MATH221-05
final examination 12/13/18
Total 200
Solutions

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Show all work legibly. **Name**:______________________

1. (20) Solve the linear system of equations

$$\begin{array}{rcrcrcr} x & - & y & & & = & 0 \\ 2x & + & y & - & 3z & = & 8 \\ x & - & 2y & + & 3z & = & -5 \end{array}$$

Solution.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 1 & -3 & 8 \\ 1 & -2 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -3 & 8 \\ 0 & -1 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 5 \\ 0 & 3 & -3 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 6 & -7 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -7/6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -7/6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -7/6 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ -7/6 \end{bmatrix}.$$

2. (40) Let $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ be the eigenvectors of a 2×2 matrix A with the corresponding eigenvalues 1 and 0.

   (a) (20) Compute A^5.

Solution. If $U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$A = U \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^{-1}, \text{ and } A^5 = U \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^{-1} = A.$$

Since $U^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, one has $A = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$, and $A^5 = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$.

(b) (20) Compute det A.

Solution. det $A = \det U \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \det U^{-1} = 0$.

3. (20) Let T be a linear transformation that rotates vector in $\mathbf{R}^2$ by 45^o clockwise.

   (a) (10) Find A the standard matrix for the linear transformation T.

Solution. $A = [T(\mathbf{e}_1)T(\mathbf{e}_2)] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

   (b) (10) True or False? T is onto.

Solution. Since columns of A are linearly independent the matrix is invertible, and T is onto.

4. (20) Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (a) (10) Find A^{-1} if exists.

Solution.

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (b) (10) If B is a 2×3 matrix so that $AB = C = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$. Find B.

Solution.

$$B = A^{-1}C = \begin{bmatrix} 6 & 5 & 4 \\ -3 & -2 & -1 \end{bmatrix}.$$

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5. (40) Let A be a 3×2 matrix with linearly independent columns.

   (a) (20) True or False? The 2×2 matrix $A^T A$ is invertible.

   Solution. Let $A^T A\mathbf{x} = 0$. Note that $|A\mathbf{x}|^2 = \mathbf{x}^T A^T A\mathbf{x} = 0$, and $A\mathbf{x} = 0$. Since columns of A are linearly independent $\mathbf{x} = 0$, and $A^T A$ is invertible.

   (b) (20) True or False? The 3×3 matrix AA^T is invertible.

   Solution. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$. Note that $AA^T = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The matrix AA^T is not invertible.

6. (20) Let A be a 2×2 matrix satisfying $\det\ (A^3) = 0$. True or False? The columns of A are linearly dependent.

   Solution. $0 = \det\ (A^3) = (\det\ A)^3$.

7. (20) Let $A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 25 \\ 1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \end{bmatrix}$. Compute $\det\ A$.

   Solution.

$$\det\ A = (-1)^{1+3} \begin{vmatrix} 2 & 4 & 8 \\ 3 & 9 & 25 \\ 5 & 0 & 0 \end{vmatrix} = (-1)^{1+3}(-1)^{1+3}5 \begin{vmatrix} 4 & 8 \\ 9 & 25 \end{vmatrix} = 5(100 - 72) = 140.$$

8. (20) Let L be a line defined by $\mathbf{u} + t\mathbf{v}$, where $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (a) (10) Find projection $\mathbf{p}$ of the vector $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ on the line L.

Solution. Let $\mathbf{p} = \mathbf{u} + t\mathbf{v}$. Note that

$$0 = \mathbf{v}^T(\mathbf{a} - \mathbf{p}) = \mathbf{v}^T(\mathbf{a} - (\mathbf{u} + t\mathbf{v})) = \mathbf{v}^T\mathbf{a} - \mathbf{v}^T\mathbf{u} - t\mathbf{v}^T\mathbf{v}$$

and $t\mathbf{v}^T\mathbf{v} = \mathbf{v}^T(\mathbf{a} - \mathbf{u})$, and $t = \dfrac{\mathbf{v}^T(\mathbf{a} - \mathbf{u})}{\mathbf{v}^T\mathbf{v}} = \dfrac{1}{2}$. Finally $\mathbf{p} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$.

(b) (10) Compute the distance d between the vector $\mathbf{a}$ and the line L.

Solution. $d = |\mathbf{a} - \mathbf{p}| = \sqrt{1/4 + 1/4} = \dfrac{1}{\sqrt{2}}$.