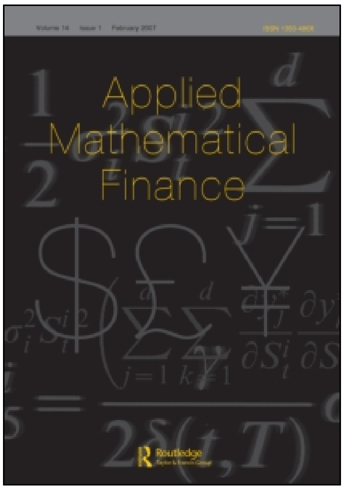

Applied Mathematical Finance
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713694021

Trader Behavior and its Effect on Asset Price Dynamics

James A. Primbs[a]; Muruhan Rathinam[b]
[a] Management Science and Engineering, Stanford University, Stanford, CA, USA [b] Mathematics and Statistics, University of Maryland, Baltimore, MD, USA

Online publication date: 24 June 2010

To cite this Article Primbs, James A. and Rathinam, Muruhan(2009) 'Trader Behavior and its Effect on Asset Price Dynamics', Applied Mathematical Finance, 16: 2, 151 — 181
To link to this Article: DOI: 10.1080/13504860802583444
URL: http://dx.doi.org/10.1080/13504860802583444

Applied Mathematical Finance,
Vol. 16, No. 2, 151–181, April 2009

Routledge
Taylor & Francis Group

Trader Behavior and its Effect on Asset Price Dynamics

JAMES A. PRIMBS* & MURUHAN RATHINAM**

**Management Science and Engineering, Stanford University, Stanford, CA, USA, **Mathematics and Statistics, University of Maryland, Baltimore, MD, USA*

(Received 23 May 2007; in revised form 21 August 2008)

ABSTRACT *In this paper, we present a natural mathematical framework to model trader behavior as a continuous time discrete event process, and derive stochastic differential equations for aggregate behavior and price dynamics by passing to diffusion limits. In particular, we model extraneous, value, momentum and hedge traders. Through analysis and numerical simulation we explore some of the effects these trading strategies have on price dynamics.*

KEY WORDS: Trader behavior, price dynamics, stock pinning, diffusion limit, Poisson random measure

1. Introduction

Market prices are determined by individual market participants that submit buy and sell orders at random times. These market participants use a range of strategies to determine the time and size of each trade. Thus, it is the interaction of many traders with diverse trading strategies that ultimately determines the price dynamics of a financial asset. However, in many areas of finance, price dynamics are modeled exogenously and traders are price takers that do not influence price dynamics. Thus, market dynamics are considered from a phenomenological viewpoint. Hence, an important area of investigation is whether phenomenological models can be consistently and profitably connected with more detailed models of price dynamics that stem from fundamental properties of trader behavior.

In this paper, our goal is to develop a framework for a first principles model of price dynamics that begins from an explicit assignment of trader behavior and yet under reasonable assumptions can be simplified to a relatively low order stochastic dynamics model. Thus, we seek a mathematical bridge between the low order phenomenological models often encountered in financial engineering, and the agent-based models used in simulated markets (LeBaron, 2001). While the goal of

Correspondence Address: James A. Primbs, 444 Terman Engr. Ctr., Management Science and Engineering, Stanford University, Stanford, CA 94305-4026, USA. Email: japrimbs@stanford.edu

1350-486X Print/1466-4313 Online/09/020151–31
DOI: 10.1080/13504860802583444

providing a seamless connection between these domains is overly ambitious, this paper contributes in that direction by proposing a framework in which four different classes of trader behavior are modeled, and aggregate behavior is obtained in terms of a low dimensional diffusion process.

One approach to exploring the range of price dynamics that can arise from the complex interactions of heterogeneous traders is to use a simulated market environment in which individual agents can be assigned complex trading rules. One of the best known examples of this approach is the Santa Fe Artificial Stock Market (LeBaron, 2002). Broadly speaking, such an approach falls under the category of agent-based models of markets (Farmer and Joshi, 2002; LeBaron, 2000, 2001; Lux, 1997; Raberto *et al.*, 2001). This approach has the advantage of allowing for complicated rule-based behavior assignments, but analysis of resulting dynamics can be challenging. An example of this is the work of Chiarella and Iori (2002) where fundamentalist, chartist and noise traders are analyzed in a simulated order-driven market. In an option pricing context, Qiu *et al.* (2007b) use an agent-based market of speculators and arbitrageurs to explore the origins of volatility smiles. In certain cases, simplifications of the dynamics result from aggregation of groups of traders.

An alternate approach to agent-based simulated markets is to use low dimensional analytic models of trader behavior. This can allow for a more transparent connection between model parameters and resulting dynamics. Examples of this include the work of Farmer (2000) and Farmer and Joshi (2002), who use a discrete event model in discrete time to explore various trading strategies. Other work that explicitly considers trader feedback effects either from hedging or in illiquid market settings is that of Frey and Stremme (1997), Platen and Schweizer (1998), Sircar and Papanicolaou (1998), Schonbucher and Wilmott (2002) and Avellaneda and Lipkin (2003). Our work starts from a discrete event model similar to Farmer (2000), however we allow these events to happen at random times. We then make use of simplifying assumptions to obtain lower order stochastic differential equation models via diffusion limits.

Our market is populated with four classes of traders. The first class is referred to as *extraneous* traders, and are those who trade for reasons not based upon market conditions or prices. The next class is *value* traders. These traders refer to a perceived 'true value' to determine their buy and sell decisions. Thus, value traders are likely to purchase a stock that they perceive to be 'undervalued' and sell a stock that they perceive to be 'overvalued'. The third class of traders are *momentum* traders. Momentum traders use past price action to compute a measure of the 'trend' of prices. They then seek to take advantage of the trend by purchasing when price momentum is positive, and selling when price momentum is negative. The final class of traders are known as *hedge traders*. These are traders that hold a position in options, and trade the underlying stock in order to hedge their option holdings. We assume they trade the stock according to a Black–Scholes-based delta hedging strategy.

In our discrete event continuous time market model, each trader's demand process is modeled mathematically as driven by Poisson random measures, and each buy and sell event has the effect of moving the price proportional to the size of the trade. Hence, we use a linear price formation rule governed by a liquidity parameter as in Avellaneda and Lipkin (2003) and Farmer (2000). An alternative would be to use an explicit market clearing condition as in Frey and Stremme (1997) or Schonbucher and Wilmott (2000). We favor the linear price formation rule since market makers are not explicitly modeled as a class of traders. We also note that, for mathematical tractability, we use a Poissonian model, but a growing body of research is indicating that the time between human events is not Poissonian (Johansen, 2004; Vazquez *et al.*, 2006) and this includes inter-trade times in financial markets (Ivanov *et al.*, 2004; Masoliver *et al.*, 2003; Palatella *et al.*, 2004; Scalas *et al.*, 2004, 2005; Silva and Yakovenko, 2007). From our discrete event model for price dynamics, we proceed by aggregating trading classes and taking diffusion limits that simplify the model to a set of Ito diffusions. While in principle our framework can accommodate heterogeneous agents within a class of traders, to aggregate and obtain diffusion limits we make assumptions of homogeneity. With the resulting diffusion model, we analyze and simulate the effects of trading strategies on price dynamics using parameters calibrated to market data. We use a mix of both analytics and simulations to explore the effects of value, momentum and hedge trading. In particular, we consider the stability and stochastic volatility consequences of value and momentum traders, and the stock pinning phenomenon that arises due to hedging.

The paper is organized as follows. Section 2 develops the basic mathematical model, including models of extraneous, value, momentum and hedge traders. It also justifies the diffusion limit and resulting Ito diffusion equations. Section 2 analyzes the effects of trading strategies in the diffusion model. Both analytic techniques and simulation are used to explore price dynamics. Finally, Section 3 provides conclusions.

2. Mathematical Model

2.1 Discrete Event Model

Our basic model involves discrete trade events of a continuous range of sizes happening in continuous time, all in a stochastic manner. Suppose there are a total of n traders excluding the market maker. We assume that all traders buy or sell shares of a given stock from or to the market maker. Let N_t^i be the process counting the number of trades by the ith trader. The trading activity of the ith trader is uniquely characterized by a sequence $\left(T_j^i, C_j^i\right)$, $j=1, 2, \ldots$, of random trade times T_j^i and random trade sizes C_j^i. T_j^i is the time at which the jth trade by the ith trader occurs. C_j^i stands for the number of units (shares) that the ith trader purchased (or sold if negative) at time T_j^i. We assume without loss of generality that $T_j^i < T_{j+1}^i$ for all i and j. We shall also assume that $T_{j_1}^{i_1} \neq T_{j_2}^{i_2}$ almost surely for all j_1, j_2 when $i_1 \neq i_2$. In other words, at any given time instant, at most one trader is involved in a trade. The aggregate trade process, which we also call the aggregate demand process for the ith trader, is denoted by X_t^i and stands for the number of units (shares) that the ith trader has purchased (or sold if negative) by time t in the net balance:

$$X_t^i = X_0^i + \sum_{j=1}^{N_t^i} C_j^i.$$

We shall denote by p_t the log of the stock price. In addition to X_t^i and p_t, we shall introduce trader-specific processes Z_t^i which carry information about a given trader's strategy.

The processes N_t^i, X_t^i, Z_t^i and p_t are considered to be carried by and adapted to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

We shall assume the processes X_t^i to be conditionally independent compound Poisson processes. By this we mean that, conditioned on $\mathcal{F}_{t-}$ (all events prior to time t), X_t^i for $i=1,\ldots,n$ are independent, and each process X_t^i has a stochastic jump intensity and a jump size distribution (a probability measure on $\mathbb{R}$ of trade sizes) both of which are assumed to be functions determined by time t, the log stock price p_{t-} just prior to time t and the value of the trader-specific variable Z_{t-}^i just prior to t.

This situation may be stated precisely in terms of a *Poisson random measure*. We refer the reader to Appendix A for some basics on Poisson random measures. More details may be found in Bitchteler (2002), Jacod and Shiryaev (2003) and Applebaum (2004). Let μ^i for $i=1, \ldots, n$ be independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ with intensity measures of the form $\mathrm{d}t \times \mathrm{d}a \times \nu^i$, respectively, where $\mathrm{d}t$ and $\mathrm{d}a$ indicate the Lebesgue measures on the time and jump rate spaces and ν^i are probability measures on $\mathbb{R}$ describing the jump size distribution. The X_t^i are then assumed to be given by

$$X_t^i = X_0^i + \int_{s\in(0,\,t]} \int_{a\in[0,\,\infty)} \int_{y\in\mathbb{R}} 1_{\{0<a\le\rho^i(p_{s-},\, Z_{s-}^i)\}} \times K^i\left(s, p_{s-}, Z_{s-}^i, y\right)\mu^i(\mathrm{d}s, \mathrm{d}a, \mathrm{d}y), \quad i=1, \ldots, n. \tag{1}$$

These describe what we call conditional compound Poisson processes with stochastic intensities $R_t^i = \rho^i\left(p_{t-}, Z_{t-}^i\right)$ which determine the rates at which trades occur, and the kernel functions $K^i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (assumed to be non-vanishing except on a set of measure zero and continuous on the first argument) which determine the trade size.

