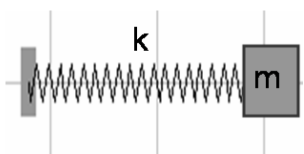
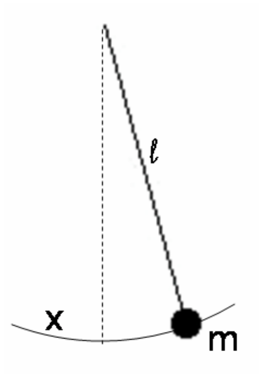
Name:

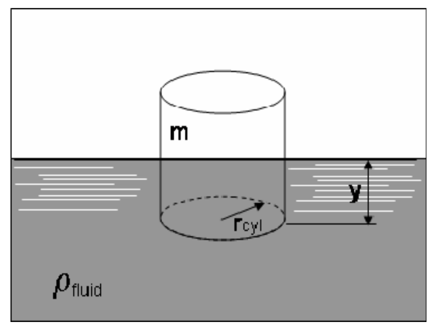
PHYS 224 - Introductory Physics III
Mid-Term Exam 1

(3 points) **P1 –** The equations of motion for the three systems below can be written as:

a) $m\frac{d^2x}{dt^2} + kx = 0$; b) $m\frac{d^2x}{dt^2} + \frac{mg}{l}x = 0$; c) $m\frac{d^2y}{dt^2} + \rho Agy = 0$







where m=1.00kg; k = 80.00N/m; ℓ = 0.50m; ρ_{fluid} = 0.90kg/m3 ; $r_{cylinder}$ = 0.20m ;
A = cross section of the base of the cylinder.

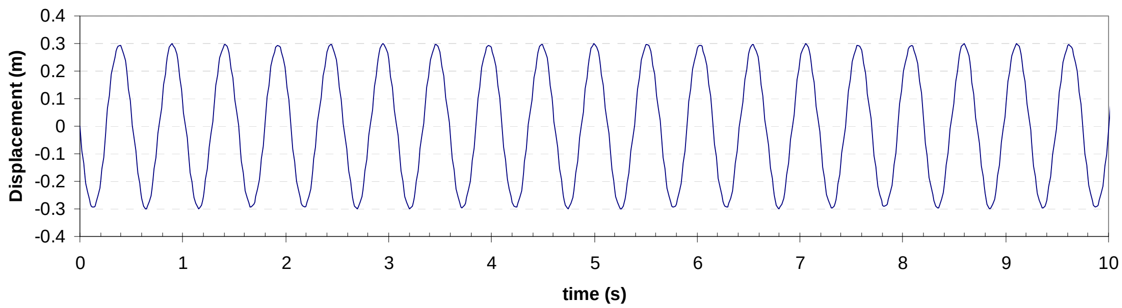
1.1 – Determine the frequency (Hz) and the period of oscillation T in each case.

1.2 – If the mass of the oscillating object is doubled in each of the systems in figures a), b) and c), determine the new period (T_{new}) of oscillation in each case.

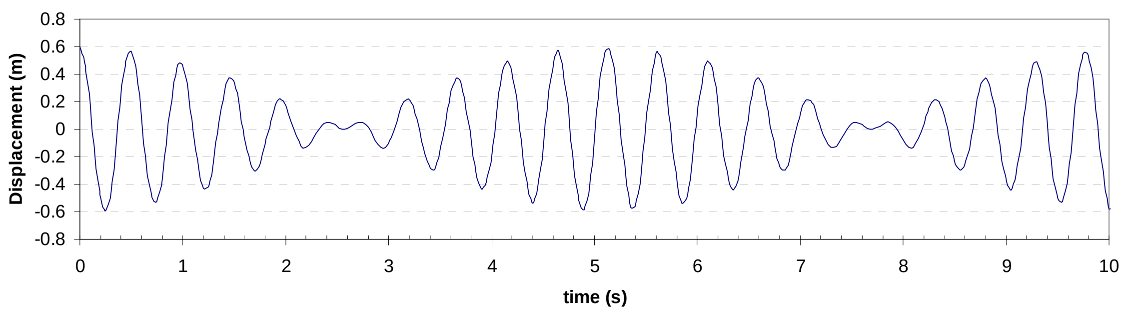
1.3 – What is the value of the restoring constant (k) in each case for the original mass (m) and for the case of mass = 2m.

