $Z(\tau)$ that matches its moments (and cumulants). This corresponds to the following recombining lattice model.

Lattice Model

Let S_0 be the initial underlying price. Then a lattice model approximating $S_t = S_0 e^{X_t}$ is given by $S_n(\tau) = S_0 \exp\left(\sum_{k=1}^{n} Z_k(\tau)\right)$ where n denotes the number of time steps of size τ and the $Z_k(\tau)$ are iid random variables distributed as

$$Z(\tau) = \begin{cases} c_1\tau - 4\alpha & p_1(\tau) = \frac{\left(c_4\tau + 3c_2^2(\tau)^2 - 4\alpha^2 c_2\tau - 4\alpha c_3\tau\right)}{384\alpha^4} \\ c_1\tau - 2\alpha & p_2(\tau) = \frac{\left(-\left(c_4\tau + 3c_2^2(\tau)^2\right) + 16\alpha^2 c_2\tau + 2\alpha c_3\tau\right)}{96\alpha^4} \\ c_1\tau & p_3(\tau) = 1 + \frac{\left(-20\alpha^2 c_2\tau + \left(c_4\tau + 3c_2^2(\tau)^2\right)\right)}{64\alpha^4} \\ c_1\tau + 2\alpha & p_4(\tau) = \frac{-2\alpha c_3\tau - \left(c_4\tau + 3c_2^2(\tau)^2\right) + 16\alpha^2 c_2\tau}{96\alpha^4} \\ c_1\tau + 4\alpha & p_5(\tau) = \frac{\left(\left(c_4\tau + 3c_2^2(\tau)^2\right) - 4\alpha^2 c_2\tau + 4\alpha c_3\tau\right)}{384\alpha^4} \end{cases} \tag{5}$$

Figure 1 shows one step of length τ of the lattice.

Since the lattice model above is stated in terms of cumulants of X_t, we also restate the positivity condition of Proposition 1 in terms of cumulants.

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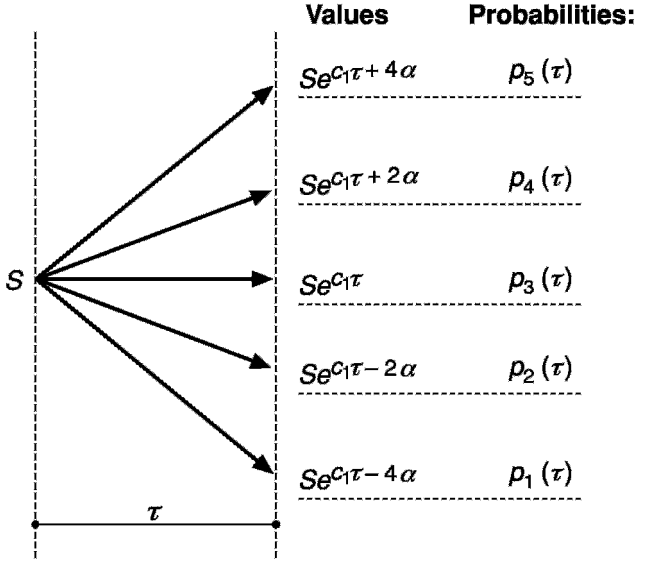

Figure 1. One step of the pentanomial lattice

Proposition 2. Provided

$$c_4 c_2 \geq 3c_3^2 - 3c_2^3\tau \quad \text{and} \quad c_4 \geq -\frac{23}{16}c_2^2\tau \tag{6}$$

there exists a range of values of α given by

$$\frac{1}{16c_2\tau}\left(c_3\tau + \left(c_3^2\tau^2 + 16c_2\tau\left(c_4\tau + 3c_2^2\tau^2\right)\right)^{\frac{1}{2}}\right) \leq \alpha \leq \frac{1}{4c_2\tau}\left(-2c_3\tau + 2\left(c_3^2\tau^2 + c_2\tau\left(c_4\tau + 3c_2^2\tau^2\right)\right)^{\frac{1}{2}}\right) \tag{7}$$

which includes

$$\hat{\alpha} = \frac{1}{2}\sqrt{c_2\tau + \frac{c_4}{3c_2}} \tag{8}$$

for which the probabilities p_l, $l=1, \ldots, 5$, are non-negative.

By stating these results in terms of the cumulants of the process X_t, this shows how the conditions scale with the time step τ. In particular, we are interested in the limit as $\tau \to 0$.