For the rest of our discussion in this paper, we shall assume that $\rho^i(p, Z) = R^i$ are constants for $i=1, \ldots, n$. This lets us simplify the model, allowing us to consider the Poisson random measures μ^i to be measures on $\mathbb{R}_+ \times \mathbb{R}$ (eliminating the 'jump rate' variable a) with intensity measures $R^i\,\mathrm{d}t \times \nu^i(\mathrm{d}y)$, and Equation (1) can be written as

$$X_t^i = X_0^i + \int_0^t \int_{\mathbb{R}} K^i\left(s, p_{s-}, Z_{s-}^i, y\right)\mu^i(\mathrm{d}s, \mathrm{d}y), \quad i=1, \ldots, n. \tag{2}$$

Note that $\int_0^t$ shall stand for $\int_{(0,\,t]}$ throughout this paper. This definition implies that X_t^i are *cadlag* (sample paths are continuous from the right with left-hand limits), and, furthermore, X_t^i are *pure jump* processes in the sense that sample paths are constant between their jumps. Also note that the Poisson random measure μ^i with intensity measure $R^i\,\mathrm{d}t \times \nu^i(\mathrm{d}y)$ can be equivalently characterized by the *point process* $\left(T_j^i, Y_j^i\right)$, $j \in \mathbb{N}$, where the time between trades $T_{j+1}^i - T_j^i$ is exponentially distributed with rate parameter R^i and the random variables Y_j^i for $j \in \mathbb{N}$ are independent and identically distributed real valued random variables with associated probability measure v^i on $\mathbb{R}$. It may be noted that (2) may be written in summation form

$$X_t^i = X_0^i + \sum_{j=1}^{N_t^i} K^i\left(T_j^i, p_{T_j^i-}, Z_{T_j^i-}^i, Y_j^i\right), \quad i=1, \ldots, n, \tag{3}$$

where

$$N_t^i = \int_0^t \int_{y \in \mathbb{R}} \mu^i(\mathrm{d}s, \mathrm{d}y), \quad i=1, \ldots, n, \tag{4}$$

which is also equivalently written as

$$N_t^i = \max\left\{ j | T_j^i \leq t \right\}.$$

The trader-specific processes Z_t^i follow a dynamics that is dependent on the process p_t and possibly on an external source of information which we model by processes which are also assumed adapted to the same filtration.

2.2 Types of Traders

Now we shall describe certain types of traders that we study in this paper, and the corresponding appropriate forms for Z_t^i, the equations driving them as well as the functional forms for the trade size functions K^i. We consider four types of traders: extraneous, value, momentum and hedge. We shall see that suitable assumptions allow us to aggregate all the trader-specific variables Z^i into one variable within each category of traders leading to a fewer number of variables for our model.

2.2.1 Extraneous Traders. The simplest case of traders is a collection of traders, all of whom trade in a manner independently of the log asset price p_t, time t and of each other. This may be captured by setting $K^i(t, p, z, y)=y$, which leads to X_t^i being compound Poisson processes. Thus the aggregate demand of all such traders $X_t^e = X_t^1 + \cdots + X_t^{n_e}$ (where n_e is the number of extraneous traders) is given by

$$X_t^e = X_0^e + \sum_{i=1}^{n_e} \int_0^t \int_{\mathbb{R}} y\, u^i(\mathrm{d}s, \mathrm{d}y),$$

where μ^i for $i=1, \ldots, n_e$ are independent Poisson random measures with intensity R^i $\mathrm{d}t \times v^i(\mathrm{d}y)$. Thus the ith extraneous trader trades at a Poisson rate R^i and his or her trade sizes are all independent and identically distributed with probability measure v^i. The aggregate demand can be rewritten as

$$X_t^e = X_0^e + \int_0^t \int_{\mathbb{R}} y\, \mu^e(\mathrm{d}s, \mathrm{d}y), \tag{5}$$

where $\mu^e = \mu^1 + \cdots + \mu^{n_e}$ is the aggregate Poisson random measure, which has intensity measure $n_e R^e$ $\mathrm{d}t \times v^e(\mathrm{d}y)$, where

$$R^e = \frac{\sum_{i=1}^{n_e} R^i}{n_e},$$

and

$$\nu^e = \sum_{i=1}^{n_e} \frac{R^i}{R^e} \nu^i,$$

which follows from the independence assumption on $\mu^1, \ldots, \mu^{n_e}$. Thus we note that it is possible to aggregate the extraneous traders into one equation purely based on the independence assumption and also note that there is no trader-specific Z variable for this case.

Note that ν^e is the probability measure on $\mathbb{R}$ which describes the trade size distribution of a randomly chosen extraneous trade event and R^e is the average rate of trading for a randomly chosen extraneous trader. For later use we shall denote the first moment of ν^e by C_1^e and the square root of the second moment by C_2^e. Thus

$$C_1^e = \int_{\mathbb{R}} y\, \nu^e(\mathrm{d}y), \qquad \left(C_2^e\right)^2 = \int_{\mathbb{R}} y^2\, \nu^e(\mathrm{d}y). \tag{6}$$

2.2.2 Value Traders. Value traders are those traders who believe that the asset has a true value V_t which may or may not be equal to its price P_t. These traders believe that eventually the asset price P_t shall revert to its perceived to be true value V_t. The perceived value V_t^i may vary trader to trader. But often it is based on some common source of information. Thus we make the simplifying assumption that V_t^i is independent of i.

Furthermore, we model them as having trade sizes in proportion to the discrepancy between the log market price p_t and the log perceived price $v_t = \log(V_t)$. To capture this, we choose $K^i(t, p, z, y) = C^v z$ where $z = v - p$ is the discrepancy and C^v is some proportionality constant independent of i. The assumption of a common proportionality constant leads to K^i being independent of i and allows us to aggregate the Poisson random measures. We may write the aggregate demand X_t^v of value traders as

$$X_t^v = X_0^v + \int_0^t \int_{y \in \mathbb{R}} C^v (v_{s-} - p_{s-}) \mu^v(\mathrm{d}s, \mathrm{d}y),$$

where μ^v is the aggregate Poisson random measure corresponding to the value traders and we take it to have intensity measure $n_v R^v\, \mathrm{d}t \times \nu^v(\mathrm{d}y)$ where n_v is the total number of value traders and R^v is the average rate of trading of a randomly chosen value trader. Define N_t^v to be the Poisson process that counts the total number of trades by all value traders. Thus

$$N_t^v = \int_0^t \int_{y \in \mathbb{R}} \mu^v(\mathrm{d}s, \mathrm{d}y),$$

and we may rewrite the equation for X_t^v as

$$X_t^v = X_0^v + \int_0^t C^v (v_{s-} - p_{s-}) \mathrm{d}N_s^v(n_v R^v), \tag{7}$$

where we have used the notation $N_t^v(n_v R^v)$ to denote the fact that N_t^v has (constant) intensity $n_v R^v$. The integral in (7) is to be interpreted as the Lebesgue–Stieltjes integral.

It is possible to explore different forms for the true value process v_t. In this paper we shall model the movement of v_t by a Brownian motion for simplicity. Thus we set

$$v_t = v_0 + \sigma_0 W_t,$$

where W_t is a standard Brownian motion independent of N_t^v. Hence, the perceived value $V_t = e^{v_t}$ follows a geometric Brownian motion and is independent of N_t^v.

2.2.3 Momentum Traders or Trend Followers. Momentum traders or trend followers are traders who look to past price action as an indication of future price movement. These traders base current trades on the perceived *momentum* of past prices. We set up a basic model of momentum trading. We assume that the momentum traders will determine their demand for the asset by computing an exponentially weighted average of past increments in log prices. This weighted average, called momentum, serves as an indicator of the trend.

We begin by defining a quantity ξ_t^i that serves as the Z^i variable in (2) and captures the momentum by

$$\xi_t^i = \int_{-\infty}^t G_i \gamma_i e^{-\gamma_i (t-s)} \, \mathrm{d}p_s.$$

In other words, the price momentum is an exponentially weighted average of past jumps in log prices. The quantity G_i is a proportionality constant and $1/\gamma_i$ is the effective time window of averaging. The ξ_t^i may be written in terms of values at $t=0$ as

$$\xi_t^i = \xi_0^i e^{-\gamma_i t} + \int_0^t G_i \gamma_i e^{-\gamma_i (t-s)} \, \mathrm{d}p_s. \tag{8}$$

This equation in turn could be rewritten in a more dynamic form as

$$\xi_t^i = \xi_0^i + G_i \int_0^t \gamma_i \, \mathrm{d}p_s - \int_0^t \gamma_i \xi_s^i \, \mathrm{d}s. \tag{9}$$

In order to see this, define r_t and q_t by

$$r_t = \int_0^t \gamma_i \, e^{\gamma_i s} \, \mathrm{d}p_s, \qquad q_t = e^{-\gamma_i t} r_t.$$

Thus we may rewrite (8) as

$$\xi_t^i = \xi_0^i e^{-\gamma_i t} + G_i q_t. \tag{10}$$

Since $e^{-\gamma_i t}$ and r_t do not have common jumps (in fact $e^{-\gamma_i t}$ is continuous), we may use the integration by parts formula for Lebesgue–Stieltjes integrals to obtain

$$\begin{aligned} q_t - q_0 &= \int_0^t e^{-\gamma_i s} \, \mathrm{d}r_s - \int_0^t \gamma_i \, e^{-\gamma_i s} r_s \, \mathrm{d}s \\ &= \int_0^t \gamma_i \, \mathrm{d}p_s - \int_0^t \gamma_i q_s \, \mathrm{d}_s. \end{aligned}$$

Using (10) and substituting for q_t and q_s and noting $q_0=0$ we obtain

$$\xi_t^i - \xi_0^i e^{-\gamma_i t} = G_i \int_0^t \gamma_i \, \mathrm{d}p_s - \int_0^t \gamma_i \xi_s^i \, \mathrm{d}s + \int_0^t \gamma_i \, e^{-\gamma_i s} \xi_0^i \, \mathrm{d}s.$$

Then (9) follows at once.

Note that the momentum ξ_t^i is an indication of the direction of movement of the asset price. The parameter γ_i reflects the time scale of the exponentially weighted moving average. Specifically, $1/\gamma_i$ may be considered to be the effective length of the time window of the moving average used by the ith momentum trader. In order to be able to aggregate the demands X_t^i of individual momentum traders we shall make the simplifying assumption that both G_i and γ_i are independent of i which allows us to assume the ξ^i are independent of i. Letting $\gamma_i=\gamma$, $G_i=G$ and $\xi^i=\xi$, we obtain

$$\xi_t = \xi_0 + G \int_0^t \gamma \, \mathrm{d}p_s - \int_0^t \gamma \xi_s \, \mathrm{d}s. \tag{11}$$

Furthermore, for the momentum traders, we assume the $K^i(t, p, \xi, y)$ have a common form

$$K^m(t, p, \xi, y) = C^m \xi,$$

where C^m is a common proportionality constant independent of i. The proportionality model makes sense since the larger the momentum variable ξ, the more a momentum trader wants to purchase. If the momentum is negative, the trader will be selling. The assumption of a common proportionality constant C^m is for convenience and enables us to aggregate all the momentum traders.

This leads to the following aggregate demand process X_t^m for momentum traders:

$$X_t^m = \int_0^t \int_{\mathbb{R}} C^m \xi_{s-} \, \mu^m(\mathrm{d}s, \mathrm{d}y), \tag{12}$$

where μ^m is the aggregate Poisson random measure corresponding to the momentum traders and we take it to have intensity $n_m R^m \, \mathrm{d}t \times v^m(\mathrm{d}y)$, where n_m is the total number of momentum traders and R^m is the average rate of trading for a randomly chosen momentum trader. As in the case of value traders we may rewrite this in terms of an integral with respect to the Poisson process $N_t^m(n_m R^m)$ which counts the total number of trades by momentum traders:

$$X_t^m = \int_0^t C^m \xi_{s-} \, \mathrm{d}N_s^m(n_m R^m). \tag{13}$$

2.2.4 Hedge Traders. Hedge traders are those that have taken positions in options and trade in the underlying stock in order to hedge. Hence, their trading is tied to their position in options, and, in the simplest model, is guided by the Black–Scholes delta hedging approach.

To model this, recall that according to the Black–Scholes model, the value of a European call option is

$$c(P, t, K, \sigma) = PN(d_1) - K\,e^{-r(T-t)}N(d_2),$$

$$d_1 = \frac{\ln(P/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}, \quad d_2 = d_1 - \sigma\sqrt{(T-t)},$$

where $N(\cdot)$ is the cumulative distribution function of a standard normal random variable. Note that $P{=}e^p$.