(3.5 points) **P2 –** Consider the following plots, the corresponding equations, and the additional parameters below:

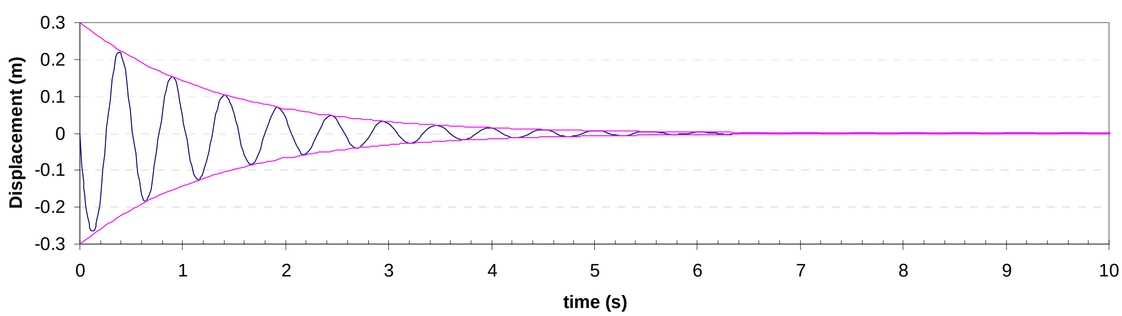
a) $z = A_o e^{j(\omega_o t + \phi)}$



b) $x = 2A_o \cos\left(\frac{\omega_o - \omega_2}{2} t\right) \cos\left(\frac{\omega_o + \omega_2}{2} t\right)$ where: $\omega_2 = 1.1 \cdot \omega_o$



c) $z = A_o e^{-\frac{\gamma}{2} t} e^{j(\omega t + \phi)}$ where: $\omega^2 = [k/m - b^2/(4m^2)]$ and $\gamma = b/m$



Other system's parameters: $\phi = \pi/2$ rad
$b = 0.30$ kg/s
$k = 30.00$ N/m
$m = 0.20$ kg

2.1 - Based on the topics discussed in class, what is the appropriate name for each one of the oscillatory motions represented in the plots above?

2.2 – The maximum amplitude (A_o) and the angular frequency of undamped oscillations (ω_o) are the same in all the systems above. Based on the information presented in the plots plus your actual calculations, determine the numerical values of A_o, and ω_o.

2.3 – For the case plotted in figure b), determine the two frequencies and the two periods of the resulting motion.

2.4 – For the case plotted in figure c), and a value of b = 0.30 Kg/s, determine the frequency and the period of the resulting motion.

2.5 – What value of b would make this system critically damped? Does the system oscillate in this condition?

2.6 - Which values of b would make the system over-damped? Using complex exponential arguments and the general solution $z = Ae^{-\frac{\gamma}{2}t}e^{j(\omega t+\phi)}$ demonstrate (analytically) if this over-damped system will oscillate or not.

(3.5 points) **P3 -** Consider a mass-spring system with periodic forcing and damping, described by the following equation of motion:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_o\cos(\omega t)$$ where: $F_o = 2N$ and the other constants of the system are specified in problem P1-a) above.

1.1 – Using **complex exponentials** for the case of **NO DAMPING**, try a solution $z = Ae^{j(\omega t+\phi)}$ in the equation of motion and determine the analytical equation for the amplitude and the phase of the system as a function of the forcing frequency ω.