Limits of the Lattice Model

In this section we consider the limits in continuous time of the lattice model as $\tau \to 0$. For this purpose, we assume that the third and fourth cumulants c_3 and c_4 are not both zero. If they are zero then it is well known that the lattice will converge to a geometric Brownian motion.

For this discrete model to have a limit in continuous time, the positivity condition must be feasible as $\tau \to 0$. In the limit, (6) becomes

$$c_4 c_2 \geq 3c_3^2 \quad \text{and} \quad c_4 \geq 0 \tag{9}$$

Hence, we further assume that (9) holds. Note that the requirement of $c_4 \geqslant 0$ is equivalent to the assumption of nonnegative excess kurtosis.

We would like this lattice to have a well defined limit as the step size approaches zero. Therefore, we also assume that α has a limit as $\tau \to 0$. Let us denote this limit by $\alpha_0 = \lim_{t \to 0} \alpha$ where it must fall in the range specified by Equation 7.

To proceed, let us define λ and the new probabilities q_1, q_2, q_4, q_5 in terms of the branch probabilities as follows:

$$\begin{cases} \lim_{\tau \to 0}(1/\tau)p_1(\tau) = \frac{\left(-4\alpha_0^2 c_2 - 4\alpha_0 c_3 + c_4\right)}{384\alpha_0^4} = \lambda q_1 \\ \lim_{\tau \to 0}(1/\tau)p_2(\tau) = \frac{16\alpha_0^2 c_2 + 2\alpha_0 c_3 - c_4}{96\alpha_0^4} = \lambda q_2 \\ \lim_{\tau \to 0}(1/\tau)(p_3(\tau) - 1) = \frac{\left(-20\alpha_0^2 c_2 + c_4\right)}{64\alpha_0^4} = -\lambda \\ \lim_{\tau \to 0}(1/\tau)p_4(\tau) = \frac{16\alpha_0^2 c_2 - 2\alpha_0 c_3 - c_4}{96\alpha_0^4} = \lambda q_4 \\ \lim_{\tau \to 0}(1/\tau)p_5(\tau) = \frac{\left(-4\alpha_0^2 c_2 + 4\alpha_0 c_3 + c_4\right)}{384\alpha_0^4} = \lambda q_5 \end{cases}$$

Therefore,

$$q_1 + q_2 + q_4 + q_5 = 1$$

If we use $\hat{\alpha}$ as given in (8) then these quantities simplify to

$$\begin{cases} \lambda = \dfrac{3c_2^2}{2c_4} \\ q_1 = \dfrac{1}{6}\left(1 - c_3\sqrt{\dfrac{3}{c_2 c_4}}\right) \\ q_2 = \dfrac{1}{3}\left(1 + c_3\sqrt{\dfrac{3}{c_2 c_4}}\right) \\ q_4 = \dfrac{1}{3}\left(1 - c_3\sqrt{\dfrac{3}{c_2 c_4}}\right) \\ q_5 = \dfrac{1}{6}\left(1 + c_3\sqrt{\dfrac{3}{c_2 c_4}}\right) \end{cases} \tag{10}$$

In the limit one might guess that $Z(\tau)$ approximates the increment of a compound Poisson process given by

$$c_1 t + \sum_{k=0}^{N_t} W_k$$

where N_t is a Poisson process with intensity λ and the W_k are iid random variables with the distribution

$$W_k = \begin{cases} -4\alpha_0 & \text{with probability} \quad q_1 \\ -2\alpha_0 & \text{with probability} \quad q_2 \\ 2\alpha_0 & \text{with probability} \quad q_4 \\ 4\alpha_0 & \text{with probability} \quad q_5 \end{cases} \tag{11}$$

This is indeed the case, as presented in the following:

Proposition 3. Consider a fixed interval $[0, T]$ of time in which the number of steps n may be increased. Thus the step size is $\tau = T/n$. We are interested in the random variable at time T given by

$$X_n = \sum_{k=1}^{n} Z_k(\tau)$$

Then as $n \to \infty$, X_n converges in distribution to

$$c_1 T + \sum_{k=0}^{N_T} W_k \tag{12}$$

where N_T is a Poisson random variable with mean λT and the W_k are iid random variables given by (11).
For the proof of this proposition, see Appendix B.