Suppose that, collectively, hedge traders have written α^h call options on the underlying stock. Recall that delta hedging ideally involves maintaining

$$\alpha^h \Delta(p, t, K, \sigma)$$

shares of the underlying stock. Here, $\Delta{=}\partial c/\partial P$, where c the price of a call option, P the stock price, K the strike price and σ the volatility of the stock. It follows that the delta of the option is given by

$$\Delta = N(d_1).$$

Suppose that the hedgers collectively hold H_t shares. The variable H_t shall play the role of Z_t in (2). The discrepancy between their holdings and the ideal position is $\alpha^h\Delta(p_t, t) - H_t$.

We model the purchase size $K^i(t, p, H, y)$ of each hedge trader to be

$$K^h(t, p, H, y) = C^h\left(\alpha^h \Delta(p, t) - H\right),$$

a quantity independent of i. The constant C^h represents the proportion of the discrepancy that is traded and we shall take it to be $1/n_h$, where n_h is the number of hedge traders. See Remark 1.

This common functional form allows us to aggregate all the hedge traders. This leads to the aggregate demand process X_t^h of hedge traders

$$X_t^h = \int_0^t \int_{\mathbb{R}} C^h\left(\alpha^h \Delta(p_{s-}, s) - H_{s-}\right)\mu^h(\mathrm{d}s, \mathrm{d}y), \tag{14}$$

where μ^h is the aggregate Poisson random measure corresponding to the hedge traders and we take it to have intensity $n_h R^h\ \mathrm{d}t \times v^h(\mathrm{d}y)$, where R^h is the average rate of trading for a randomly chosen hedge trader. As in the case of value and momentum traders we may rewrite this in terms of $N_t^h\left(n_h R^h\right)$, the Poisson process with intensity $n_h R^h$ that counts the total number of trades by all the hedge traders:

$$X_t^h = \int_0^t C^h\left(\alpha^h \Delta(p_{s-}, s) - H_{s-}\right)\mathrm{d}N_s^h\left(n_h R^h\right). \tag{15}$$

It also follows that the aggregate holding H_t is adjusted by this same amount, and thus

$$H_t = H_0 + \int_0^t C^h\left(\alpha^h \Delta(p_{s-}, s) - H_{s-}\right)\mathrm{d}N_s^h\left(n_h R^h\right). \tag{16}$$

Remark 1. The constant $C^h{=}1/n_h$ represents the proportion of the discrepancy that is traded whenever a given hedge trader trades. Note that if there was only a single hedge trader (n_h=1), then each time (s)he trades (s)he would likely trade the entire discrepancy $\alpha^h\Delta - H$ in order to restore his or her portfolio to a fully delta hedged position. In such a case, we would use C^h=1. However, in order to obtain a diffusion limit we must have a large number (n_h is much larger than 1) of hedge traders. Realistically speaking, it will be impossible to aggregate all the hedge traders because the holdings H^i_t of individual traders cannot be identical since at any instant at most one trader shall trade and only the holdings of that trader shall jump at that instant. However, in order to enable aggregation, we assume a somewhat collective social behavior on the part of hedge traders. We assume that the hedge traders look at the collective holdings H_t to make trade decisions rather than keep track of their individual holdings; in fact, the individual holdings are considered equal and hence H_t/n_h. We also assume that each trader trades independently (with a Poisson rate R^i), and, whenever (s)he trades, (s)he only trades $1/n_h$ of the actual adjustment $\alpha^h\Delta - H$ needed to the total holdings H_t, which in effect only adjusts his or her share of the holdings. While this collective social behavior is not representative of the real market, we believe that the diffusion limit obtained is likely to capture some of the key features of a more detailed model that necessarily involves a large number of variables $H^1_t, \ldots, H^{n_h}_t$ and some form of reduced order modeling and diffusion limit to simplify the dynamics.

2.3 Price Dynamics

So far we have derived models for the aggregate demand processes of different types of traders. These aggregate demands X^e, X^v, X^m, X^h evolve based on the log asset price p_t and some external processes.

In order to close the dynamics, we need to describe how the aggregate demands affect the asset price. We introduce the price formation rule

$$p_t - p_0 = \frac{1}{\lambda}\sum_{i=1}^{n}\left(X^i_t - X^i_0\right). \tag{17}$$

In other words we assume that the log asset price jumps are proportional to the jumps in the aggregate demand (Avellaneda and Lipkin, 2003; Farmer, 2000; Farmer and Joshi, 2002). The quantity λ is the market liquidity parameter and is assumed to be a constant. Thus after each trade of size K (in number of shares) the log asset price jumps by K/λ. Thus we are not including the market maker in the price dynamics, but rather assuming an algebraic relationship between changes in demand and changes in prices. A more detailed model would involve extra dynamic variables that account for the market maker. We simply assume that this unmodeled dynamics is much faster and either equilibrates rapidly to the price formation rule (17) or its oscillations lead to (17) after suitable averaging.

Throughout the rest of the paper we shall adopt the differential notation and suppress the time dependence (subscripts t, s or $s-$) to keep the presentation compact.

Following the price formation rule, we obtain

$$\mathrm{d}p = \frac{1}{\lambda}\left[\mathrm{d}X^e + \mathrm{d}X^v + \mathrm{d}X^m + \mathrm{d}X^h\right].$$

Combining this with the dynamics describing the demand processes of each strategy leads to the coupled system dynamics

$$\mathrm{d}p=\frac{1}{\lambda}\left[\int_{y\in\mathbb{R}} y\ \mu^e(\mathrm{d}t,\mathrm{d}y)+C^v(v-p)\mathrm{d}N^v(n_vR^v)+C^m\xi\ \mathrm{d}N^m(n_mR^m)\right.$$
$$\left.+C^h\left(\alpha^h\Delta-H\right)\mathrm{d}N^h\left(n_hR^h\right)\right], \tag{18}$$

$$\mathrm{d}v=\sigma_0\,\mathrm{d}W, \tag{19}$$

$$\mathrm{d}\xi=G\gamma\,\mathrm{d}p=\gamma\xi\,\mathrm{d}t, \tag{20}$$

$$\mathrm{d}H=C^h\left(\alpha^h\Delta-H\right)\mathrm{d}N^h\left(n_hR^h\right). \tag{21}$$

Equations (18), (19), (20) and (21) describe a coupled system for the variables p, v, ξ and H driven by independent driving processes μ^e, N^v, N^m, N^h, and W. In the next section we shall see that, under suitable scaling assumptions, if the number of traders of each category is very large we may obtain a diffusion approximation for this system of equations.

2.4 Diffusion Limit

Equations (18), (19), (20) and (21) describe a system of jump and diffusion Markov dynamics.

An important characteristic of a Markov process is its generator $\mathcal{A}$. Intuitively, given any function $f(t, x)$ of time t and process value $X_t=x$ at time t, $(\mathcal{A}f)(t,x)$ is the rate of change of the expected value of $f(t, X_t)$ at time t. The reader is referred to Appendix B for a definition and some details.

First we shall write down the time-dependent generator for this system:

$$\begin{aligned}\mathcal{A}f=&\frac{\partial f}{\partial t}+n_eR^e\int_{\mathbb{R}}\left[f\left(t,p+\frac{y}{\lambda},v,\xi+\frac{G\gamma y}{\lambda},H\right)-f(t,p,v,\xi,H)\right]\nu^e(\mathrm{d}y)\\
&+n_vR^v\left[f\left(t,p+\frac{C^v(v-p)}{\lambda},v,\xi+\frac{G\gamma C^v(v-p)}{\lambda},H\right)-f(t,p,v,\xi,H)\right]\\
&+n_mR^m\left[f\left(t,p+\frac{C^m\xi}{\lambda},v,\xi+\frac{G\gamma C^m\xi}{\lambda},H\right)-f(t,p,v,\xi,H)\right]\\
&+n_hR^h\left[f\left(t,p+\frac{C^h\left(\alpha^h\Delta-H\right)}{\lambda},v,\xi+\frac{G\gamma C^h\left(\alpha^h\Delta-H\right)}{\lambda},\right.\right.\\
&\qquad\left.\left.H+C^h\left(\alpha^h\Delta-H\right)\right)-f(t,p,v,\xi,H)\right]\\
&-\gamma\xi\frac{\partial f}{\partial\xi}+\frac{1}{2}(\sigma_0)^2\frac{\partial^2 f}{\partial v^2},\end{aligned} \tag{22}$$

where $f(t, p, v, \xi, H)$ is assumed to be twice differentiable in arguments p, v, ξ and H and once differentiable in t with bounded continuous derivatives.

In order to obtain a diffusion limit, we will scale various parameters in the model with the total number of traders in the market n. First we shall assume that the fraction of each category of traders is denoted by β with appropriate superscripts. Thus $\beta^e = n_e/n$, $\beta^v = n_v/n$, $\beta^m = n_m/n$ and $\beta^h = n_h/n$. Hence the number of traders within the different categories is given by $n_e = \beta^e n$, $n_v = \beta^v n$, $n_m = \beta^m n$ and $n_h = \beta^h n$. We shall be concerned with the asymptotic behavior as $n \to \infty$ with β^* parameters being held constant. We shall assume that the trading intensity of each trader remains constant as n increases, thus the R^* parameters are independent of n. Furthermore, we also assume that all trade size parameters C^* scale proportional to $1/n$. The interpretation of this scaling is as follows. If we assume that there is a fixed amount of the asset available for trading, then as the number of traders increases, the amount of the asset held by each trader scales inversely with n. Therefore, the trade size of each trader should also scale inversely with n. Hence, C^* scales as $1/n$. Note that, in the case of hedge traders, it was already explained that $C^h = 1/n_h = 1/(\beta^h n)$. In the case of extraneous traders, only the first two moments of the trade size distribution become relevant in the diffusion limit. We denoted these by C_1^e and $\left(C_2^e\right)^2$, respectively (6). The scaling assumption then simply states that both C_e^1 and C_2^e are proportional to $1/n$. Thus the second moment actually scales as $1/n^2$. We shall rename all the trade size parameters C^* so that, in all equations written so far, C^* is replaced by C^*/n_*, which is equal to $C^*/\beta^* n$. Thus we rewrite (22) for the generator $\mathcal{A}$ showing the dependence on n as

$$
\begin{aligned}
\mathcal{A}_n f = \frac{\partial f}{\partial t} &+ n\beta^e R^e \int_{\mathbb{R}} \left[f\left(t, p + \frac{y}{\lambda}, v, \xi + \frac{G\gamma y}{\lambda}, H \right) - f(t, p, v, \xi, H) \right] \nu^e(\mathrm{d}y) \\
&+ n\beta_v R^v \left[f\left(t, p + \frac{C^v(v-p)}{n\beta_v \lambda}, v, \xi + \frac{G\gamma C^v(v-p)}{n\beta_v \lambda}, H \right) - f(t, p, v, \xi, H) \right] \\
&+ n\beta^m R^m \left[f\left(t, p + \frac{C^m \xi}{n\beta^m \lambda}, v, \xi + \frac{G\gamma C^m \xi}{n\beta^m \lambda}, H \right) - f(t, p, v, \xi, H) \right] \\
&+ n\beta^h R^h \left[f\left(t, p + \frac{C^h(\alpha^h \Delta - H)}{n\beta^h \lambda}, v, \xi + \frac{G\gamma C^h(\alpha^h \Delta - H)}{n\beta^h \lambda}, \right. \right. \\
&\qquad\qquad \left. \left. H + \frac{C^h}{n\beta^h}(\alpha^h \Delta - H) \right) - f(t, p, v, \xi, H) \right] \\
&- \gamma \xi \frac{\partial f}{\partial \xi} + \frac{1}{2}(\sigma_0)^2 \frac{\partial^2 f}{\partial v^2}.
\end{aligned} \tag{23}
$$

We also rewrite (6) in terms of the renamed parameters:

$$
C_1^e = \beta^e n \int_{\mathbb{R}} y\, \nu^e(\mathrm{d}y), \qquad \left(C_2^e\right)^2 = (\beta^e n)^2 \int_{\mathbb{R}} y^2 \nu^e(\mathrm{d}y). \tag{24}
$$

Now we consider the sequence of generators $\mathcal{A}_n$ indexed by n, and show that this sequence is asymptotic to another sequence $\mathcal{B}_n$ of generators which correspond to Ito diffusion equations. More specifically, for each function $f(t, p, v, \xi, H)$ with the differentiability conditions stated above we shall show that

$$\mathcal{A}_n f \sim \mathcal{B}_n f, \quad n \to \infty,$$

pointwise. We take this as justification for using the Ito diffusion equations corresponding to $\mathcal{B}_n$ in place of the jump Markov model described by (18), (19), (20) and (21). See Remark 2.

