Name:

MATH221

quiz #4, 12/04/08

Sections 4.3–4.6

Total 100

Solutions

Show all work legibly.

1. (20) True or False? If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent in $\mathbf{R}^3$, then $\mathcal{B}$ is a basis for $\mathbf{R}^3$.

   Solution.
   A set of linearly independent vectors in $\mathbf{R}^3$ may not contain more than 3 vectors. If $b \in \mathbf{R}^3$, then the vector set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{b}\}$ is linearly dependent and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{b} = 0$ not all c_i zeros. This implies $c_4 \neq 0$, $-c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 = c_4\mathbf{b}$, and $\mathbf{b} \in \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Hence $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbf{R}^3$.

   Mark one and explain.
   □ True □ False

2. (20) Find dimension of the space spanned by $\{(1+x)^2, x, 1+x\}$.

Solution.

If $\mathcal{B} = \{1, x, x^2\}$, then the matrix of coordinates for the three vectors is $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

This matrix is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ with three linearly independent columns.

dim Span$\{(1+x)^2, x, 1+x\} = 3$
dim Span$\{(1+x)^2, x, 1+x\} =$

3. (20) Find the change–of–coordinate matrix from $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$ to the standard basis in $\mathbf{R}^2$.

Solution.

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \text{ where } P_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}, \text{ and } [\mathbf{x}]_{\mathcal{B}} = (P_{\mathcal{B}})^{-1}\mathbf{x} = \frac{1}{25}\begin{bmatrix} 8 & -1 \\ 9 & 2 \end{bmatrix}\mathbf{x}.$$

The matrix is:

4. (20) Find the dimension of the subspace spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 4 \\ 2 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 4 \\ 0 & -5 & -20 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The dimension is: 2

5. (20) If the null space of a 5×6 matrix is 4−dimensional, what is the dimension of the row space of A?

Solution.
Due to The Rank Theorem one has

$$\text{Rank } A + \dim \text{ Null } A = n,$$

where A is an $m \times n$ matrix. Hence the dimension of the row space of A is 2.

The dimension is: 2

6. (20) **Bonus Problem**.
Let $\mathcal{B} = \{1, 1+t, (1+t)^2\}$. Find coordinates $x_1,\ x_2,\ x_3$ of $\mathbf{p}(t) = 7 - 8t + 3t^2$ relative to $\mathcal{B}$.

Solution.

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $P_{\mathcal{B}}\mathbf{x} = \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}$, where $P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

The coordinates are: $x_1 = 18$, $x_2 = -14$, $x_3 = 3$

$x_1 =$ $\qquad$ $x_2 =$ $\qquad$ $x_3 =$