This proposition provides us with the distribution to use to price European options that is consistent with the lattice model. Next, we employ Fourier transform techniques as in Carr and Madan (1999) to efficiently compute the price of European options. For that purpose, the characteristic function of the distribution must be known. For the random variable given by (12), the characteristic function or the Fourier transform (Breiman, 1992) is given by

$$\phi_T(u) = e^{iuc_1 T} \exp\left(\lambda T \sum_{l \in \{1,\,2,\,4,\,5\}} q_l \left(e^{iu(2l-6)\alpha} - 1 \right) \right) \tag{13}$$

This will be used in the following section on pricing European calls and puts.

Pricing European Calls and Puts with the Fourier Transform

In this section, we follow the work of Carr and Madan (1999) on option pricing using the Fourier transform. The Fourier transform approach is particularly useful when the Fourier transform of the risk neutral probabilities is known. Our case only differs slightly from Carr and Madan (1999) in that we have a discrete distribution and therefore use the discrete Fourier transform.

Let $q_T(n)$ denote the discrete risk neutral probability distribution of the random variable given in (12). Using the limiting distribution in (12) as the risk neutral probabilities, the value of a call option is simply given as a discounted expectation of the payoff. This is given by

$$C_T(\bar{k}, K) = e^{-rT} \sum_{n=\bar{k}}^{\infty} \left(e^{c_1 T + 2\alpha_0 n} - K \right) q_T(n) \tag{14}$$

where $\bar{k}$ is the smallest integer greater than

$$\left(\frac{ln(K/S) - c_1 T}{2\alpha_0} \right)$$

Since the approach is similar to Carr and Madan (1999), we present the pricing formula below and leave the details of the derivation to Appendix C.

Fourier Transform Pricing Formula

The value of a non-dividend paying European call option on an underlying modeled as $S_0 e^{X_t}$ with initial price S_0, distribution of X_t at expiration T of (12), and strike price K may be computed as

$$C_T(\bar{k}, K) = \frac{e^{-\beta \bar{k}}}{2\pi} \int_{-\pi}^{\pi} \psi(u, K) e^{-iu\bar{k}} du \tag{15}$$

where $\beta>0$ is a parameter used to make the Fourier transform well defined,

$$\psi(u, K)=e^{-rT}\left(\frac{1}{1-e^{-(\beta+iu)}}\right)\left[S_0e^{c_1T}\phi\left(-i(\frac{\beta+2\alpha_0+iu}{2\alpha_0}\right)-K\phi\left(-i(\frac{\beta+iu}{2\alpha_0}\right)\right] \quad (16)$$

and

$$\phi(u)=E\left[e^{iuX}\right]=\sum_{n=-\infty}^{\infty}e^{iu2\alpha_0n}q_T(n)=\exp\left(\lambda T\sum_{l\in\{1,\,2,\,4,\,5\}}q_l\left(e^{(2l-6)\alpha_0iu}-1\right)\right) \quad (17)$$

is the moment generating function of the limiting distribution in (12).

Remark: Since the formula is given in terms of a Fourier transform, the fast Fourier transform algorithm may be used to efficiently evaluate the price. Using the fast Fourier transform in this context was introduced by Carr and Madan (1999). In the above formula, we need to input the drift (c_1) of the Levy process X_t in a risk neutral world. To determine this value when the underlying asset does not pay a dividend, note that c_1 should be chosen so that the following risk neutral condition is satisfied

$$e^{rT}=E\left[e^{X_T}\right]=\phi(-i)=\exp(c_1T)\exp\left(\lambda T\sum_{l\in\{1,\,2,\,4,\,5\}}q_l\left(e^{(2l-6)\alpha_0}-1\right)\right)$$

which, upon solving for c_1, gives

$$c_1=r-\lambda\sum_{l\in\{1,\,2,\,4,\,5\}}q_l\left(e^{(2l-6)\alpha_0}-1\right)$$

Numerical Examples

In this section we demonstrate features of the pentanomial lattice model for option pricing. In particular we use the model to generate volatility smiles and smirks for European call options.

We consider two cases. The first involves a non-dividend paying underlying asset with daily skewness and excess kurtosis values of

$$s=0, \quad \kappa=3$$

The yearly volatility is $\sigma=0.2$, the risk-free rate is assumed to be zero, and we used times to expiration of 20, 50, and 100 days with a convention of 250 trading days in a year. The resulting volatility smiles are shown in Figure 2.

