Math 600: Real Analysis

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November 26, 2016

1 Differentiability of functions

We first state a lemma about continuity of linear maps between normed vector spaces.

Lemma 1 Let $(V_i, \|.\|_i)$ be normed vector spaces for $i = 1, 2$ and let $L : V_1 \to V_2$ be linear. Then the following are equivalent:

1. L is continuous on V_1.
2. L is continuous at 0.
3. $L(S)$ is bounded where
$$S = \{x \in V_1 \,|\, \|x\|_1 = 1\},$$
is the unit sphere in V_1.
4. There exists M such that for all $x \in V_1$ it holds that
$$\|L(x)\|_2 \leq M\|x\|_1.$$

Given a continuous linear map $L : V_1 \to V_2$, its *induced norm* $\|L\|$ is defined by
$$\|L\| = \sup\{\|L(x)\|_2 \,|\, x \in V_1, \|x\|_1 \leq 1\}.$$

Let $(V_i, \|.\|_i)$ be normed vector spaces. Let $U \subset V_1$ be an open set, $x \in U$ and let $F : U \subset V_1 \to V_2$ be a map. We say that F is *Frechet differentiable* at x provided there exists a continuous linear map $L : V_1 \to V_2$ such that
$$\lim_{y \to 0} \frac{\|F(x+y) - F(x) - L(y)\|_2}{\|y\|_1} = 0.$$

Intuitively, L provides a linear approximation for the map $y \mapsto F(x+y) - F(x)$ for small y. The above limit states that the error $F(x+y)-F(x)-L(y)$ in the approximation vanishes faster that $\|y\|_1$ as $y \to 0$.

With F as above, we state the lemma on the uniqueness of such L when it exists.

Lemma 2 Suppose $L_1, L_2 : V_1 \to V_2$ be both continuous linear maps that satisfy

$$\lim_{y \to 0} \frac{\|F(x+y) - F(x) - L_i(y)\|_2}{\|y\|_1} = 0 \quad i = 1, 2.$$

Then $L_1 = L_2$.

When F is Frechet differentiable at x, the corresponding unique linear map L in the above definition is said to be the *derivative of* F *at* x, and is denoted by $DF(x)$. Thus $DF(x) : V_1 \to V_2$ is a continuous linear map.

Given $F : U \subset V_1 \to V_2$ as above, we have the basic lemma.

Lemma 3 Suppose F is differentiable at x. Then F is continuous at x.

Suppose F as above is differentiable at each point on a set $A \subset U$. Then we say that F is differentiable on A.

Suppose $F : U \subset V_1 \to V_2$ is differentiable on U. Then we have the map $DF : U \subset V_1 \to \mathcal{L}(V_1, V_2)$, where $\mathcal{L}(V_1, V_2)$ denotes the vector space of all continuous linear maps from V_1 into V_2. It is straightforward to verify that $\mathcal{L}(V_1, V_2)$ is a vector space and we may equip it with the induced norm defined above. Then, we may define the continuity of DF in the usual manner. If DF is continuous on U then we say that F is continuously differentiable on U.

An alternative notion of derivative is that of a *directional derivative*. With F as above and $x \in U$, and given $v \in V_1$, the *directional derivative of* F *at* x *along* v is given by

$$\lim_{h \to 0} \frac{F(x+hv) - F(x)}{h},$$

provided the limit exists. If the directional derivative of F at $x \in U$ exists along all $v \in V_1$, then we say that F is *Gateaux differentiable* at x.

We have the following lemma relating Frechet and Gateaux derivatives.

Lemma 4 Let F be as above and $x \in U$. Suppose F is Frechet differentiable at x. Then F is Gateaux differentiable at x and the directional derivative at x along $v \in V_1$ is given by $DF(x)(v)$.

Proof We have that

$$\frac{\|F(x+hv)-F(x)-DF(x)(hv)\|_2}{\|hv\|_1} \to 0,$$

as $h \to 0$. Hence

$$\left\|\frac{F(x+hv)-F(x)}{h} - DF(x)(v)\right\|_2 \to 0,$$

which implies the result. ∎

Theorem 5 (Chain rule) Let $(V_i, \|.\|_i)$ be normed vector spaces for $i = 1,2,3$. Let $F : U \subset V_1 \to V_2$ and $G : W \subset V_2 \to V_3$ where U, W are open and $F(U) \subset W$. Suppose F is Frechet differentiable at $x \in U$ and G is Frechet differentiable at $F(x)$. Then $G \circ F : U \to V_3$ is Frechet differentiable at x and

$$D(G \circ F)(x) = DG(F(x)) \circ DF(x).$$

1.1 Partial derivatives and Frechet differentiability

We take $V_1 = \mathbb{R}^n$ and $V_2 = \mathbb{R}^m$ and any norms on both spaces. With F as before, and writing $x = (x_1, \ldots, x_n)$, we define the partial derivative of F at x

$$\frac{\partial F}{\partial x_i}(x) = \lim_{h \to 0} \frac{F(x+he_i)-F(x)}{h},$$

where $e_i \in \mathbb{R}^n$ is the vector with all zero entrees except for a 1 in the ith component. Thus a partial derivative is simply the directional derivative along a standard basis direction.

Theorem 6 Suppose $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $x_0 \in U$. Suppose that there exists an open neighborhood $V \subset U$ of x_0 such that the partial derivatives $\frac{\partial F}{\partial x_i}(x)$ exist for all $i = 1, \ldots, n$ and are continuous at all $x \in V$. Then F is Frechet differentiable on V and DF is continuous on V.

Let V be a vector space over $\mathbb{R}$. Given $x, y \in V$ we define $[x,y]$ and (x,y) by

$$[x,y] = \{\alpha y + (1-\alpha)x \in V_1 \,|\, \alpha \in [0,1]\},$$

and

$$(x,y) = \{\alpha y + (1-\alpha)x \in V_1 \,|\, \alpha \in (0,1)\}.$$

Theorem 7 (Mean Value Theorem) Let $(V, \|.\|)$ be a normed vector space (over $\mathbb{R}$) and $U \subset V$ be open. Suppose $F : U \subset V \to \mathbb{R}$ be differentiable on U. Suppose $x, y \in U$ and the line segment $[x, y] \subset U$. Then, there exists $z \in (x, y)$ such that

$$F(x) - F(y) = DF(z)(x - y).$$

Proof Define $\phi : [0, 1] \to V$ by

$$\phi(t) = tx + (1 - t)y,$$

and noting that $\phi([0, 1]) \subset U$, we define $g : [0, 1] \to \mathbb{R}$ be defined by $g = F \circ \phi$. By the chain rule g is differentiable on $[0, 1]$ and $g'(t) = DF(\phi(t))(x - y)$. Hence by the Mean Value Theorem in one dimensional domains, there exists $t_0 \in (0, 1)$ such that

$$g(1) - g(0) = g'(t_0).$$

Setting $z = F(\phi(t_0))$ we obtain the result. ∎