To obtain the appropriate diffusion generator $\mathcal{B}_n$ we simply truncate to terms of order $O(1)$ and $O(n^{-1})$ after applying Talyor's theorem to $\mathcal{A}_n f$ to expand around (t, p, v, ξ, H). This leads to the equation for $\mathcal{B}_n f$ where all derivatives of f are evaluated at (t, p, v, ξ, H):

$$\begin{aligned}
\mathcal{B}_n f = {} & \frac{\partial f}{\partial t} + R^e \left[\frac{\partial f}{\partial p} \frac{C_1^e}{\lambda} + \frac{\partial f}{\partial \xi} \frac{G\gamma C_1^e}{\lambda} + \frac{1}{2} \frac{1}{n\beta^e} \frac{\partial^2 f}{\partial p^2} \frac{\left(C_2^e\right)^2}{\lambda^2} \right. \\
& \left. + \frac{1}{2} \frac{1}{n\beta^e} \frac{\partial^2 f}{\partial \xi^2} \frac{G^2\gamma^2 \left(C_2^e\right)^2}{\lambda^2} + \frac{1}{n\beta^e} \frac{\partial^2 f}{\partial p \partial \xi} \frac{G\gamma \left(C_2^e\right)^2}{\lambda^2} \right] \\
& + R^v \left[\frac{\partial f}{\partial p} \frac{C^v(v-p)}{\lambda} + \frac{\partial f}{\partial \xi} \frac{G\gamma C^v(v-p)}{\lambda} + \frac{1}{2} \frac{1}{n\beta^v} \frac{\partial^2 f}{\partial p^2} \left(\frac{C^v(v-p)}{\lambda} \right)^2 \right. \\
& \left. + \frac{1}{2} \frac{1}{n\beta^v} \frac{\partial^2 f}{\partial \xi^2} \left(\frac{G\gamma C^v(v-p)^2}{\lambda} \right) + \frac{1}{n\beta^v} \frac{\partial^2 f}{\partial p \partial \xi} G\gamma \left(\frac{C^v(v-p)}{\lambda} \right)^2 \right] \\
& + R^m \left[\frac{\partial f}{\partial p} \frac{C^m \xi}{\lambda} + \frac{\partial f}{\partial \xi} \frac{G\gamma C^m \xi}{\lambda} + \frac{1}{2} \frac{1}{n\beta^m} \frac{\partial^2 f}{\partial p^2} \left(\frac{C^m \xi}{\lambda} \right)^2 \right. \\
& \left. + \frac{1}{2} \frac{1}{n\beta^m} \frac{\partial^2 f}{\partial \xi^2} \left(\frac{G\gamma C^m \xi}{\lambda} \right)^2 + \frac{1}{n\beta^m} \frac{\partial^2 f}{\partial p \partial \xi} G\gamma \left(\frac{C^m \xi}{\lambda} \right)^2 \right] \\
& + R^h \left[\frac{\partial f}{\partial p} \frac{C^h\left(\alpha^h \Delta - H\right)}{\lambda} + \frac{\partial f}{\partial H} C^h\left(\alpha^h \Delta - H\right) + \frac{\partial f}{\partial \xi} \frac{G\gamma C^h\left(\alpha^h \Delta - H\right)}{\lambda} \right. \\
& + \frac{1}{2} \frac{1}{n\beta^h} \frac{\partial^2 f}{\partial p^2} \left(\frac{C^h\left(\alpha^h \Delta - H\right)}{\lambda} \right)^2 + \frac{1}{2} \frac{1}{n\beta^h} \frac{\partial^2 f}{\partial \xi^2} \left(\frac{G\gamma C^h\left(\alpha^h \Delta - H\right)}{\lambda} \right)^2 \\
& + \frac{1}{2} \frac{1}{n\beta^h} \frac{\partial^2 f}{\partial H^2} \left(C^h\left(\alpha^h \Delta - H\right) \right)^2 + \frac{1}{n\beta^h} \frac{\partial^2 f}{\partial p \partial \xi} \frac{G\gamma \left(C^h\left(\alpha^h \Delta - H\right) \right)^2}{\lambda^2} \\
& \left. + \frac{1}{n\beta^h} \frac{\partial^2 f}{\partial p \partial H} \frac{\left(C^h\left(\alpha^h \Delta - H\right) \right)^2}{\lambda} + \frac{1}{n\beta^h} \frac{\partial^2 f}{\partial \xi \partial H} \frac{G\gamma \left(C^h\left(\alpha^h \Delta - H\right) \right)^2}{\lambda} \right] \\
& - \gamma \xi \frac{\partial f}{\partial \xi} + \frac{1}{2} (\sigma^v)^2 \frac{\partial^2 f}{\partial v^2}.
\end{aligned} \tag{25}$$

By the assumed continuity of the second partial derivatives of f and Taylor's theorem it follows that $\mathcal{B}_n f$ and $\mathcal{A}_n f$ differ by terms of order $o(n^{-1})$ as $n \to \infty$, and hence $\mathcal{B}_n f \sim \mathcal{A}_n f$ as $n \to \infty$.

We observe that $\mathcal{B}_n$ corresponds to the following Ito–SDE model (see Appendix B or Oksendal, 1998):

$$\begin{aligned} \mathrm{d}p = {} & \left[\frac{R^e C_1^e}{\lambda} + \frac{R^v C^v (v-p)}{\lambda} + \frac{R^m C^m \xi}{\lambda} + \frac{R^h C^h \left(\alpha^h \Delta - H\right)}{\lambda}\right] \mathrm{d}t \\ & + \sqrt{\frac{R^e}{n\beta^e}} \frac{C_2^e}{\lambda} \mathrm{d}B^e + \sqrt{\frac{R^v}{n\beta^v}} \frac{C^v (v-p)}{\lambda} \mathrm{d}B^v + \sqrt{\frac{R^m}{n\beta^m}} \frac{C^m \xi}{\lambda} \mathrm{d}B^m \\ & + \sqrt{\frac{R^h}{n\beta^h}} \frac{C^h \left(\alpha^h \Delta - H\right)}{\lambda} \mathrm{d}B^h, \end{aligned} \tag{26}$$

$$\mathrm{d}v = \sigma_0 \, \mathrm{d}W, \tag{27}$$

$$\begin{aligned} \mathrm{d}\xi = {} & \left[\frac{G\gamma R^e C_1^e}{\lambda} + \frac{G\gamma R^v C^v (v-p)}{\lambda} + \frac{G\gamma R^m C^m \xi}{\lambda} \right. \\ & \left. + \frac{G\gamma R^h C^h \left(\alpha^h \Delta - H\right)}{\lambda} - \gamma\xi\right] \mathrm{d}t + \sqrt{\frac{R^e}{n\beta^e}} \frac{G\gamma C_2^e}{\lambda} \mathrm{d}B^e \\ & + \sqrt{\frac{R^v}{n\beta^v}} \frac{G\gamma C^v (v-p)}{\lambda} \mathrm{d}B^v + \sqrt{\frac{R^m}{n\beta^m}} \frac{G\gamma C^m \xi}{\lambda} \mathrm{d}B^m \\ & + \sqrt{\frac{R^h}{n\beta^h}} \frac{G\gamma C^h \left(\alpha^h \Delta - H\right)}{\lambda} \mathrm{d}B^h, \end{aligned} \tag{28}$$

$$\mathrm{d}H = R^h C^h \left(\alpha^h \Delta - H\right) \mathrm{d}t + \sqrt{\frac{R^h}{n\beta^h}} C^h \left(\alpha^h \Delta - H\right) \mathrm{d}B^h, \tag{29}$$

where B^e, B^v, B^m, B^h and W are independent standard Brownian motions. We will use this diffusion model (26), (27), (28) and (29) to explore the behavior of different trading strategies on price dynamics. It must be noted that (28) may be written as $\mathrm{d}\xi = G\gamma \ \mathrm{d}p - \gamma\xi \ \mathrm{d}t$, as in (20). It must also be noted that $C^h = 1$ from our modeling assumptions.

Remark 2. A rigorous proof of the diffusion limit is beyond the scope of this paper and is a subject of ongoing work. For a detailed study of diffusion limits where the leading order behavior is the solution to a deterministic ODE and the correction of $O(1/\sqrt{n})$ includes the diffusion terms, the reader is referred to the study of 'density dependent processes' (Ethier and Kurtz, 1986; Kurtz, 1981). Our result cannot be obtained by direct application of the results of Kurtz (1981) and Ethier and Kurtz (1986) as our jump sizes are not restricted to lie in a countable set of lattice points.

3. Analysis of Trader Behavior

In this section we investigate the effects of trading behavior by exploring the dynamics in (26)–(29) both analytically and through simulation. We begin by calibrating parameters in (26)–(29) to a market of only extraneous traders using NYSE data. Next, we include value and momentum traders in the dynamics. With this model, we analytically characterize the mean stability properties of the price dynamics when value and momentum traders interact. In addition, we are able to analytically explore the stochastic volatility properties of the resulting dynamics. This is followed with numerical simulations. Finally, we consider a market with hedge traders and extraneous traders. In this setting, we investigate the stock pinning phenomena due to hedging feedback through numerical simulations.

3.1 Calibration of Model Parameters to Extraneous Traders

We begin by calibrating (26)–(29) to market data when the market is only populated by extraneous traders. We used data from six different NYSE stocks of various sizes (General Electric (GE), 3M (MMM), Hewlett-Packard (HPQ), Airtran Holdings (AAI), OSI Restaurant Partners (OSI), Oxford Industries (OXM)).

The results of the calibration are shown in Table 1. The first two rows under the stock price symbols tabulate the shares outstanding and yearly volatility for each company. The yearly stock price volatility was estimated using daily historical data from January 3, 2000 to February 23, 2007.[1] Table 1 also summarizes data downloaded from the NYSE website (nyse.com) that provides the number and volume of trades that took place on April 1, 2004 for trade sizes in the ranges 0–2000 shares, 2001–5000 shares, 5001–10,000 shares and 10,000+ shares.

The next portion of the table presents parameter values in the dynamics. The parameter β_e is chosen to be 1, indicating a market of only extraneous traders. The next value tabulated ($\int_{\mathbb{R}} |y|v^e(dy)/(n\beta^e)$) is the average trade size per extraneous trade event. This value is estimated from the NYSE trade and volume data by dividing the total volume by the total number of trades. The next quantity $C_2^e/(n\beta^e)$ is the square root of the second moment of the trade size per extraneous trade event. This was estimated by computing the square of the average trade size in each of the ranges, 0–2000, 2001–5000, 5001–10,000 and 10,000+, weighting these numbers by the number of trades in that particular trade range, and then computing the square root of the average. Thus, this estimate is smaller than the true value, but we assume it to be close enough for our purposes.

In the following row, the total number of extraneous traders n is estimated by dividing the shares outstanding by the average trade size $\int_{\mathbb{R}} |y|v^e(dy)/(n\beta^e)$. Thus, we have assumed that the average trader holds the average trade size, and trades everything (s)he owns at each trade event. Using n, one may then easily compute C_2^e/β^e from values above. Note that since $C_2^e/(n\beta^e)$ is related to the average trade size per trade event, we assume that its value should be constant regardless of the actual fraction β^e of traders that are extraneous traders. Since n is a fixed constant, $C_2^e/(n\beta^e)$ can only be constant if C_2^e/β^e is constant. Thus, we tabulate a value for the ratio C_2^e/β^e that is assumed to be constant regardless of the value of β^e. Note that

Table 1. Extraneous parameter calibration for GE, MMM, HPQ, AAI, OSI and OXM (note: calculations were done to many digit precision in Excel but only up to two digits beyond the decimal point is shown in the table).