The underlying limit of the lattice corresponds to the compound Poisson process given in (12). Proposition 2 provides an allowable range for α. For these parameter values, α may lie between

$$\alpha\in[0.0055,\ 0.0110] \quad (18)$$

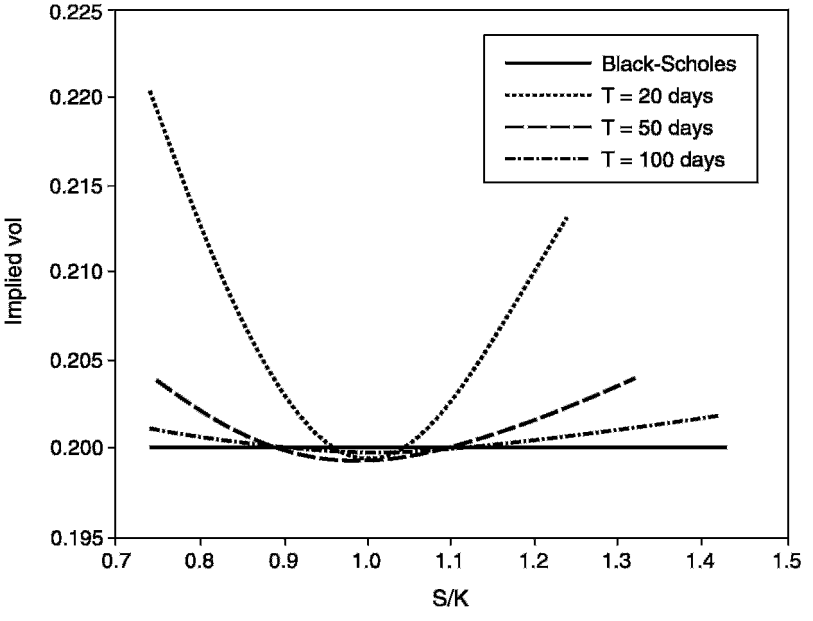

Figure 2. Implied volatility plots under excess kurtosis. Parameter values: $r_f=0$, $\sigma=0.2$, $s=0$, and $\kappa=3$

and $\hat{\alpha}=0.0063$. Additionally, the intensity of the limiting process is $\lambda=125$, and the probabilities are $q_1=0.1667$, $q_2=0.3333$, $q_4=0.1667$, $q_5=0.3333$.

The second case considered uses a daily skewness and excess kurtosis of

$$s=0.5, \quad \kappa=3$$

The rest of the parameter values were kept the same. These results are shown in Figure 3. In this case, the allowable range for α is given by

$$\alpha\in[0.0059, 0.0082] \tag{19}$$

and $\hat{\alpha}=0.0063$. The intensity of the limiting process is $\lambda=125$, and the jump probabilities are $q_1=0.0833$, $q_2=0.5000$, $q_4=0.1667$, $q_5=0.2500$.

These examples demonstrate that this simple lattice model is able to model a wide range of volatility smiles and smirks. However, some features of the model also emerge. An important artefact of the model and its limit as a compound Poisson process is that the minimum change in value of the underlying asset is determined by the allowable range of α. In these two numerical examples, this range is given in (18) and (19). This puts an explicit limit on the fidelity of the model, and this can be a limitation especially in cases that exhibit large kurtosis.

Conclusions

In this paper we have analysed a pentanomial lattice model that incorporated skewness and kurtosis. We also determined conditions on skewness and kurtosis

