Name:

MATH221
quiz #3, 11/01/07
Sections 2.3, 4.1–4.3
Total 100
Solutions

Show all work legibly.

1. (20) Let H and K be subspaces of a vector space V. The *intersection* of H and K, $H \cap K$, is the set of $\mathbf{v} \in V$ that belong to both H and K. True or False? $H \cap K$ is a subspace of V.

   Solution:

   (a) If $\mathbf{v} \in H \cap K$, then $\mathbf{v} \in H$ and $\mathbf{v} \in K$, hence $c\mathbf{v} \in H$ and $c\mathbf{v} \in K$, and $c\mathbf{v} \in H \cap K$.

   (b) Same argument as above shows $\mathbf{v},\ \mathbf{u} \in H \cap K$ implies $\mathbf{v} + \mathbf{u} \in H \cap K$.

   The two conditions above show that $H \cap K$ is a subspace of V.

2. (20) True or False? The range of a linear transformation is a vector subspace.

   Solution:

   (a) If $\mathbf{b} = T(\mathbf{x})$, then $c\mathbf{b} = T(c\mathbf{x})$.

   (b) If $\mathbf{b}_1 = T(\mathbf{x}_1)$, and $\mathbf{b}_2 = T(\mathbf{x}_2)$, then $\mathbf{b}_1 + \mathbf{b}_2 = T(\mathbf{x}_1 + \mathbf{x}_2)$.

   The range of a linear transformation is, therefore, a vector subspace.

3. (20) Let $M_{2\times 2}$ be a vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A^T$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. True or False? T is a linear transformation.

   Solution: A straightforward verification of linearity shows that T is a linear transformation.

4. (20) Find a basis $\mathcal{B}$ for the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbf{R}^3$ satisfying $x_1 + x_2 + x_3 = 0$.

   Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

5. (20) True or False? If Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathbf{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbf{R}^4$.

   <u>Solution</u>:

   Four vectors spanning $\mathbf{R}^4$ must be linearly independent, hence they form a basis for the space.

6. (20) (**extra credit**) True or False? If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly independent in $\mathbf{R}^4$, then $\mathcal{B}$ is a basis for $\mathbf{R}^4$.

   <u>Solution</u>: Four linearly independent vectors in $\mathbf{R}^4$ must span $\mathbf{R}^4$, hence they form a basis for the space.