Stock	GE	MMM	HPQ	AAI	OSI	OXM
Shrs. outstd.	10,310,000,000	733,890,000	2,720,000,000	91,050,000	73,950,000	17,780,000
Yearly vol.	0.29255	0.244098	0.444440461	0.584288327	0.330121618	0.337124611
# of trades						
0–2k	4552	4300	3559	1073	1369	334
2–5k	784	96	489	85	34	1
5–10k	579	20	193	24	19	0
10k+	846	19	150	23	15	1
Volume						
0–2k	3,238,100	1,646,300	1,971,800	561,800	537,100	60,600
2–5k	2,492,600	290,300	1,513,600	268,800	107,500	2100
5–10k	4,014,700	127,000	1,242,300	140,000	129,400	0
10k+	22,287,400	377,100	3,850,000	588,200	351,100	10,300
Parameters						
β^e	1	1	1	1	1	1
$\int_{\mathbb{R}} \|y\| \nu^e(\mathrm{d}y)/(n\beta^e)$	4737.88	550.33	1953.47	1293.61	782.95	217.26
$C_2^e/(n\beta^e)$	9616.33	1486.43	5063.72	3756.43	2591.40	601.33
n	2,176,079.21	1,333,552.73	1,392,391.90	70,384.43	94,450.40	81,836.71
C_2^e/β^e	20,925,885,490	1,982,236,132	7,050,676,303	264,393,842.7	244,758,896.8	49,210,513.68
R^e	0.776741028	0.83142569	0.788391544	4.280065898	3.803583502	1.026434196
λ	42,735,079.41	6,412,063.14	11,937,336.11	3,528,670.08	4,704,994.15	516,962.34

later in Section 3.2.3 where value and momentum traders are calibrated, similar reasoning will lead us to assume the ratios C^v/β^v and C^m/β^m should also be constant across values of β^v and β^m, respectively.

Next, the intensity of the average trader R^e is provided. This is computed by first considering that the shares outstanding divided by the daily volume gives the length of time in years for the entire shares outstanding to be traded. This is also the period of time within which every trader will trade once on average. Therefore, R^e is the reciprocal of this number. Finally, to estimate the liquidity parameter λ, Equation (26) indicates that, in the presence of only extraneous traders, the stock's volatility is given by

$$\sigma = \sqrt{\frac{R^e}{n\beta^e}}\frac{C_2^e}{\lambda}. \tag{30}$$

Using the values in the table, one can then solve for the liquidity parameter.

The parameter values in Table 1 allow us to describe completely the price dynamics in a market populated only with extraneous traders. In fact, it is clear from our model that a market populated with extraneous traders alone will produce price dynamics in the form of geometric Brownian motion. Thus, to create more interesting dynamics we must include other classes of traders.

Next, we will consider a market with extraneous, value and momentum traders. Before performing numerical simulations, we analytically explore the stability effects of the interaction of value and momentum traders. Additionally, we compute the stochastic volatility properties implied by the model. After this analysis, we simulate price paths from this market using parameters in line with those calibrated in Table 1.

3.2 *Value and Momentum Dynamics*

3.2.1 Stability. In this subsection we shall explore the mean stability of our diffusion model described by (26), (27), (28) and (29) in the absence of hedge traders.

For simplicity we shall assume that the perceived true log value v is a constant (equivalently $\sigma_0 = 0$). Also we assume that $C_1^e = 0$, which is equivalent to assuming that the extraneous traders on average purchase or sell equal amounts. Let us define the variable $q = v - p$, which is the discrepancy between the market log price and the perceived true log price. These assumptions lead to the coupled system of two affine SDEs for q and ξ:

$$\begin{aligned} \mathrm{d}q = &\left[-\frac{R^v C^v q}{\lambda} + \frac{R^m C^m \xi}{\lambda}\right]\mathrm{d}t + \sqrt{\frac{R^e}{n\beta^e}}\frac{C_2^e}{\lambda}\mathrm{d}B^e \\ &- \sqrt{\frac{R^v}{n\beta^v}}\frac{C^v q}{\lambda}\mathrm{d}B^v + \sqrt{\frac{R^m}{n\beta^m}}\frac{C^m \xi}{\lambda}\mathrm{d}B^m, \end{aligned} \tag{31}$$

$$\begin{aligned}\mathrm{d}\xi = & \left[-\frac{G\gamma R^v C^v q}{\lambda} + \frac{G\gamma R^m C^m \xi}{\lambda} - \gamma\xi\right]\mathrm{d}t + \sqrt{\frac{R^e}{n\beta^e}}\frac{G\gamma C_2^e}{\lambda}\mathrm{d}B^e \\ & - \sqrt{\frac{R^v}{n\beta^v}}\frac{G\gamma C^v q}{\lambda}\mathrm{d}B^v + \sqrt{\frac{R^m}{n\beta^m}}\frac{G\gamma C^m \xi}{\lambda}\mathrm{d}B^m.\end{aligned} \tag{32}$$

If we define

$$a_v = \frac{R^v C^v}{\lambda}, \qquad a_m = \frac{R^m C^m}{\lambda},$$

for stability of the mean, it suffices to investigate the eigenvalues of the drift matrix

$$\begin{bmatrix} -a_v & a_m \\ -G\gamma a_v & (Ga_m - 1)\gamma \end{bmatrix}. \tag{33}$$

The characteristic equation is

$$z^2 + (a_v - \gamma(Ga_m - 1))z + \gamma a_v = 0.$$

Since $\gamma\ a_v < 0$, asymptotic stability (of the mean) is equivalent to

$$a_v > \gamma(Ga_m - 1). \tag{34}$$

If $Ga_m < 1$, then we always have asymptotic stability of the mean. Even when Ga_m is larger than 1, the presence of value traders can create a stabilizing effect as predicted by (34). Also note that the shorter the effective length $1/\gamma$ of the time window the more likely the system to be unstable.

3.2.2 Stochastic Volatility. In the absence of hedge traders we obtain the equations

$$\begin{aligned}&\mathrm{d}p = a_v(v - p)\mathrm{d}t + a_m\xi\,\mathrm{d}t + \sigma_e\,\mathrm{d}B^e + b_v(v - p)\mathrm{d}B^v + b_m\xi\,\mathrm{d}B^m, \\ &\mathrm{d}v = \sigma_0\,\mathrm{d}W, \\ &\mathrm{d}\xi = G\gamma\,\mathrm{d}p - \gamma\xi\,\mathrm{d}t,\end{aligned} \tag{35}$$

where

$$b_v = \frac{a_v}{\sqrt{n\beta^v R^v}}, \quad b_m = \frac{a_m}{\sqrt{n\beta^m R^m}}, \quad \sigma_e = \sqrt{\frac{R^e}{n\beta^e}}\frac{C_2^e}{\lambda}.$$

If value and momentum traders are absent and only the extraneous traders are present, we obtain the geometric Brownian motion model for stock price $P_t = e^{p_t}$:

$$\mathrm{d}P_t = \sigma_e P_t\,\mathrm{d}B_t^e,$$

with constant volatility σ_e. With the value traders present, we may derive the SDE for P_t. Using the Ito formula we obtain

$$\begin{aligned}\mathrm{d}P_t &= a_v P_t(v_t - p_t)\mathrm{d}t + a_m P_t \xi_t \,\mathrm{d}t + \frac{1}{2} P_t \left(\sigma_e^2 + b_v^2 (v_t - p_t)^2 + b_m^2 \xi_t^2\right)\mathrm{d}t \\ &\quad + \sigma_e P_t \,\mathrm{d}B_t^e + b_v P_t (v_t - p_t)\mathrm{d}B_t^v + b_m P_t \xi_t \,\mathrm{d}B_t^m.\end{aligned} \tag{36}$$

This equation shows that both the diffusion and drift terms are nonlinear in P_t (bearing in mind $p_t = \ln(P_t)$), and also depend on v_t, ξ_t. The instantaneous square volatility L_t of P_t is given by

$$L_t = \sigma_e^2 + b_v^2 (v_t - p_t)^2 + b_m^2 \xi_t^2. \tag{37}$$

Thus the instantaneous square volatility itself is a stochastic process. Using the Ito formula we obtain

$$\begin{aligned}\mathrm{d}L_t &= 2b_v^2 (v_t - p_t)\mathrm{d}v_t + b_v^2 (\mathrm{d}v_t)^2 - 2b_v^2 (v_t - p_t)\mathrm{d}p_t \\ &\quad + 2b_m^2 \xi_t \,\mathrm{d}\xi_t + b_v^2 (\mathrm{d}p_t)^2 + b_m^2 (\mathrm{d}\xi_t)^2.\end{aligned}$$

Substituting $\mathrm{d}\xi_t = G\gamma \,\mathrm{d}p_t - \gamma \xi_t \,\mathrm{d}t$ we obtain

$$\begin{aligned}\mathrm{d}L_t &= 2b_v^2 (v_t - p_t)\mathrm{d}v_t + b_v^2 (\mathrm{d}v_t)^2 + 2\left(-b_v^2 (v_t - p_t) + b_m^2 G\gamma \xi_t\right)\mathrm{d}p_t \\ &\quad - 2b_m^2 \gamma \xi_t^2 \,\mathrm{d}t + \left(b_v^2 + b_m^2 G^2 \gamma^2\right)(\mathrm{d}p_t)^2.\end{aligned}$$

Several stochastic volatility models have been proposed in the literature (Heston, 1998; Hull and White, 1987; Stein and Stein, 1991). A quantity of interest is the conditional covariance of price increments and square volatility increments, i.e. $\mathrm{Cov}(\mathrm{d}P_t, \mathrm{d}L_t | \mathcal{F}_t)$. More precisely, this is the differential of the cross variation process $<P_t, L_t>$ (Oksendal, 1998). To compute $\mathrm{Cov}(\mathrm{d}P_t, \mathrm{d}L_t | \mathcal{F}_t)$, we first observe that $\mathrm{Cov}(\mathrm{d}v_t, \mathrm{d}p_t | \mathcal{F}_t) = 0$ due to the independence of W_t, B_t^e, B_t^m and B_t^v. Thus

$$\mathrm{Cov}(\mathrm{d}P_t, \mathrm{d}L_t | \mathcal{F}_t) = 2P_t\left(b_m^2 G\gamma \xi_t - b_v^2 (v_t - p_t)\right)\mathrm{Var}(\mathrm{d}p_t | \mathcal{F}_t).$$

Noting that $\mathrm{Var}(\mathrm{d}p_t | \mathcal{F}_t) = (\mathrm{d}p_t)^2$, the quadratic variation process of p_t is given by $\left(\sigma_e^2 + b_v^2 (v_t - p_t)^2 + b_m^2 \xi_t^2\right)\mathrm{d}t$, we obtain

$$\mathrm{Cov}(\mathrm{d}P_t, \mathrm{d}L_t | \mathcal{F}_t) = 2P_t\left(b_m^2 G\gamma \xi_t - b_v^2 (v_t - p_t)\right)\left(\sigma_e^2 + b_v^2 (v_t - p_t)^2 + b_m^2 \xi_t^2\right)\mathrm{d}t.$$

Thus the covariance is positive (negative) if the price momentum and the mismatch $p_t - v_t$ are both positive (negative respectively).

3.2.3 Calibration of Value and Momentum Traders. We consider the stock MMM in Table 1. To calibrate the parameters for value traders, we began by assuming that value traders transact at the same intensity as extraneous traders. Thus, we chose $R^v = R^e = 0.83142569$ from Table 1.

Next, we calibrated the ratio C^v/β^v by considering markets with different (but roughly realistic) proportions of value traders (i.e. different values of β^v) and chose the value for C^v/β^v that best preserved the total volume traded in the entire market relative to a market of only extraneous traders. That is, in a market of only extraneous traders, the expected yearly volume is given by the average trade size from Table 1 ($\int_{\mathbb{R}} |y|\nu^e(\mathrm{d}y)/(n\beta^e) = 550.33$) multiplied by the number of extraneous traders (n=1,333,552.73) multiplied by the intensity (R^e=0.83142569). This gives a total volume of 610,178,386.8.

When the market is made up of a fraction β^v of value traders and $\beta^e = 1-\beta^v$ of extraneous traders, then we expect the total volume in the market to be roughly the same as the total volume in the market with only extraneous traders. Thus, the value traders should make up a fraction β^v of the overall volume of 610,178,386.8.