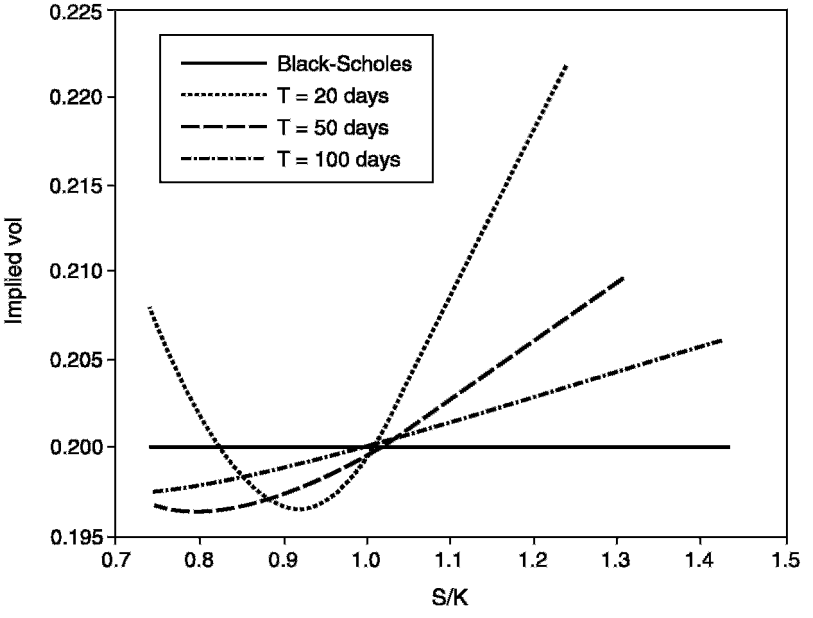

Figure 3. Implied volatility plots under skewness and excess kurtosis. Parameter values: $r_f=0$, $\sigma=0.2$, $s=0.5$, and $\kappa=3$

under which this model could be taken to a limit in continuous time. We derived the limiting distribution to be a compound Poisson distribution. Finally, we presented a formula using Fourier transform techniques to more efficiently and consistently compute European option prices under the limiting distribution. Hence, this provides a consistent model for computing American and European option prices under skewness and kurtosis. These results also indicate that a compound Poisson process may be a reasonable choice for a model of an underlying asset where it is desired to have American and European options priced simply, efficiently, and consistently.

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Appendix A. Positivity Condition

In this section we prove Proposition 1 by determining the range of α over which all of the branch probabilities are nonnegative. Clearing denominators in (2) leads to the following conditions

$$\mu_4 - 4\alpha^2\mu_2 - 4\alpha\mu_3 \geq 0 \tag{20}$$

$$-\mu_4 + 16\alpha^2\mu_2 + 2\alpha\mu_3 \geq 0 \tag{21}$$

$$64\alpha^4 - 20\alpha^2\mu_2 + \mu_4 \geq 0 \tag{22}$$

$$-\mu_4 + 16\alpha^2\mu_2 - 2\alpha\mu_3 \geq 0 \tag{23}$$

$$\mu_4 - 4\alpha^2\mu_2 + 4\alpha\mu_3 \geq 0 \tag{24}$$

First, consider condition (22) and note that it does not depend on μ_3. Minimizing (22) over α leads to $\alpha^2 = \frac{5}{32}\mu_2$. Plugging this back into (22) gives the condition $\frac{\mu_4}{\mu_2} \geq \frac{25}{16}$ for nonnegativity. Therefore, requiring that $\frac{\mu_4}{\mu_2} \geq \frac{25}{16}$ ensures that p_3 is nonnegative.

Now, let's consider the other conditions, (20), (21), (23), and (24). Without loss of generality, we assume skewness (hence μ_3) is positive (otherwise, we simply flip the distribution). The roots of these equations lead to

$$\alpha \le \left[\frac{1}{4\mu_2}\left(-2\mu_3 + 2\left(\mu_3^2 + \mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \ge \left[\frac{1}{4\mu_2}\left(-2\mu_3 - 2\left(\mu_3^2 + \mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \ge \left[\frac{1}{16\mu_2}\left(-\mu_3 + \left(\mu_3^2 + 16\mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \ge \left[\frac{1}{16\mu_2}\left(-\mu_3 - \left(\mu_3^2 + 16\mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \ge \left[\frac{1}{16\mu_2}\left(\mu_3 + \left(\mu_3^2 + 16\mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \ge \left[\frac{1}{16\mu_2}\left(\mu_3 - \left(\mu_3^2 + 16\mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \le \left[\frac{1}{4\mu_2}\left(2\mu_3 + 2\left(\mu_3^2 + \mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

$$\alpha \ge \left[\frac{1}{4\mu_2}\left(2\mu_3 - 2\left(\mu_3^2 + \mu_2\mu_4\right)^{\frac{1}{2}}\right)\right]$$

