Name:

**MATH221**
quiz #2, 10/23/08
Sections 1.7–1.9, 2.1–2.3
Total 100
**Solutions**

Show all work legibly.

1. (20) True or False? If for each $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the columns of $A$ are linearly independent.

   **Solution**.
   Note that $A\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$ where $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$. $\mathbf{x} = 0$, solves the equation $A\mathbf{x} = 0$, and is the **only** solution for this equation. This shows that the only solution for for the equation $x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = 0$ is $\mathbf{x} = 0$, and the vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ are linearly independent.

   Mark one and explain.
   ▫ True ▫ False

2. (20) True or False? Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects a vector $\mathbf{x}$ through the line $x_2 = 0$, i.e. $T(x_1, x_2) = (-x_1, x_2)$. $T$ is a linear transformation.

**Solution**.

(a) $T(a\mathbf{x}) = (-ax_1, ax_2) = a(-x_1, x_2) = aT(\mathbf{x})$.

(b) $T(\mathbf{x} + \mathbf{y}) = (-(x_1 + y_1), x_2 + y_2) = (-x_1, x_2) + (-y_1, y_2) = T(\mathbf{x}) + T(\mathbf{y})$.

Mark one and explain.

□ True □ False

3. (20) Let $\mathbf{e}_i$ be a vector in $\mathbf{R}^n$ with $i^{\text{th}}$ entry 1, and all the other entries 0. For a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ with $T(\mathbf{e}_i) = \mathbf{e}_1$ find $T(\mathbf{x})$, where $\mathbf{x} = (n, n-1, \ldots, 2, 1)^T$.

**Solution**.
$\mathbf{x} = n\mathbf{e}_1 + (n-1)\mathbf{e}_2 + \ldots + 2\mathbf{e}_{n-1} + \mathbf{e}_n$, and

$$T(\mathbf{x}) = nT(\mathbf{e}_1)+(n-1)T(\mathbf{e}_2)+\ldots+2T(\mathbf{e}_{n-1})+T(\mathbf{e}_n) = [n + (n-1) + \ldots + 2 + 1]\,\mathbf{e}_1 = \frac{n(n+1)}{2}\mathbf{e}_1.$$

$$T(\mathbf{x}) = \frac{n(n+1)}{2}\mathbf{e}_1$$

4. (20) Find $A^{-1}$, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

**Solution**.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -7 & 0 & -4 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & -7 & 0 & -4 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -7 & 3 & -7 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3/7 & 1 & -1/7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4/7 & 0 & -1/7 \\ 0 & 0 & 1 & -3/7 & 1 & -1/7 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 13/7 & -2 & 2/7 \\ 0 & 1 & 0 & 4/7 & 0 & -1/7 \\ 0 & 0 & 1 & -3/7 & 1 & -1/7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 9/7 & -2 & 3/7 \\ 0 & 1 & 0 & 4/7 & 0 & -1/7 \\ 0 & 0 & 1 & -3/7 & 1 & -1/7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 9/7 & -2 & 3/7 \\ 4/7 & 0 & -1/7 \\ -3/7 & 1 & -1/7 \end{bmatrix}$$

5. (20) Let $T(x_1, x_2, x_3) = (x_2 + 2x_3, x_1 + 3x_3, 4x_1 - 3x_2 - 7x_3)$. Find a formula for $T^{-1}$.

**Solution**.

$$T(\mathbf{x}) = A\mathbf{x}, \text{ where } A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & -7 \end{bmatrix}, \text{ and } T^{-1}\mathbf{x} = A^{-1}\mathbf{x}.$$

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & -7 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & -7 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -19 & 0 & -4 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -13 & 3 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3/13 & 4/13 & -1/13 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 9/13 & 1/13 & 3/13 \\ 0 & 1 & 0 & 19/13 & -8/13 & 2/13 \\ 0 & 0 & 1 & -3/13 & 4/13 & -1/13 \end{bmatrix}, \text{ and } A^{-1} = \begin{bmatrix} 9/13 & 1/13 & 3/13 \\ 19/13 & -8/13 & 2/13 \\ -3/13 & 4/13 & -1/13 \end{bmatrix}.$$

$$T^{-1}(y_1, y_2, y_3) = \frac{1}{13}(9y_1 + y_2 + 3y_3, 19y_1 - 8y_2 + 2y_3, -3y_1 + 4y_2 - y_3)$$

6. (20) **Bonus Problem**.

True or False? If the columns of $B$ are linearly dependent, then so are the columns of $AB$.

**Solution**.

If the columns of $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ are linearly dependent, then there are scalars $c_1, \ldots, c_n$ not all zeros so that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = 0.$$

Note that

$$AB = A[\mathbf{b}_1, \ldots, \mathbf{b}_n] = [A\mathbf{b}_1, \ldots, A\mathbf{b}_n],$$

and

$$c_1A\mathbf{b}_1 + \ldots + c_nA\mathbf{b}_n = A\left(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n\right) = A\mathbf{0} = \mathbf{0}.$$

Mark one and explain.

▫ True ▫ False