MATH221-05
quiz #2, 11/01/18
Total 100
Solutions

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Show all work legibly. **Name**:______________________

1. (20) True or False? If for each $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the columns of A are linearly independent.

   Solution. Consider the linear equation $A\mathbf{x} = 0$. Clearly $\mathbf{x} = 0$ is a solution. Since this is the only solution for the system columns of A are linearly independent.

   Mark one and explain.
   □ True □ False

2. (20) Let $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$. Find A^{-1} if exists.

   Solution.

$$\begin{bmatrix} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 1/2 & 0 \\ 3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1/2 & 0 \\ 0 & 0 & 5 & 1 & -3/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 5 & 1 & -3/2 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1/5 & 3/10 & 1 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 1/5 & 1/5 & 0 \\ 0 & 1 & 0 & -1/5 & 3/10 & 1 \\ 0 & 0 & 1 & 1/5 & -3/10 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/5 & 1/5 & 0 \\ -1/5 & 3/10 & 1 \\ 1/5 & -3/10 & 0 \end{bmatrix}.$$

3. (20) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. If possible find a 2×2 matrix X so that $AX = B$.

Solution. Assume that $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ is a solution. Then

$$x_{11} + 2x_{21} = 5, \text{ and } 2x_{11} + 4x_{21} = 2(x_{11} + 2x_{21}) = 2 \times 5 = 7.$$

This contradiction completes the proof.

$X =$

4. (20) Let A be an invertible matrix. True or False? If $AB = CA$, then $B = C$.

Solution. Since $AB = CA$ one has $ABA^{-1} = C$. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, and

$$C = ABA^{-1} = \begin{bmatrix} -2 & 1 \\ -6 & -3/2 \end{bmatrix} \neq B.$$

Mark one and explain.
▫ True ▫ False

5. (20) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a linearly independent set in $\mathbf{R}^n$. True or False? The set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbf{R}^n$.

Solution. Let $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$. Since columns of U are linearly independent the reduced row echelon form contains n linearly independent vectors, hence it is I, and U^{-1} exits. This shows that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ spans $\mathbf{R}^n$.

Mark one and explain.
▫ True ▫ False

6. (20) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V. True or False? If $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_n]_{\mathcal{B}}\}$ is a linearly dependent set, then the vector set $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly dependent.

Solution. Let $[\mathbf{u}_i]_{\mathcal{B}} = \begin{bmatrix} c_{i1} \\ \ldots \\ c_{in} \end{bmatrix}$. Since the vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_n]_{\mathcal{B}}\}$ are linearly depen-dendent the equation

$$x_1 \begin{bmatrix} c_{11} \\ \ldots \\ c_{1n} \end{bmatrix} + \ldots + x_n \begin{bmatrix} c_{n1} \\ \ldots \\ c_{nn} \end{bmatrix} = 0$$

has infinetly many solutions. Consider now equation

$$0 = x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n.$$

Note that

$$\begin{aligned} 0 &= x_1\mathbf{u}_1 + \ldots + x_n\mathbf{u}_n \\ &= x_1(c_{11}\mathbf{b}_1 + \ldots + c_{1n}\mathbf{b}_n) \\ &+ \ldots \\ &+ x_n(c_{n1}\mathbf{b}_1 + \ldots + c_{nn}\mathbf{b}_n) \\ &= (x_1c_{11} + \ldots + x_nc_{n1})\mathbf{b}_1 \\ &+ \ldots \\ &+ (x_1c_{1n} + \ldots + x_nc_{nn})\mathbf{b}_n \end{aligned}$$

Since the vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are linearly independent this yielsd a system of linear equations

$$\begin{aligned} x_1c_{11} + \ldots + x_nc_{n1} &= 0 \\ &\ldots \\ x_1c_{1n} + \ldots + x_nc_{nn} &= 0 \end{aligned}$$

If C is an $n \times n$ matrix with entries c_{ij}, and $\mathbf{x}$ is a vector in $\mathbf{R}^n$ with entries x_i then the system of equations above becomes $C\mathbf{x} = 0$. Since the columns of C are linearly independent $\mathbf{x} = 0$.

Mark one and explain.

□ True □ False