Now, the volume of the value traders varies over time depending on the difference between the market price p_t and the value price v_t. However, given a constant value of $(v_t - p_t)$, the yearly expected volume for value traders would be

$$\begin{aligned}\text{Expected Trade Volume of Value Traders} &= C^v|v_t - p_t|(R^v) \\ &= \left(\frac{C^v}{\beta^v}\right)\beta^v|v_t - p_t|(R^v).\end{aligned} \tag{38}$$

This should be equal to the corresponding fraction of the total volume from an all extraneous trader market, $\beta^v \times (610{,}178{,}386.8)$. Setting these values equal gives

$$\left(\frac{C^v}{\beta^v}\right)|v_t - p_t|(R^v) = 610{,}178{,}386.8, \tag{39}$$

where we assume the quantity C^v/β^v is a constant for all values of β^v (just as C_2^e/β^e is assumed constant for extraneous traders across values of β^e (see the discussion of Table 1 values in Section 3.1 for clarification)). The left-hand side of (39) can only be computed by simulations that depend upon the value of β^v. Thus, we selected various values of β^c and attempted to find a reasonable value of C^v/β^v that approximately satisfied Equation (39). Furthermore, we represented C^v/β^v relative to C_2^e/β^e as $C^v/\beta^v = \kappa(C_2^e/\beta^e)$ and equivalently searched for a value of κ.

Figure 1 shows the left-hand side of (39) versus the right-hand value for β^v=0.1, 0.25 and 0.5 under κ=35 and the volatility of the value price as σ_0=0.12. As the figure shows, this value of κ=35 is a slight underestimate across those values of β^v, but is of the correct order of magnitude. Furthermore, when momentum traders are added to the market that increases the volume of value traders, thus correcting for

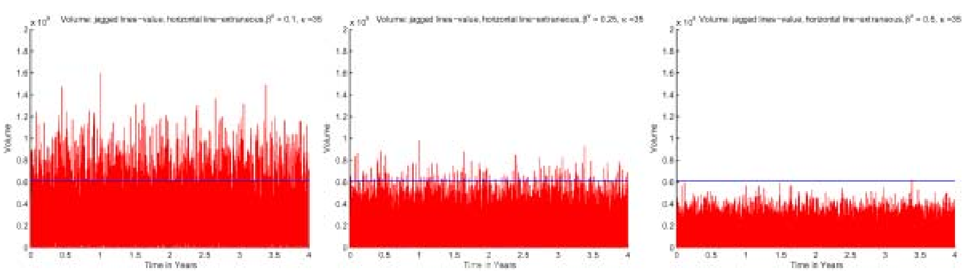

Figure 1. Volume of extraneous traders (horizontal line) and volume of value traders (jagged lines) with κ=35 corresponding to Equation (39) for three values of β^v. Left, β^v=0.1; middle, β^v=0.25; right, β^v=0.5.

the underestimation. Thus, $\kappa=35$ is a reasonable choice and we use it across all simulations in this section.

For momentum traders, parameter values were chosen to explore a range of phenomena rather than to fit a single specific market scenario. Furthermore, calibration as was done with the value traders is difficult due to the destabilizing effect that momentum traders have on the market. Thus, for simplicity with momentum traders we used $C^m/\beta^m = C_2^e/\beta^e = 1{,}982{,}236{,}132$. Furthermore, we selected $G=1$ and set the intensity of the momentum traders equal to the time scale of their trend following, $R^m=\gamma$. Thus, momentum traders that look at short-time-scale trends use a larger value of γ and are assumed to trade more often with an intensity matching the time-scale.

3.2.4 Simulations. With the parameter selections guided by the previous subsection, we ran simulation results of a market with extraneous, value and momentum traders.

Figure 2 provides simulations of a market for 1000 days. Plots on the left show the value price and the market price, the middle plots provide the corresponding daily returns, and the plots on the right show the volatility of the market price averaged over a 20 day window. The top plot is a market with 40% value traders ($\beta^v=0.4$) and 60% extraneous traders ($\beta^e=0.6$). With value traders in the market, the market price closely follows the value price. Furthermore, the volatility of the market price is higher than the value price (see the right-hand plot) and no volatility spikes or clustering is present. As the proportion of value traders is increased, the market volatility decreases due to the increased action of value traders holding the market price tighter to the value price. Overall, the top plot of Figure 2 is qualitatively representative of the return and volatility characteristics seen for a wide range of β^v values.

Next, we performed simulations with 0.4 of the market as value traders ($\beta^v=0.4$) and 0.3 as momentum traders ($\beta^m=0.3$). The rest of the traders in the market were assumed to be extraneous traders ($\beta^e=1-\beta^v-\beta^m=0.3$). The middle and bottom plots in Figure 2 show the results. In the middle plot, the momentum traders use a parameter value of $\gamma=2$ and a trading intensity of $R^m=2$. Roughly speaking, this corresponds to a half year time scale for both the frequency of their trades and the time scale of the trends they attempt to capture. From the figure, one can see the emergence of infrequent periods of slightly increased volatility. The bottom portion of Figure 2 shows the results for $\gamma=4$ and $R^m=4$. Thus, the momentum traders trade more frequently and based upon a shorter time scale. In this case, periods of increased and clustered volatility are clearly present. From Equation (37), this clustering is predicted to occur at times when the market price is trending (large ξ_t) and/or is different from the value price (large v_t-p_t). These conditions induce increased trading by momentum traders and/or value traders, respectively, leading to periods of increased volatility. It is interesting to note that this volatility clustering appears before the dynamics enters a region with complex eigenvalues (cf. Equation (33)). In fact, in the complex eigenvalue region the market price dynamics exhibits unrealistic oscillations around the value price.

Finally, we compared our simulation results against the actual price data from MMM between 2002 and 2006. The data show periodically occurring price and

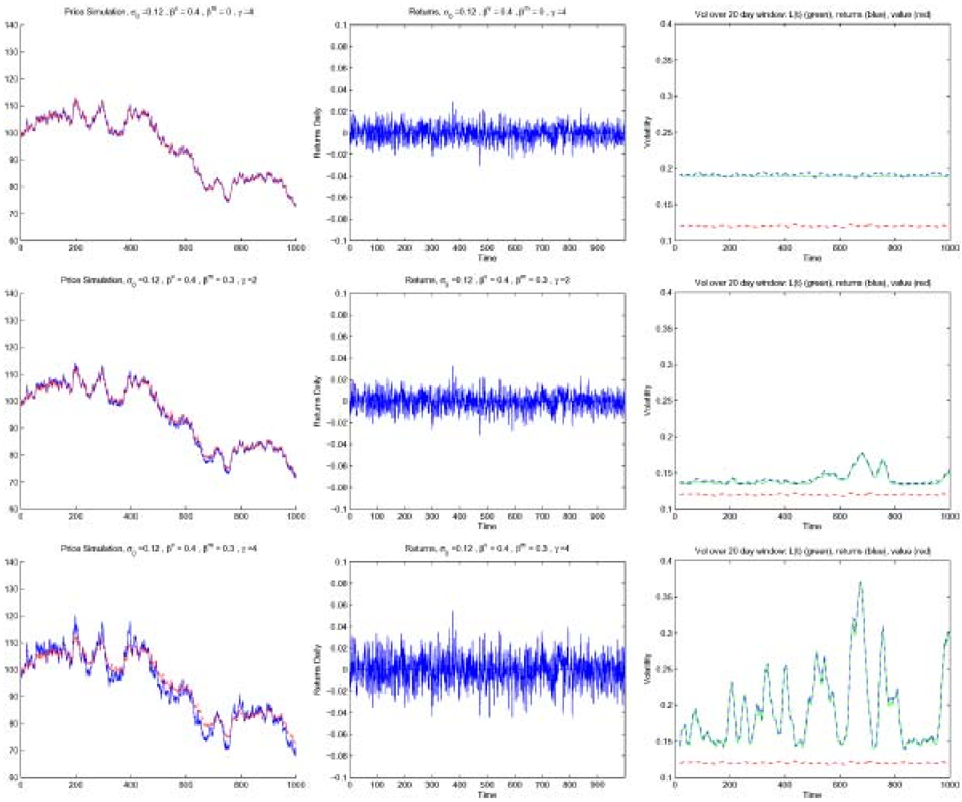

Figure 2. Extraneous, value and momentum traders. Plots on the left show the value price (light line) and the market price (dark line). Plots in the middle are the returns corresponding to the market price. Plots on the right are the volatility of the value price (lower line), market price (dashed upper line) and theoretical volatility $\sqrt{L(t)}$ averaged over a 20 day window (light upper line). Top, β^v=0.4 and β^m=0; middle, β^v=0.4 and β^m=0.3, γ=2; bottom, β^v=0.4 and β^m=0.3, γ=4.

return spikes that are of a different nature than the volatility clustering seen in our simulations. We hypothesized that these spikes could be due to periodic information shocks to the value price, and come in the form of earnings announcements or the like. Therefore, we attempted to mimic this by placing a 3% jump in the value price (with equal probability of being positive or negative) at quarterly intervals.

The simulated results appear in the top plots of Figure 3 for β^v=0.6, β^m=0.3 and γ=4 and show a qualitative resemblance to the actual data shown in the bottom plots. The plots on the left show the price, while the middle plots show returns and the plots on the right are an estimate of the volatility using daily data over a 20 day window. Note that, given only daily data, one cannot accurately estimate the volatility inside a 20 day window. Thus, in our simulation in the top right volatility plot, for consistency we also only used daily data to estimate the volatility. The resulting plots of volatility (right-hand side of Figure 3) are qualitatively quite similar. A perceptible difference is that increases in volatility in the simulation tend

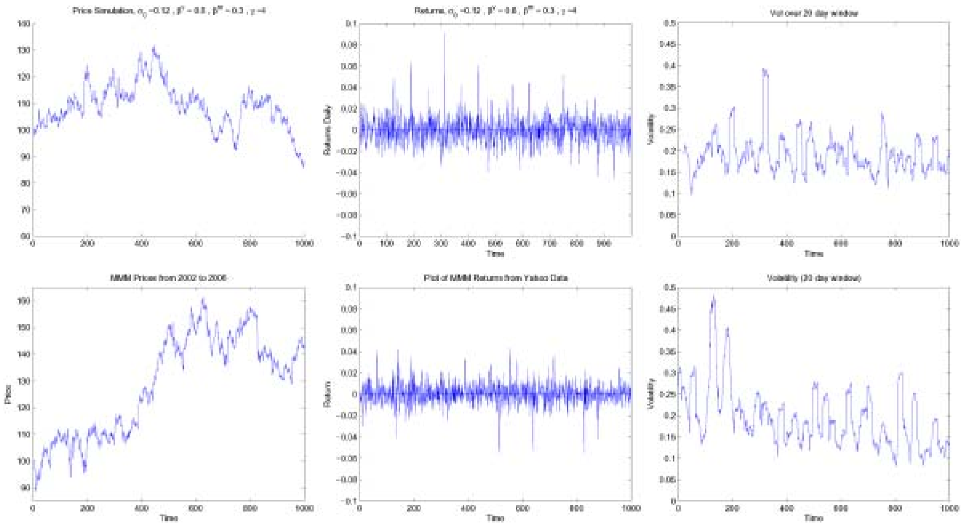

Figure 3. Top: simulation of the value price with shocks (left), its returns (middle) and estimated volatility using a 20 day window (right). Parameters: $\beta^v=0.4$, $\beta^m=0.3$ and $\gamma=4$. Bottom: data from MMM for the period 2002 into 2006.

to happen more abruptly than in the actual data, where a volatility spike can increase over a couple of days.

Our calibrated results are able to reproduce price dynamics that show a qualitative resemblance to the actual market data, especially after including periodic jumps in the value price. Furthermore, even without shocks to the value price, volatility clustering appears due to the interaction of value and momentum traders. Thus, our model is able to qualitatively capture some features of real markets while also allowing for reasonable calibration procedures.

