Name: SOLUTION. Date:

PHYS 224 - Introductory Physics III
Mid-Term Exam 1

P1) 1.1)

a) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{80}{1}}$

$\omega = 8.94$ rad/s

$T = \frac{2\pi}{\omega} \Rightarrow T = 0.702$ s

$f = 1/T = 1.424$ Hz

b) $\omega = \sqrt{\frac{\cancel{m} g}{\ell \cancel{m}}} = \sqrt{\frac{9.8}{0.5}}$

$\omega = 4.43$ rad/s

$T = 1.42$ s

$f = 0.704$ Hz

c) $\omega = \sqrt{\frac{\rho A g}{m}}$

$\omega = \sqrt{\frac{0.9\pi \cdot 0.2^2 \cdot 9.8}{1}}$

$\omega = 1.05$ rad/s

$T = 5.97$ s

$f = 0.167$ Hz.

1.2)

a) $\omega_{new} = \sqrt{\frac{80}{2}} = 6.32$ rad/s

$T_{New} = 0.993$ s

b) $\omega_{new} = \omega$

$T_{New} = 1.42$ s

c) $\omega_{New} = \sqrt{\frac{\rho A g}{2m}}$

$\omega_{New} = 0.744$ rad/s

$T_{New} = 8.44$ s

1.3)

a) $F = k \cdot x$

$\therefore$ k does not depend on the mass.

$k_m = k_{2m} = 80$ N/m

b) $F = \frac{m \cdot g}{\ell} \cdot x$

$\Rightarrow k = \frac{m \cdot g}{\ell} = \frac{1 \cdot 9.8}{0.5}$

$k_m = 19.6$ N/m

$k_{2m} = 39.2$ N/m

c) $F = \rho A g \cdot y$.

$k = \rho A g$

$\therefore$ k does not depend on m.

$k_m = k_{2m} = 1.11$ N/m

2.1) a) Simple harmonic motion
b) beat
c) damped harmonic motion

2.2) $A_0 = 0.30\,m$

$$w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{0.2}} = 12.25\ \text{rad/s}$$

2.3)
$$w_{motion} = \frac{w_0 + w_2}{2} = \frac{12.25 + 1.1 \times 12.25}{2} = 12.86\ \frac{\text{rad}}{s}$$

$$w_{beat} = \frac{w_2 - w_0}{2} = \frac{1.1 \times 12.25 - 12.25}{2} = 0.612\ \frac{\text{rad}}{s}$$

$$f_{motion} = \frac{w_{motion}}{2\pi} = \frac{12.86}{2\pi} = 2.047\ Hz$$

$$f_{beat} = \frac{w_{beat}}{2\pi} \cdot 2 = \frac{0.612}{2\pi} \cdot 2 = 0.195\ Hz$$

$$T_{motion} = \frac{1}{f_{motion}} = 0.489\ s$$

$$T_{beat} = \frac{1}{f_{beat}} = 5.13\,s$$

2.4) $b = 1.0\ Kg/s$

$$w^2 = \left(w_0^2 - \frac{b^2}{4m^2}\right) = \left(12.25^2 - \frac{1^2}{4 \cdot 0.2^2}\right) \Rightarrow w = 11.99\ \text{rad/s}$$

$$f = \frac{w}{2\pi} = 1.91\ Hz \qquad T = \frac{1}{f} = 0.524\,s$$

2.5) Critically Damped System $\Rightarrow w_0^2 = \frac{b^2}{4m^2} \Rightarrow b = \sqrt{4m^2 w_0^2}$

$\Rightarrow b = 2 m w_0 = 4.90\ Kg/s$

The system will not oscillate. It will decay fast to the equilibrium position without any overshoot.

2.6) Overdamped system $\Rightarrow \omega_0^2 - \frac{b^2}{4m^2} < 0 \Rightarrow \boxed{b > 2m\omega_0}$

Solution $z = A e^{-\frac{\gamma}{2}t} e^{j(\omega t + \phi)}$

$\omega^2 = \omega_0^2 - \frac{b^2}{4m^2}$ for $b > 2m\omega_0 \Rightarrow \omega^2 < 0$

$\Rightarrow \omega = \pm j\beta$ where $\beta = \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$

Substituting back in the solution:

$z = A e^{-\gamma/2\, t} e^{j(\pm j\beta t + \phi)} = A e^{-\gamma/2\, t} e^{\mp\beta t + j\phi}$

$z = A e^{-(\gamma/2 \pm \beta)t + j\phi}$

$\therefore \boxed{x = Re(z) = A e^{-(\gamma/2 \pm \beta)t}}$ Which has no oscillatory component. Only exponential decay.

P3) case a) $E = K + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \text{const}$

Since there is no damping or forcing in the system.

For $v = 0 \Rightarrow x = A \Rightarrow E = \frac{1}{2} k A^2 = \frac{1}{2} 80 \cdot 0.3^2 = 3.6\,J$

case b) System has damping: $A = A_0 e^{-\frac{\gamma}{2} t}$

$$E = \frac{1}{2} k A^2 = \underbrace{\frac{1}{2} k A_0^2}_{E_0} e^{-\frac{2\gamma}{2} t} = 3.6\, e^{-\gamma t}$$

In case a) the energy is conserved as a function of time because the system has no damping.

In case b) the mechanical energy of the system decays exponentially due to damping.

④ a) $\dfrac{d^2z}{dt^2} + \dfrac{k}{m} z = \dfrac{F_o}{m} e^{j\omega t}$

$z = A e^{j(\omega t + \phi)}$

$\dfrac{dz}{dt} = A j\omega e^{j(\omega t + \phi)}$ | $\dfrac{d^2z}{dt^2} = -A\omega^2 e^{j(\omega t + \phi)}$

$\left(-A\omega^2 + \overbrace{\dfrac{k}{m}}^{\omega_o^2} A\right) e^{j(\omega t + \phi)} = \dfrac{F_o}{m} e^{j\omega t}$

$(\omega_o^2 - \omega^2) A = \dfrac{F_o}{m} \dfrac{e^{j\omega t}}{e^{j(\omega t + \phi)}} = \dfrac{F_o}{m} e^{-j\phi}$

$\Rightarrow (\omega_o^2 - \omega^2) A = \dfrac{F_o}{m} \cos\phi - j \dfrac{F_o}{m} \sin\alpha$

$\Rightarrow \boxed{A = \dfrac{F_o}{m(\omega_o^2 - \omega^2)} \cos\phi}$

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b) $\omega = 6$ rad/s $\quad F_o = 2N \quad m = 1$ kg.

$\omega_o = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{80}{1}} = 8.94$ rad/s

$A = \dfrac{2}{1(80 - 36)} \overbrace{\cos\phi}^{=1.0} = 0.0455$ m