Name:

MATH221
quiz #2, 09/20/07
Sections 1.7–1.9, 2.1-2.2
Total 100
Solutions

Show all work legibly.

1. (20) True or False? If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{R}^4$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly independent.

   Solution: There are three numbers c_1, c_2, and c_3 not all zeros so that $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = 0$. Hence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 = 0$, and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

   Answer: False.

2. (20) True or False? If for each $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the columns of A are linearly independent.

   Solution: Let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, then $A\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$. Since $x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = 0$ has a unique solution, $x_1 = \ldots = x_n = 0$, and the columns of A are linearly independent.

   Answer: True

3. (20) Let $\mathbf{e}_i$ be a vector in $\mathbf{R}^n$ with i^{th} entry 1, and all the other entries 0. For a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ with $T(\mathbf{e}_i) = \mathbf{e}_1$ find $T(\mathbf{x})$, where $\mathbf{x} = (1, 2, \ldots, n)^T$.

   Solution:

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \ldots x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \ldots + x_nT(\mathbf{e}_n) = [x_1 + \ldots + x_n]\mathbf{e}_1 = \begin{bmatrix} x_1 + \ldots + x_n \\ 0 \\ \cdots \\ 0 \end{bmatrix}.$$

   Answer: $T(\mathbf{x}) = \begin{bmatrix} x_1 + \ldots + x_n \\ 0 \\ \cdots \\ 0 \end{bmatrix}.$

4. True or False? If the columns of B are linearly dependent, then so are the columns of AB.

Solution: If $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$, and $c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = 0$ (not all c_i zeros).

$$AB = A[\mathbf{b}_1, \ldots, \mathbf{b}_n] = [A\mathbf{b}_1, \ldots, A\mathbf{b}_n],$$

and

$$c_1(A\mathbf{b}_1) + \ldots + c_n(A\mathbf{b}_n) = A(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = 0.$$

Answer: True.

5. (20) Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$, and compute $\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

True or False? $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 2 & 5 & 1 & 0 \\ -3 & -7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5/2 & 1/2 & 0 \\ -3 & -7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 5/2 & 1/2 & 0 \\ 0 & 1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & 3 & 2 \end{bmatrix}, \text{ and } A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}.$$

Answer: $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$, True.

6. (20) (**extra credit**) Let $T(\mathbf{x}) = T(x_1, x_2, x_3) = (x_2 + 2x_3, x_1 + 3x_3, 4x_1 - 3x_2 + 8x_3)$. Find a formula for $T^{-1}(\mathbf{y})$.

Solution: $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$. First compute A^{-1}.

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & -7 & -12 & -4 & -4 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 5 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -13/2 & 11 & -5/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

and $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$.

$$T^{-1}(\mathbf{y}) = A^{-1}\mathbf{y} = (-9/2y_1 + 7y_2 - 3/2y_3, -2y_1 + 4y_2 - y_3, 3/2y_1 - 2y_2 + 1/2y_3).$$

Answer: $T^{-1}(y_1, y_2, y_3) = (-9/2y_1 + 7y_2 - 3/2y_3, -2y_1 + 4y_2 - y_3, 3/2y_1 - 2y_2 + 1/2y_3)$.