Our model and results should be contrasted with existing literature that also creates realistic price dynamics. For example, in the work of Bak *et al.* (1997), Lux and Marchesi (2000) and Qiu *et al.* (2007a), volatility clustering and heavy tails are exhibited, but through a different mechanism. Those models allow agents to switch between behaviors, giving rise to crowd or imitation effects that lead to heavy tails and volatility clustering. In our simulations, agents are not allowed to switch between strategies. Furthermore, our model uses a simple linear price formation rule depending on a single liquidity parameter and does not capture effects at the level of the order book. For example, Chiarella and Iori (2002) show through the simulation of an order driven double auction market that order book effects among traders can create significant volatility clustering and heavy tails.

Thus, our model is not alone in being able to mimic market price dynamics. Nevertheless, our ability to calibrate to actual market data and provide low order stochastic differential equation models is perhaps a useful link between the effects of trader behavior and phenomenological models used in areas of finance such as option pricing.

3.3 Hedge Traders and Stock Pinning

In this section we explore the effects of hedging options on the price process of the underlying asset in a market populated with only extraneous and hedge traders.

In the context of option pricing and agent-based simulations, Qiu *et al.* (2007b) have used market simulations involving speculators and arbitrageurs to generate the well-known implied volatility smile phenomenon. Qiu *et al.*'s volatility smile paper models the buying and selling of options themselves to explain the volatility smile. While we study hedge traders, we only model the buying/selling of the underlying stock and not that of the option. While our framework could be applied to the modeling of buying/selling of options we do not pursue that in this paper. Instead, we explore a different phenomenon, namely stock pinning.

Stock pinning is a phenomenon in which a stock with high open interest in an option is drawn to the strike price of the option at expiration. This phenomenon has been empirically observed (Krishnan and Nelken, 2001) and studied in the context of hedging (Avellaneda and Lipkin, 2003). In addition to the pinning phenomenon, our model also predicts reverse pinning when the hedgers are short options and hence replicating a long position. In this case, the hedge traders drive prices away from the strike price of the option.

To calibrate parameters for hedge traders, we used option data from Yahoo!Finance. In Table 2 we have collected data on March 1, 2007 for the option strike and expiration with the largest open interest values corresponding to each stock. The first four rows of the table give the option specifications.

The next two rows present our assumptions about the fraction of the open interest that is being hedged, and the average number of trades in a day that a hedger makes. In this table, we have assumed that half of the open interest is being hedged, and that hedgers trade twice a day on average. From these assumptions, the parameter values can be computed. For hedge traders, C^h is always set to 1. The parameter α^h represents the size of the option position held by the hedgers and indicates the direction of their hedging strategy. Thus, the negative value of α^h in the table indicates that hedgers hold a long position in options equal to one half the total open interest and are replicating an offsetting short position. An R^h value of 500 comes

Table 2. Hedge parameter calibration data for GE, MMM, HPQ, AAI, OSI and OXM.

Underlying	GE	MMM	HPQ	AAI	OSI	OXM
Call/put	Call	Put	Call	Call	Call	Put
Strike	37.5	70	45	12.5	40	40
Expiration	3/16/2007	4/20/2007	5/18/2007	3/16/2007	5/18/2007	4/20/2007
Open interest	73,672	15,880	22,531	7776	13,416	386
Fraction of hedgers	0.5	0.5	0.5	0.5	0.5	0.5
Trades in day	2	2	2	2	2	2
C^h	1	1	1	1	1	1
α^h	−36,836	−7940	−11,265.5	−3888	−6708	−193
R^h	500	500	500	500	500	500
β^h	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001

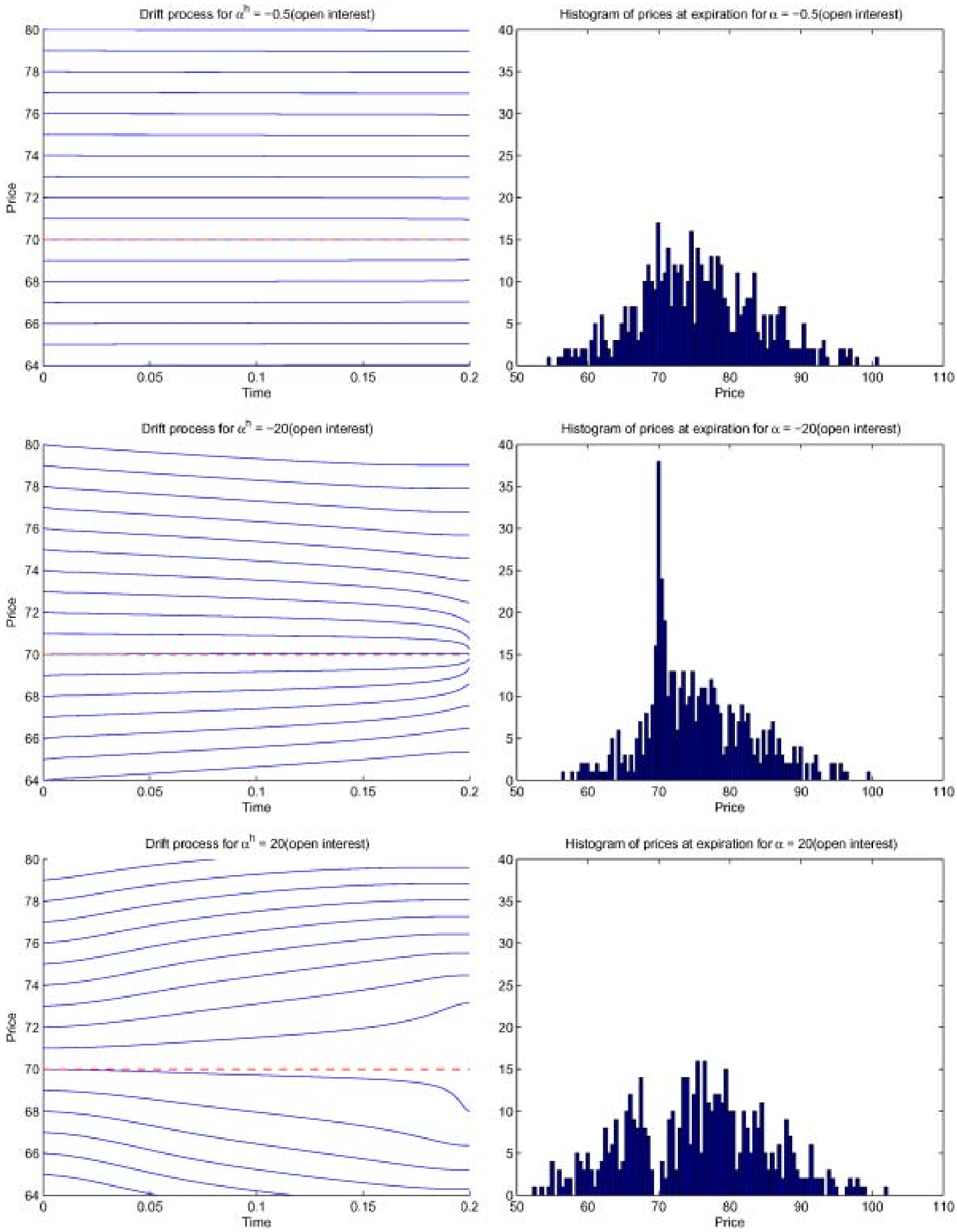

Figure 4. Hedge and extraneous traders. Left-hand plots show simulations of the drift term in the dynamics for MMM with $\alpha^h = -0.5$ (top), $\alpha^h = -20$ (middle) and $\alpha^h = 20$ (bottom). Right-hand plots are histograms of the price at expiration for 1000 simulations of the full stochastic dynamics from the initial condition \$75 ($T=0.2$, $K=70$).

from our assumption of 250 trading days in a year and that hedgers trade twice a day. Finally, we chose β^h, which is the fraction of hedgers in the market, to be 0.0001, representing a very small fraction of all traders.

All six of the stocks and option values produce roughly similar hedge trading dynamics. For example, the quantity $(R^hC^h\alpha^h/\lambda)$ that appears in the drift term of the price dynamics (26) corresponding to hedge traders ranges from -0.187 for OXM to -0.713 for OSI, and the quantity $(\left(\sqrt{R^h/n\beta^h}\right)(C^h\alpha^h/\lambda))$ corresponding to the volatility produced by hedge traders ranges from -0.0013 for GE to -0.0104 for OSI.

Since the six stocks in Table 2 produced similar parameter values, to explore the price dynamics caused by hedging, again we ran simulations only on the stock MMM. In addition to the parameter values in the table, we varied the parameter α^h that represents the open interest of the hedgers in the market. Avellaneda and Lipkin (2003) report that pinning often occurs at times when the open interest is unusually large. In particular, they give an example where open interest for a single strike in the front month expiration was more than 25,000 on a stock that averaged a few hundred contracts per strike! To capture this range of potential situations, we varied α^h from half of the open interest as given in Table 2 to 20 times the open interest, and considered both positive and negative values.

First, using MMM parameter values and a strike price of \$70, we simulated only the drift terms of the dynamics, ignoring the diffusion terms, for different initial conditions. The results using α^h equal to -0.5, -20 and 20 times the open interest are shown in the left-hand side of Figure 4. Next, we simulated the full stochastic dynamics starting from an initial price of \$75 for the same values of α^h. The results of 1000 sample paths at expiration are shown on the right-hand side of Figure 4. With parameter values corresponding to the open interest data observed on March 1, 2007 in Table 2 and α^h equal to -0.5 the open interest, hedging activity is not large enough to produce a noticeable pinning effect. Thus, assuming that Table 2 represents typical market values, we are unlikely to observe pinning under typical option and hedging activity. However, when α^h is chosen as -20 times the open interest, pinning is clearly observed. In fact, at an α^h value of -5 times the open interest, pinning becomes perceptible in the simulations. Thus, the situation described by Avellaneda and Lipkin (2003) with nearly a 100-fold increase in open interest is likely to induce severe pinning. In the bottom plots of Figure 4 we assume that hedgers were replicating a positive position in options equal to 20 times the open interest. In this case, we observe a reverse pinning phenomena in which the underlying stock price is driven away from the strike price. Thus, the histogram shows a strong dip at the strike price of \$70.

4. Conclusions

In this paper we built a model of price dynamics beginning from the trading strategies and discrete buy and sell orders of individual agents. We considered four classes of traders: extraneous, value, momentum and hedgers. The demand process of each trader was modeled to capture their trading style using Poisson random measures, resulting in a discrete event model of price movement. To simplify this model, we aggregated trading classes and took diffusion limits to arrive at a coupled set of Ito diffusions. We analyzed these equations both analytically and through simulation to explore the effects of trading strategies on price dynamics. In particular, we considered the stability and stochastic volatility consequences of value and momentum trading, and the stock pinning phenomenon associated with hedge trading. Overall, we produced a consistent model beginning from first principles that allows one to gain a better understanding of the role of trader behavior in price dynamics.

Future work will include a detailed model of the dynamics that underlies the price formation rule. This will involve including a model of the market maker and the order book. Additionally, a rigorous proof of the diffusion limit is a subject of ongoing research.

Acknowledgement

The research of M.R. was supported by NSF grant DMS-0610013.

Note

[1] Data from Yahoo!Finance.

References

Applebaum, D. (2004) *Levy Processes and Stochastic Calculus* (Cambridge: Cambridge University Press).

Avellaneda, M. and Lipkin, M. D. (2003) A market-induced mechanism for stock pinning. *Quantitative Finance*, 3(6), pp. 417–425.

Bak, P., Paczuski, M. and Shubik, M. (1997) Price variations in a stock market with many agents. *Physica A*, 246(3–4), pp. 430–453.

Bitchteler, K. (2002) *Stochastic Integration with Jumps* (Cambridge: Cambridge University Press).

Chiarella, C. and Iori, G. (2002) A simulation analysis of the microstructure of double auction markets. *Quantitative Finance*, 2(5), pp. 346–353.

Ethier, S. N. and Kurtz, T. G. (1986) *Markov Processes: Characterization and Convergence* (New York: Wiley).

Farmer, J. D. (2000) Market force, ecology, and evolution. Working Paper, Santa Fe Institute.

