1. True of False? If $A = [a_1, \ldots, a_m]$ is an $n \times m$ matrix and $\forall x \in \mathbb{R}^m$ one has $Ax = 0$, then $A = 0$.

2. True of False? If $\{a_1, \ldots, a_n\}$ is a linearly dependent vector set, then one of the vectors is a linear combination of the others.

3. True of False? If $\{a_1, \ldots, a_n\}$ is a linearly independent vector set, then one of the vectors is a linear combination of the others.

4. Let $\{a_1, \ldots, a_n\}$ be a linearly independent vector set. If the the entries $b_{ij}$ of the matrix $B$ are defined by $b_{ij} = a_i^T a_j$, then $\det B \neq 0$ (i.e. rank $B = n$, and $n$ columns of $B$ are linearly independent).

5. Let $\{a_1, \ldots, a_n\}$ be a linearly independent vector set in a vector space $\mathbb{R}^m$. If $x \in \mathbb{R}^m$, and $x \not\in \text{span} \{a_1, \ldots, a_n\}$, then $x = a + b$ such that $a \in \text{span} \{a_1, \ldots, a_n\}$, and $b^T a_i = 0$, $i = 1, \ldots, n$.

6. If $A = [a_1, \ldots, a_m]$ is an $n \times m$ matrix and $x \in \mathbb{R}^m$, then $Ax = x_1 a_1 + \ldots + x_m a_m$.

7. If $A$ is an $n \times m$ matrix and $B = [b_1, \ldots, b_k]$, $b_i \in \mathbb{R}^m$, then $AB = [Ab_1, \ldots, Ab_k]$.

8. True of False? If $A = [a_1, \ldots, a_m]$, $B = [b_1, \ldots, b_n]$, and the vector sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ are linearly independent, then the set $\{Ab_1, \ldots, Ab_n\}$ is linearly independent.

9. True of False? If $A = [a_1, \ldots, a_m]$, $B = [b_1, \ldots, b_n]$, and the vector set $\{b_1, \ldots, b_n\}$ is linearly dependent, then the set $\{Ab_1, \ldots, Ab_n\}$ is linearly dependent.

10. If $A = [a_1, \ldots, a_m]$, then $AA^T = a_1 a_1^T + \ldots + a_m a_m^T$.

11. True of False? If $A$ and $P$ are $n \times n$ matrices so that $AP = A$, then $P = I$.

12. Let $u, v \in \mathbb{R}^n$ so that $v^T u \neq 1$. For the matrices $I - uv^T$ and $I - \frac{uv^T}{v^T u - 1}$ compute the product $(I - uv^T) \left( I - \frac{uv^T}{v^T u - 1} \right)$.

13. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   - For $u = e_1 - e_2$ compute $E_1 = I - uu^T$ and $(I - uu^T)A = E_1 A$.
   - Compute $E_2 = I - (1 - \alpha)ee_2^T$, and $E_2 A$.
   - Compute $E_3 = I - \alpha ee_2^T$, and $E_3 A$.

14. Elementary row operations:
   - multiplication of a row by a nonzero number,
   - switching two different rows,
• addition of a multiple of row $i$ to row $j$.

(a) Reduced row echelon form of $A$.

(b) If $B$ is obtained from $A$ by a single elementary row operation, and $k$ rows of $A$ are linearly independent, then there are $k$ linearly independent rows of $B$.

(c) If $B$ is obtained from $A$ by a single elementary row operation, then $R(A^T) = R(B^T)$.

(d) If $B$ is obtained from $A$ by a single elementary row operation, then there is an invertible matrix $E$ such that $EA = B$ (the matrix $E$ is called an elementary matrix).

(e) For each $A$ there is a finite set of elementary matrices $E_1, E_2, \ldots, E_n$ so that $E_nE_{n-1} \ldots E_2E_1A$ is in a row echelon form.

(f) If $B$ is obtained from $A$ by a single elementary row operation, then $N(B) = N(A)$.

(g) Apply elementary row operations to an $n \times m$ matrix $A$ and reduce it to

$$\begin{bmatrix} I_k & B \\ 0_{(n-k)\times k} & 0_{(n-k)\times (m-k)} \end{bmatrix}$$

$B = [b_1, \ldots, b_{m-k}]$ is $k \times (m-k)$. Let $\hat{B} = [\hat{b}_1, \ldots, \hat{b}_{m-k}]$, where $\hat{b}_i = (b_i^T, e_i^T)^T$, and $e_i$ is the $i^{th}$ column of $I_{m-k}$. True or False? The set $\{b_1, \ldots, b_{m-k}\}$ is a basis for $N(A)$.

(h) If $B$ is obtained from $A$ by a single elementary row operation, and $\{a_i, \ldots, a_k\}$ are linearly independent, then $\{b_i, \ldots, b_k\}$ are linearly independent.

(i) The number of linearly independent columns of a matrix $A$ is the same as the number of linearly independent rows of $A$ (rank $A$).

(j) If $B$ is nonsingular, then $N(BA) = N(A)$ (in particular $N(A)$ and the null space of the reduced row echelon of $A$ are identical).

15. $\text{tr}(AB) = \text{tr}(BA)$.

16. If $A = A^T$, then the eigenvalues $\lambda_i$ are all real, and $v_i^T v_j = \delta_{ij}$.

17. If $A = A^T$, then $\text{tr}(A) = \lambda_1 + \ldots + \lambda_n$.

18. If $A^{-1}$ exits, then $Ax = \lambda x$ yields $\lambda \neq 0$.

19. If $A^{-1}$ exits, $A = A^T$, and $Ax = \lambda x$, then $A^{-1}x = \lambda^{-1}x$.

20. If $A = A^T$, and $A^{-1}$ exits, then $A^{-1}$ is symmetric.

21. If $A$ is an $n \times n$ matrix such that $I + A + A^2 + \ldots + A^n + \ldots$ converges, then

$$(I - A)^{-1} = I + A + A^2 + \ldots + A^n + \ldots$$

22. If $S = -S^T$ (is skew–symmetric), then $I + S$ is nonsingular.
23. If $A$ and $B$ are nonsingular, and $AB = BA$, then $AB^{-1} = B^{-1}A$.

24. If $A$ is a matrix, $Ax_i = y_i, i = 1, \ldots, n$ and \{$y_1, \ldots, y_n$\} is a linearly independent set, then \{${x}_1, \ldots, x_n$\} is a linearly independent set.

25. True or False? If $\{x_1, \ldots, x_n\}$ is a linearly independent set, and $X = [x_1, \ldots, x_n]$, then $XX^T$ is nonsingular.

26. True or False? If $\{x_1, \ldots, x_n\}$ is a linearly independent set, and $X = [x_1, \ldots, x_n]$, then $X^TX$ is nonsingular.

27. Let $A$ and $B$ be $n \times m$ matrices. True or False? If there are matrices $P$ and $Q$ such that $AP = B$ and $A = BQ$, then $P$ and $Q$ are nonsingular.

28. If $E$ is an elementary matrix, then rank $(EA) = \text{rank } A$. If $B, C$ are invertible, then rank $(BA) = \text{rank } (AC) = \text{rank } A$.

29. True or False? If $A, B$ are $n \times n$ matrices, then rank $(AB) = \text{rank } (BA)$.

30. Let $A$ be an $n \times m$ matrix. If $A = [a_1, \ldots, a_p, a_{p+1}, \ldots, a_m]$, and dim $R(A) = p$, then dim $N(A^T) = m - p$. 