Name:

MATH221
quiz #3, 11/20/09
Sections 4.1-4.3
Total 100
Solutions

Show all work legibly.

1. (20) Let H and K be subspaces of a vector space V. The *union* of H and K, $H \cup K$, is the set of vectors $\mathbf{v} \in V$ that belong either to H, or K, or both H and K. True or False? $H \cup K$ is a subspace of V.

   Solution.
   Let $V = \mathbf{R}^2$, $H = \left\{ \mathbf{x} \ : \ \mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\}$, $K = \left\{ \mathbf{x} \ : \ \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right\}$.

   Note $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ belongs to $H \cup K$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ belongs to $H \cup K$, but $\mathbf{v} + \mathbf{w}$ does not belong to $H \cup K$.

   Mark one and explain.
   □ True □ False

2. (20) The *kernel* of a linear transformation T is the set of all vectors that satisfy $T(\mathbf{x}) = 0$. True or False? The kernel of a linear transformation is a vector space.

Solution.

(a) If c is a scalar, and $\mathbf{0} = T(\mathbf{x})$, then $\mathbf{0} = c\mathbf{0} = T(c\mathbf{x})$.

(b) If $\mathbf{0} = T(\mathbf{x}_1)$, and $\mathbf{0} = T(\mathbf{x}_2)$, then $\mathbf{0} = \mathbf{0} + \mathbf{0} = T(\mathbf{x}_1) + T(\mathbf{x}_2) = T(\mathbf{x}_1 + \mathbf{x}_2)$.

Mark one and explain.

□ True　　□ False

3. Let $M_{2\times 2}$ be a vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A - A^T$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. True or False? T is a linear transformation.

Solution.

(a) If c is a scalar, then $T(cA) = cA - (cA)^T = cA - cA^T = cT(A)$.

(b) $T(A + B) = (A + B) - (A + B)^T = A + B - A^T - B^T = T(A) - T(B)$.

Mark one and explain.

□ True □ False

4. (20) Find a basis $\mathcal{B}$ for the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in $\mathbf{R}^4$ satisfing $x_1 + x_2 + x_3 = 0$.

Solution.

$$x_1 = -x_2 - x_3, \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

5. (20) True or False? If Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbf{R}^2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbf{R}^2$.

Solution.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

span $\mathbf{R}^2$ and are linearly dependent.

Mark one and explain.

□ True □ False

6. (20) **Bonus Problem**.

Let V be a vector space of functions defined on the interval $[0, 1]$. Consider a transformation $T : V \to R$ defined by $T(f) = f(0)$. True or False? T is a linear transformation.

Solution.

(a) If c is a scalar, then $T(cf) = cf(0) = cT(f)$.

(b) $T(f + g) = (f + g)(0) = f(0) + g(0) = T(f) + T(g)$.

Mark one and explain.

□ True □ False