The most constraining conditions are

$$\frac{1}{16\mu_2}\left(\mu_3 + \left(\mu_3^2 + 16\mu_2\mu_4\right)^{\frac{1}{2}}\right) \le \alpha \le \frac{1}{4\mu_2}\left(-2\mu_3 + 2\left(\mu_3^2 + \mu_2\mu_4\right)^{\frac{1}{2}}\right) \tag{25}$$

which defines an allowable range of the parameter α. Now, let us employ the following parameterization. Take $\beta^2\mu_2\mu_4 = \mu_3^2$. If we substitute this parameterization into both sides of (25) we obtain

$$\frac{(\mu_2\mu_4)^{1/2}}{16\mu_2}\left(\beta + \left(\beta^2 + 16\right)^{\frac{1}{2}}\right) \le \alpha \le \frac{(\mu_2\mu_4)^{1/2}}{4\mu_2}\left(-2\beta + 2\left(\beta^2 + 1\right)^{\frac{1}{2}}\right)$$

Both sides are equal when $\beta = \frac{1}{\sqrt{3}}$, and the inequality is true for $\beta \le \frac{1}{\sqrt{3}}$. Hence, we have feasibility if $\beta \le \frac{1}{\sqrt{3}}$ or $\mu_2\mu_4 \ge 3\mu_3^2$. Furthermore $\alpha = \sqrt{\frac{\mu_4}{12\mu_2}}$ is a feasible choice. This proves Proposition 1.

Appendix B. Convergence Proof

Here we prove a theorem which is stronger than Proposition 3. We consider a pentanomial lattice model in the time interval [0, T]. Denote by $\mathcal{T}$ a 'mesh' of a finite number of time steps in the interval [0, T]:

$$\mathcal{T} : 0 = s_0 < s_1 \cdots < s_N = T$$

Let $\tau_j = s_j - s_{j-1}$ for $j=1, \ldots, N$ denote the step sizes. We shall assume s_j and hence τ_j are all deterministic. We shall denote by $|\mathcal{T}|$ the largest step size:

$$|\mathcal{T}| = \max\{\tau_j | j = 1, \ldots, N\}$$

For a given mesh $\mathcal{T}$ define n_t by

$$n_t = \max\{n | s_n \leq t\}$$

We shall define the lattice process corresponding to mesh $\mathcal{T}$ to be

$$\hat{X}(t) = \sum_{j=1}^{n_t} Z_j$$

for all $t \in [0, T]$, where the increments Z_j are given by

$$P\{Z_j = c_1\tau_j + (2l - L - 1)\alpha(\tau_j))\} = p_l(\tau_j), \quad l = 1, \ldots, L$$

Thus the process $\hat{X}(t)$ by definition has sample paths that are continuous from the right with left-hand limits.

Let $X(t)$ be the process with sample paths that are continuous from the right with left-hand limits, defined by

$$X(t) = X(0) + c_1 t + \sum_{j=1}^{N_t} W_j$$

where N_t is the Poisson process with intensity λ defined by (10) and W_j are iid random variables defined by (11).

Consider any finite set of time points

$$0 = t_0 < t_1 < t_2 < \cdots < t_m \leq T$$

Define the increments

$$Y_i = X(t_i) - X(t_{i-1}), \hat{Y}_i = \hat{X}(t_i) - \hat{X}(t_{i-1}), \quad i = 1, \ldots, m$$

For an R-valued random variable A we shall denote its characteristic function by f_A. Thus

$$f_A(u) = E(e^{iuA})$$

Lemma 1. For each $u \in \mathrm{R}$ and $i=1, \ldots, m$ there exist $\delta_i(u)>0$ and $C_i(u)>0$ such that

$$\left| f_{\hat{Y}_i}(u) - f_{Y_i}(u) \right| \leq C_i \tau$$

for all meshes $\mathcal{T}$ such that $\tau = |\mathcal{T}| < \delta_i$.

Proof. For brevity we denote f_{Y_i} by f_i and $f_{\hat{Y}_i}$ by $\hat{f}_i$. From the definition, Y_i is the sum of $c_1(t_i - t_{i-1})$ and four independent Poisson random variables with means $q_l\lambda$ for $l=1, 2, 4, 5$ where q_l are as defined in (10). It follows that

$$\log f_i(u) = iuc_i(t_i - t_{i-1}) + \sum_{l=1,\, l\neq 3}^{5} q_l \lambda (t_i - t_{i-1}) \left(e^{(2l-6)i\alpha_0 u} - 1 \right)$$