You must show clearly the steps you took in this deduction and not only the final result.

1.2 – Make a sketch of A(ω) and ϕ(ω) for the case of NO DAMPING

1.3 – Determine the amplitude of the system with **NO DAMPING,** when the forcing angular frequency is $\omega=6\ rad/s$.

1.4 – What is the main change when DAMPING is added to the system? Make a sketch of A(ω) in this case.

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P1) 1.1)

a) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{80}{1}}$

$\omega = 8.94$ rad/s

$T = \frac{2\pi}{\omega} \Rightarrow T = 0.702$ s

$f = 1/T = 1.424$ Hz

b) $\omega = \sqrt{\frac{\cancel{m} g}{l \cancel{m}}} = \sqrt{\frac{9.8}{0.5}}$

$\omega = 4.43$ rad/s

$T = 1.42$ s

$f = 0.704$ Hz

c) $\omega = \sqrt{\frac{\rho A g}{m}}$

$\omega = \sqrt{\frac{0.9 \pi \cdot 0.2^2 \cdot 9.8}{1}}$

$\omega = 1.05$ rad/s

$T = 5.97$ s

$f = 0.167$ Hz.

1.2)

a) $\omega_{new} = \sqrt{\frac{80}{2}} = 6.32 \frac{rad}{s}$

$T_{New} = 0.993$ s

b) $\omega_{new} = \omega$

$T_{New} = 1.42$ s

c) $\omega_{New} = \sqrt{\frac{\rho A g}{2m}}$

$\omega_{New} = 0.744$ rad/s

$T_{New} = 8.44$ s

1.3)

a) $F = k \cdot x$

$\therefore k$ does not depend on the mass.

$k_m = k_{2m} = 80$ N/m

b) $F = \frac{m \cdot g}{l} \cdot x$

$\Rightarrow k = \frac{m \cdot g}{l} = \frac{1 \cdot 9.8}{0.5}$

$k_m = 19.6$ N/m

$k_{2m} = 39.2$ N/m

c) $F = \rho A g \cdot y$.

$k = \rho A g$

$\therefore k$ does not depend on m.

$k_m = k_{2m} = 1.11 \frac{N}{m}$

$\frac{kg \cdot \cancel{m}}{s^2 \cdot m^2}$

0.5 2.1) a) simple harmonic motion

b) beat

c) damped harmonic motion.

2.2) $A_0 = 0.30\ m$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{0.2}} = 12.25\ rad/s$$

2.3)

$$\omega_{motion} = \frac{\omega_0 + \omega_2}{2} = \frac{12.25 + 1.1\times 12.25}{2} = 12.86\ \frac{rad}{s}$$

$$\omega_{beat} = \frac{\omega_2 - \omega_0}{2} = \frac{1.1\times 12.25 - 12.25}{2} = 0.612\ \frac{rad}{s}$$

$$f_{motion} = \frac{\omega_{motion}}{2\pi} = 2.047\ Hz$$

$$f_{beat} = \frac{\omega_{beat}}{2\pi}\cdot 2 = \frac{0.612}{2\pi}\cdot 2 = 0.195\ Hz$$

$$T_{motion} = \frac{1}{f_{motion}} = 0.489\ s$$

$$T_{beat} = \frac{1}{f_{beat}} = 5.13\ s$$

2.4) $b = 0.3\ kg/s$

$$\omega^2 = \left(\omega_0^2 - \frac{b^2}{4m^2}\right) = \left(12.25^2 - \frac{0.3^2}{4\cdot 0.2^2}\right) \Rightarrow \omega = 12.23\ \frac{rad}{s}$$

$$f = \frac{\omega}{2\pi} = 1.95\ Hz \quad ; \quad T = 0.514\ s$$

2.5) Critically Damped System $\Rightarrow \omega_0^2 = \frac{b^2}{4m^2} \Rightarrow b = \sqrt{4m^2\omega_0^2}$

$\Rightarrow b = 4.90\ kg/s$ The system will not oscillate. It will decay fast to the equilibrium position without any overshoot.