Farmer, J. D. and Joshi, S. (2002) Master curve for price-impact function. *Journal of Economic Behavior and Organization*, 49(2), pp. 149–171.

Frey, R. and Stremme, A. (1997) Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 7(4), pp. 351–374.

Heston, S. L. (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2), pp. 327–343.

Hull, J. and White, A. (1987) The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42(2), pp. 281–300.

Ivanov, P. C., Yuen, A., Podobnik, B. and Lee, Y. (2004) Common scaling patterns in intertrade times of US stocks. *Physical Review E*, 69(5), 056107.

Jacod, J. and Shiryaev, A. N. (2003) *Limit Theorems for Stochastic Processes* (New York: Springer).

Johansen, A. (2004) Probing human response times. *Physica A*, 338(1–2), pp. 286–291.

Krishnan, H. and Nelken, I. (2001) The effect of stock pinning upon option prices. *RISK*, December, S17–S20.

Kurtz, T. G. (1981) *Approximation of Population Processes* (Philadelphia: SIAM).

LeBaron, B. (2000) Agent based computational finance: suggested readings and early research. *Journal of Economic Dynamics and Control*, 24(5–7), pp. 679–702.

LeBaron, B. (2001) A builder's guide to agent based financial markets. *Quantitative Finance*, 1(2), pp. 254–261.

LeBaron, B. (2002) Building the SFI artificial stock market. Working Paper, Brandeis University.

Lux, T. (1997) Time variation of second moments from a noise trader/infection model. *Journal of Economic Dynamics and Control*, 22(1), pp. 1–38.

Lux, T. and Marchesi, M. (2000) Volatility clustering in financial markets: a microsimulation of interacting agents. *International Journal of Theoretical and Applied Finance*, 3(4), pp. 675–702.

Masoliver, J., Montero, M. and Weiss, G. H. (2003) Continuous-time random-walk model for financial distributions. *Physical Review E*, 67(2), pp. 1–10.

Oksendal, B. (1998) *Stochastic Differential Equations: An Introduction with Applications*, 5th edn (New York: Springer).

Palatella, L., Perello, J., Montero, M. and Masoliver, J. (2004) Activity autocorrelation in financial markets. *The European Physical Journal* B, 38(4), pp. 671–677.

Platen, E. and Schweizer, M. (1998) On feedback effects from hedging derivatives. *Mathematical Finance*, 8(1), pp. 67–84.

Qiu, G., Kandhai, D. and Sloot, P. M. A. (2007a) Understanding the complex dynamics of stock markets through cellular automata. *Physical Review E*(1), 046116.

Qiu, G., Kandhai, D. and Sloot, P. M. A. (2007b) Understanding the volatility smile of options markets through microsimulation. *arXiv.org archive* (arXiv:0709.2070v1).

Raberto, M., Cincottia, S., Focardib, S. M. and Marchesi, M. (2001) Agent based simulation of a financial market. *Physica A*, 299(1), pp. 319–327.

Scalas, E., Gorenflo, R., Luckock, H., Mainardi, F., Mantelli, M. and Raberto, M. (2004) Anomalous waiting times in high-frequency financial data. *Quantitative Finance*, 4(6), pp. 695–702.

Scalas, E., Gorenflo, R., Luckock, H., Mainardi, F., Mantelli, M. and Raberto, M. (2005) On the intertrade waiting-time distribution. *Finance Letters*, 3, pp. 38–43.

Schonbucher, P. and Wilmott, P. (2000) The feedback-effect of hedging in illiquid markets. *SIAM Journal of Applied Mathematics*, 61(1), pp. 232–272.

Silva, A. C. and Yakovenko, V. M. (2007) Stochastic volatility of financial markets as the fluctuating rate of trading: an empirical study. *Physica A*, 382(1), pp. 278–285.

Sircar, R. and Papanicolaou, G. (1998) General Black–Scholes models accounting for increased market volatility from hedging strategies. *Applied Mathematical Finance*, 5(1), pp. 45–82.

Stein, E. M. and Stein, J. C. (1991) Stock price distribution with stochastic volatility: an analytic approach. *Review of Financial Studies*, 4(4), pp. 727–752.

Vazquez, A., Oliveira, J. G., Dezso, Z., Goh, K.-L., Kondor, I. and Barabasi, A.-L. (2006) Modeling bursts and heavy tails in human dynamics. *Physical Review E*, 73(3), pp. 1–19.

Appendix A. Poisson Random Measures

Poisson random measures provide a convenient way to describe jump processes that have prescribed jump rates and jump size distributions. A Poisson random measure μ on $\mathbb{R}_+ \times \mathbb{R}$ is a random assignment of markers (isolated points) on $\mathbb{R}_+ \times \mathbb{R}$ with the following three properties. Firstly, the number of markers in a measurable subset $A \subset \mathbb{R}_+ \times \mathbb{R}$ is Poisson distributed with mean $m(A)$ where m is an underlying measure on $\mathbb{R}_+ \times \mathbb{R}$. This measure m is called the intensity measure of the Poisson random measure. Secondly, given two disjoint subsets $A, B \subset \mathbb{R}_+ \times \mathbb{R}$ the number of markers in A and B are independent. The third condition requires that the number of markers in any set of the form $\{t\} \times A$ where $t \in \mathbb{R}_+$ and $A \subset \mathbb{R}$ is either 0 or 1 with probability 1. This ensures that there is no more than one jump at any given time instant.

In the models encountered in this paper the intensity measure m takes the form $m = R\, \mathrm{d}t \times v$ where $\mathrm{d}t$ is the Lebesgue measure on $\mathbb{R}_+$ and v is a probability measure on $\mathbb{R}$. Given a Poisson random measure μ on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $R\, \mathrm{d}t \times v$,

$$Z_t = \int_{s \in (0,\, t]} \int_{y \in \mathbb{R}} y\ \mu(\mathrm{d}s, \mathrm{d}y)$$

describes a process that has a jump rate R and jump size distribution ν on $\mathbb{R}$. In fact, Z_t is a process with independent increments and the probability that exactly one jump with jump size belonging to a subset $A \subset \mathbb{R}$ occurs in an interval $(t, t+h]$ is given by $R\nu(A)h+o(h)$ as $h \to 0+$ and the probability of more than one jump occurring during $(t, t+h]$ is $o(h)$ as $h \to 0+$. By definition, Z_t is a process with right continuous paths with left-hand limits.

One way to look at the infinitesimal description of the process Z_t is to consider how the expected value of any function of Z_t changes in an infinitesimal time interval. Given a function $f : \mathbb{R} \to \mathbb{R}$, and given $Z_t = z$,

$$E(f(Z_{t+h})) = \int_{y \in \mathbb{R}} Rhf(z+y)\nu(\mathrm{d}y) + o(h).$$

For convenience we shall write $\int_0^t$ to mean $\int_{s \in (0,\, t]}$. More complex forms of jump Markov processes X_t are modeled via an integral equation using the Poisson random measure μ by

$$X_t = X_0 + \int_0^t \int_{y \in \mathbb{R}} K(s, X_{s-}, y)\mu(\mathrm{d}s, \mathrm{d}y),$$

where the kernel function $K(t, x, y)$ is assumed to be continuous in t.

Given $X_t = x$ and any function $f : \mathbb{R} \to \mathbb{R}$,

$$E(f(X_{t+h})) = \int_{y \in \mathbb{R}} Rhf(x + K(t, x, y))\nu(\mathrm{d}y) + o(h),$$

as $h \to 0+$. A particular case is when $K(t, x, y)$ is independent of y, $K(t, x, y) = K(t, x)$. In this case, X_t is given by

$$X_t = X_0 + \int_0^t K(s, X_{s-})\mathrm{d}N_s,$$

where N_t is the Poisson process with rate R that counts the number of markers:

$$N_t = \int_0^t \int_{y \in \mathbb{R}} \mu(\mathrm{d}s, \mathrm{d}y).$$

For more details the reader is referred to Bitchteler (2002), Jacod and Shiryaev (2003) and Applebaum (2004).

Appendix B: Generators of Markov Processes

Given an $\mathbb{R}^n$ valued time non-homogeneous Markov process X_t, its time-dependent generator $\mathcal{A}$ is an operator defined by

$$(\mathcal{A}f)(t, x) = \lim_{h \to 0+} \frac{1}{h} E(f(t+h, X_{t+h}) - f(t, x))$$

for all functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ for which this limit exists, and the expected value is taken under the condition that $X_t = x$. Intuitively, given any function $f(t, x)$ of time t and process value x at time t, $(\mathcal{A}f)(t, x)$ is the rate of change of the expected value of

$f(t, X_t)$ at time t. The generator essentially captures all the relevant information about a Markov process. We refer the reader to Ethier and Kurtz (1986) and Applebaum (2004) for more details.

Now we shall outline the derivation of the form for generators of the type of Markov processes that we encounter in this paper. For simplicity we consider an $\mathbb{R}$ valued X_t which satisfies the following equation:

$$X_t = X_0 + \int_0^t \int_{y\in\mathbb{R}} F(s, X_{s-}, y)\mu(\mathrm{d}s, \mathrm{d}y) + \int_0^t G(s, X_{s-})\mathrm{d}W_s + \int_0^t H(s, X_{s-})\mathrm{d}s,$$

where μ and W are a Poisson random measure and a Brownian motion, respectively, that are independent of each other. Let $f : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be twice continuously differentiable. Then by Ito's formula (Applebaum, 2004) we obtain that

$$\begin{aligned} f(t+h, X_{t+h}) = f(t, X_t) &+ \int_t^{t+h} \frac{\partial f}{\partial t}(u, X_{u-})\mathrm{d}u \\ &+ \int_t^{t+h}\int_{y\in\mathbb{R}} (f(u, X_{u-} + F(u, X_{u-}, y)) \\ &- f(u, X_{u-}))u(\mathrm{d}u, \mathrm{d}y) \\ &+ \int_t^{t+h} \frac{\partial f}{\partial x}(u, X_{u-})G(u, X_{u-})\mathrm{d}W_u \\ &+ \int_t^{t+h} \frac{\partial f}{\partial x}(u, X_{u-})H(u, X_{u-})\mathrm{d}u \\ &+ \frac{1}{2}\int_t^{t+h} \frac{\partial^2 f}{\partial x^2}(u, X_{u-})G^2(u, X_{u-})\mathrm{d}u. \end{aligned}$$

Taking the expected value conditioned upon $X_t = x$, we obtain

$$\begin{aligned} E(f(t+h, X_{t+h}) = f(t, x) &+ \int_t^{t+h} E\left\{\frac{\partial f}{\partial t}(u, X_{u-})\right\}\mathrm{d}u \\ &+ \int_t^{t+h}\int_{y\in\mathbb{R}} E\{f(u, X_{u-} + F(u, X_{u-}, y)) \\ &- f(u, X_{u-})\}R\nu(\mathrm{d}y)\mathrm{d}u \\ &+ \int_t^{t+h} E\left\{\frac{\partial f}{\partial x}(u, X_{u-})H(u, X_{u-})\right\}\mathrm{d}u \\ &+ \frac{1}{2}\int_t^{t+h} E\left\{\frac{\partial^2 f}{\partial x^2}(u, X_{u-})G^2(u, X_{u-})\right\}\mathrm{d}u, \end{aligned}$$

where we have used the fact that the expected value of the Ito integral (w.r.t. the Brownian motion) is zero and $v \times R\ \mathrm{d}t$ is the intensity measure of μ. Applying the mean value theorem and taking the limit as $h \to 0+$, we obtain

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}E(f(t+h, X_{t+h})-f(t,x)) = \frac{\partial f}{\partial t}(t,x) &+ \int_{y\in\mathbb{R}} R(f(t, x+F(t,x,y)) \\ &- f(t,x)v(\mathrm{d}y)) + \frac{\partial f}{\partial x}(t,x)H(t,x) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x)G^2(t,x).\end{aligned}$$

The third and fourth terms involving $\partial f/\partial x$ and $\partial^2 f/\partial x^2$ are the drift and diffusion terms. The second term is the term due to the Poisson random measure.