Hwk 1 Solutions:

1. (This question was not graded. This proof is due to Geoffrey Clapp)

We will show that the \( n \) columns of \( B \) are linearly independent by contradiction. Assume not. That is, assume that \( \exists \{y_1, \ldots, y_n\} \) with at least one non-zero element, such that

\[
y_1b_1 + \cdots + y_nb_n = \vec{0},
\]

where \( B = [b_1, \ldots, b_n] \). This can be reexpressed as \( B\vec{y} \), where

\[
\vec{y} = \begin{bmatrix}
y_1 \\
\vdots \\
y_n
\end{bmatrix}.
\]

Now define \( A = [a_1, \ldots, a_n] \). By construction of \( B \), \( A^T A = B \). By substituting \( A^T A \) into \( B\vec{y} = \vec{0} \), we get \( A^T A\vec{y} = \vec{0} \). Also \( \vec{y}^T (A^T A)\vec{y} = \vec{y} \cdot (\vec{0}) = \vec{0} \). If we reposition parentheses we get

\[
(\vec{y}^T A^T) (A\vec{y}) = (A\vec{y})^T (A\vec{y}) = \vec{0}.
\]

Anything multiplied by its transpose can only be zero if it is zero itself. Therefore, \( A\vec{y} = \vec{0} \). But this contradicts that \( \{a_1, \ldots, a_n\} \) are linearly independent since \( A\vec{y} = \vec{0} \) implies

\[
y_1a_1 + y_2a_2 + \cdots + y_na_n = 0.
\]

Therefore the \( n \) columns of \( B \) are linearly independent, and we know that \( B \) is a square matrix, so \( \det B \neq 0 \).

2. write \( A = [A_{11}] \), block form with just one block \( A_{11} = A \). Write

\[
B = [B_{11}, B_{12}, \ldots, B_{1k}]
\]

where the (1, j)-block is \( B_{1j} = b_j \). Then according to the block multiplication rules on pg. 111, the (i, j)-block is

\[
A_{11}B_{1j} = A_{11}b_j.
\]

Therefore

\[
AB = [Ab_1, \ldots, Ab_k].
\]

3. True. Linear independence of \( a_1, \ldots, a_m \) implies that \( Av \neq 0 \), for all \( v \neq 0 \). Suppose that \( \{Ab_1, \ldots, Ab_n\} \) is not linearly independent. Then there must exists \( n \) scalars \( c_1, \ldots, c_n \) not all zero such that

\[
c_1Ab_1 + \cdots + c_nAb_n = 0.
\]

But this equals

\[
A(c_1b_1) + \cdots + A(c_n b_n)
\]

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\[ A(c_1b_1 + \cdots + c_nb_n) \Rightarrow c_1b_1 + \cdots + c_nb_n = 0. \]

But this is not possible since \( \{b_1, \ldots, b_n\} \) are linearly independent.

4. Suppose that the \( a_i \)'s are \( n \times 1 \) column vectors. Block decompose \( A \) as \( [a_1, \ldots, a_m] \) where each \( a_i \) is a \( n \times 1 \) block. Similarly block decompose \( A^T \) as

\[
\begin{bmatrix}
    a_1^T \\
    \vdots \\
    a_m^T
\end{bmatrix}.
\]

Then according to block matrix multiplication \( AA^T = \)

\[
[a_1, \ldots, a_m] \begin{bmatrix}
    a_1^T \\
    \vdots \\
    a_m^T
\end{bmatrix} = a_1a_1^T + \cdots + a_na_n^T.
\]

5. \( E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \quad E_2A = \begin{bmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} \end{bmatrix}, \)

\( E_3 = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}, \quad E_3A = \begin{bmatrix} a_{11} & a_{12} \\ (1-\alpha)a_{11} & (1-\alpha)a_{12} \end{bmatrix}. \)

6. Assume \( A = [a_{ij}] \) is an \( m \times n \) matrix and \( B = [b_{ij}] \) is an \( n \times m \) matrix.

\[
\text{tr}(AB) = \sum_{k=1}^m \left( \sum_{l=1}^n a_{kl}b_{lk} \right).
\]

And

\[
\text{tr}(BA) = \sum_{k=1}^n \left( \sum_{l=1}^m b_{kl}a_{lk} \right) = \sum_{k=1}^n \left( \sum_{l=1}^m a_{lk}b_{kl} \right).
\]

where in the last step we just switched the order of summation. Now just relabel \( k \) as \( l \), and \( l \) as \( k \).

7. True. Assume that \( \{x_1, \ldots, x_n\} \) is linearly dependent. Then there exist \( c_1, \ldots, c_n \) not all zero such that \( c_1x_1 + \cdots + c_nx_n = 0 \). Then

\[
0 = A(c_1x_1 + \cdots + c_nx_n) = c_1Ax_1 + \cdots + c_nAx_n = c_1y_1 + \cdots + c_ny_n.
\]
But the $c_i$'s are not all zero and this contradicts the linear independence of the $y_i$'s. Therefore $\{x_1, \ldots, x_n\}$ is a linearly independent set.

8. False. Here is one counterexample. Let

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$ 

Clearly $x_1$ and $x_2$ are linearly independent. If $X = [x_1, x_2]$, then

$$X X^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Clearly this matrix is singular.