2.6) Overdamped system $\Rightarrow \omega_0^2 - \frac{b^2}{4m^2} < 0 \Rightarrow \boxed{b > 2m\omega_0}$

Solution $z = A e^{-\frac{\gamma}{2}t} e^{j(\omega t + \phi)}$

$\omega^2 = \omega_0^2 - \frac{b^2}{4m^2}$ for $b > 2m\omega_0 \Rightarrow \omega^2 < 0$

$\Rightarrow \omega = \pm j\beta$ where $\beta = \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$

Substituting back in the solution:

$z = A e^{-\gamma/2\, t} e^{j(\pm j\beta t + \phi)} = A e^{-\gamma/2\, t} e^{\mp\beta t + j\phi}$

$z = A e^{-(\gamma/2 \pm \beta)t + j\phi}$

$\therefore \boxed{x = \mathrm{Re}(z) = A e^{-(\gamma/2 \pm \beta)t}}$ Which has no oscillatory component. Only exponential decay.

3.1)

$$\frac{d^2z}{dt^2} + \cancel{b\frac{dz}{dt}}^{\,0} + \frac{k}{m} z = \frac{F_0}{m}\cos(\omega t)$$

$$z = A\, e^{j(\omega t+\phi)}$$

$$\frac{dz}{dt} = A j\omega\, e^{j(\omega t+\phi)} \quad \Big| \quad \frac{d^2z}{dt^2} = -A\omega^2 e^{j(\omega t+\phi)}$$

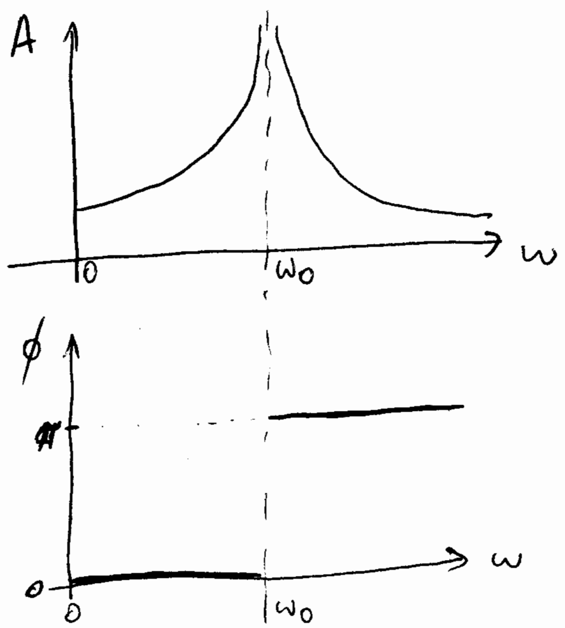
$$\left(-A\omega^2 + \frac{k}{m}A\right) e^{j(\omega t+\phi)} = \frac{F_0}{m} e^{j\omega t}$$

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m}\frac{e^{j\omega t}}{e^{j(\omega t+\phi)}} = \frac{F_0}{m} e^{-j\phi}$$

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m}\cos\phi - j\frac{F_0}{m}\sin\phi$$

$$\begin{cases} j\dfrac{F_0}{m}\sin\phi = 0 \Rightarrow \phi = 0 \text{ or } \phi = \pi \\[2ex] A = \dfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\phi \end{cases}$$

3.2)



3.3) $F_o = 2N$; $\omega = 6$ rad/s ; $m = 1.00$ kg

$k = 80$ N/m

$\omega_o = \sqrt{\frac{80}{1}} = 8.94$ rad/s.

$$A = \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos\phi = \frac{2}{1(8.94^2 - 6^2)} \cdot 1$$

$A = 0.0455m$

3.4) The amplitude at the resonance frequency is limitted by the damping.