It also follows from the definition that

$$\hat{Y}_i = \sum_{j=n_{t_{i-1}}+1}^{n_{t_i}} Z_j$$

where Z_j are the iid random variables defined earlier. Thus

$$\log \hat{f}_i(u) = \sum_{j=n_{t_{i-1}}+1}^{n_{t_i}} \log f_{Z_j}(u)$$

From the definition of Z_j it follows that

$$\log f_{Z_j}(u) = ic_1 \tau_j u + \log \left\{ p_3(\tau_j) + e^{-4i\alpha(t_j)u} p_1(\tau_j) + e^{-2i\alpha(t_j)u} p_2(\tau_j) + e^{2i\alpha(t_j)u} p_4(\tau_j) + e^{4i\alpha(t_j)u} p_5(\tau_j) \right\}$$

where $p_1, \ldots, p_5$ are as defined in (5). Note that for each fixed $u \in \mathrm{R}$, $\log f_{Z_j}(u)$ is an analytic function of τ_j in a neighbourhood of $\tau_j = 0$. First we obtain the Taylor expansion of the terms inside the log in the above equation to obtain

$$\begin{aligned}
p_3(\tau_j) &= 1 - \lambda \tau_j + O\left(\tau_j^2\right) \\
e^{-4i\alpha(\tau_j)u} p_1(\tau_j) &= \lambda q_1 e^{-4i\alpha_0 u} \tau_j + O\left(\tau_j^2\right) \\
e^{-2i\alpha(\tau_j)u} p_2(\tau_j) &= \lambda q_2 e^{-2i\alpha_0 u} \tau_j + O\left(\tau_j^2\right) \\
e^{2i\alpha(\tau_j)u} p_4(\tau_j) &= \lambda q_4 e^{2i\alpha_0 u} \tau_j + O\left(\tau_j^2\right) \\
e^{4i\alpha(\tau_j)u} p_5(\tau_j) &= \lambda q_5 e^{4i\alpha_0 u} \tau_j + O\left(\tau_j^2\right)
\end{aligned}$$

Using the fact that $\log(1+z) = z + O(z^2)$ we obtain

$$\begin{aligned}
\left| \log \hat{f}_i(u) - \log f_i(u) \right| \leq\ & |c_1 u| \left| \left(\sum_{j=n_{t_{i-1}}+1}^{n_{t_i}} \tau_j \right) - (t_i - t_{i-1}) \right| \\
& + \left| \sum_{l=1,\, l\neq 3}^{5} q_l \lambda \left(e^{(2l-6)i\alpha_0 u} - 1 \right) \right| \left| \left(\sum_{j=n_{t_{i-1}}+1}^{n_{t_i}} \tau_j \right) - (t_i - t_{i-1}) \right| \\
& + \sum_{j=n_{t_{i-1}}+1}^{n_{t_i}} K_j \tau_j^2
\end{aligned}$$

which holds for $\tau_j > 0$ small enough, and $K_j > 0$. Thus we obtain

$$\left|\log \hat{f}_i(u) - \log f_i(u)\right| \leq D_i \tau$$

for all meshes $\mathcal{T}$ for which $\tau = |\mathcal{T}|$ is sufficiently small. The quantity $D_i > 0$ depends on t_{i-1} and t_i as well as u. By continuity of the exponential function the claim follows.

Without loss of generality let us assume that $X(0) = \hat{X}(0) = 0$ with probability 1. Then using Lemma 1 we may prove that all the finite dimensional distributions of the process $\hat{X}$ converge to that of X with order $O(\tau)$. Here is a precise statement and proof.

Theorem 1. Let $0 \leqslant t_1 < t_2 < \ldots < t_m \leqslant T$ be any finite set of time points. Let $\hat{F}$ and F be the multivariate characteristic functions defined by

$$F(u_1, \ldots, u_m) = E\left(e^{i\left(\sum_{i=1}^m u_i X(t_i)\right)}\right)$$

and

$$\hat{F}(u_1, \ldots, u_m) = E\left(e^{i\left(\sum_{i=1}^m u_i \hat{X}(t_i)\right)}\right)$$

Then for each $(u_1, \ldots, u_m) \in \mathrm{R}^m$ there exist $C > 0$ and $\delta > 0$ such that

$$\left|\hat{F}(u_1, \ldots, u_m) - F(u_1, \ldots, u_m)\right| \leq C\tau$$

for all meshes $\mathcal{T}$ such that $\tau = |\mathcal{T}| < \delta$.

Proof. Assuming $X(0) = \hat{X}(0) = 1$, from the independent increment property of both X and $\hat{X}$ we see that

$$F(u_1, \ldots, u_m) = f_{Y_1}(u_1) \ldots f_{Y_m}(u_m)$$

and

$$\hat{F}(u_1, \ldots, u_m) = f_{\hat{Y}_1}(u_1) \ldots f_{\hat{Y}_m}(u_m)$$

The theorem follows by the application of Lemma 1.

It is clear that Theorem 1 implies Proposition 3.

Appendix C. Fourier Transform Pricing

In this section, we derive the Fourier transform based pricing formula given in Equations 15–17. The basic approach is to apply a discrete Fourier transform in $\bar{k}$ to Equation 14. Similar to Carr and Madan (1999), we must first multiple this by $e^{\beta \bar{k}}$ with $\beta > 0$ in order to guarantee existence of the discrete Fourier transform and write

$$c_T(\bar{k}, K) = e^{\beta\bar{k}} C_T(\bar{k}, K) = e^{\beta\bar{k}} \sum_{n=\bar{k}}^{\infty} e^{-rT} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) q_T(n) \tag{26}$$

Now, we simply compute the Fourier transform of the modified $c_T(\bar{k}, K)$. This is given by

$$\begin{aligned}
\psi(u, K) &= \sum_{\bar{k}=-\infty}^{\infty} e^{iu\bar{k}} c_T(\bar{k}, K) \\
&= \sum_{\bar{k}=-\infty}^{\infty} e^{iu\bar{k}} \sum_{n=\bar{k}}^{\infty} e^{\beta\bar{k}} e^{-rT} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) q_T(n) \\
&= e^{-rT} \sum_{n=-\infty}^{\infty} q_T(n) \sum_{\bar{k}=-\infty}^{n} e^{(\beta+iu)\bar{k}} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) \\
&= e^{-rT} \sum_{n=-\infty}^{\infty} q_T(n) \sum_{\bar{k}=0}^{\infty} e^{(\beta+iu)(n-\bar{k})} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) \\
&= e^{-rT} \sum_{n=-\infty}^{\infty} q_T(n) e^{(\beta+iu)(n)} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) \sum_{\bar{k}=0}^{\infty} e^{-(\beta+iu)\bar{k}} \\
&= e^{-rT} \sum_{n=-\infty}^{\infty} q_T(n) e^{(\beta+iu)(n)} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) \left(\frac{1}{1 - e^{-(\beta+iu)}}\right) \\
&= e^{-rT} \left(\frac{1}{1 - e^{-(\beta+iu)}}\right) \sum_{n=-\infty}^{\infty} q_T(n) e^{(\beta+iu)(n)} \left(S_0 e^{c_1 T + 2\alpha_0 n} - K\right) \\
&= e^{-rT} \left(\frac{1}{1 - e^{-(\beta+iu)}}\right) \left(\sum_{n=-\infty}^{\infty} q_T(n) S_0 e^{c_1 T} e^{(\beta+2\alpha_0+iu)(n)} - \sum_{n=-\infty}^{\infty} q_T(n) e^{(\beta+iu)(n)} K\right) \\
&= e^{-rT} \left(\frac{1}{1 - e^{-(\beta+iu)}}\right) \left[S_0 e^{c_1 T} \phi\left(-i\left(\frac{\beta + 2\alpha_0 + iu}{2\alpha_0}\right)\right) - K\phi\left(-i\left(\frac{\beta+iu}{2\alpha_0}\right)\right)\right]
\end{aligned} \tag{27}$$

where

$$\phi(u) = E\left[e^{iuX}\right] = \sum_{n=-\infty}^{\infty} e^{iu2\alpha_0 n} q_T(n) = \exp\left(\lambda T \sum_{l \in \{1, 2, 4, 5\}} q_l \left(e^{(2l-6)\alpha_0 iu} - 1\right)\right)$$

To compute the price of the option, we compute the inverse Fourier transform of the result in (27), and remove the $e^{\beta\bar{k}}$ term. We can do this using the fast Fourier transform and furthermore, since it is a discrete Fourier transform, the integral is limited to be between $[-\pi, \pi]$. Hence prices are given by

$$C_T(\bar{k}, K) = \frac{e^{-\beta\bar{k}}}{2\pi} \int_{-\pi}^{\pi} \psi(u, K) e^{-iu\bar{k}} du$$