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Korovkin-type results and doubly stochastic transformations over Euclidean Jordan algebras

M. Seetharama Gowda
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21250, USA
gowda@umbc.edu

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Abstract

A well-known theorem of Korovkin asserts that if $\{T_k\}$ is a sequence of positive linear transformations on $C[a,b]$ such that $T_k(h) \to h$ (in the sup-norm on $C[a,b]$) for all $h \in \{1, \phi, \phi^2\}$, where $\phi(t) = t$ on $[a,b]$, then $T_k(h) \to h$ for all $h \in C[a,b]$. In particular, if T is a positive linear transformation on $C[a,b]$ such that $T(h) = h$ for all $h \in \{1, \phi, \phi^2\}$, then T is the Identity transformation. In this paper, we present some analogs of these results over Euclidean Jordan algebras. We show that if T is a positive linear transformation on a Euclidean Jordan algebra $\mathcal{V}$ such that $T(h) = h$ for all $h \in \{e, p, p^2\}$, where e is the unit element in $\mathcal{V}$ and p is an element of $\mathcal{V}$ with distinct eigenvalues, then $T = T^* = I$ (the Identity transformation) on the span of the Jordan frame corresponding to the spectral decomposition of p; consequently, if a positive linear transformation coincides with the Identity transformation on a Jordan frame, then it is doubly stochastic. We also present sequential and weak-majorization versions.

Key Words: Korovkin's theorem, Euclidean Jordan algebra, positive linear transformation, unital, stochastic, doubly stochastic

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1 Introduction

A well-known theorem of Korovkin [15] asserts that if $\{T_k\}$ is a sequence of positive linear transformations on the space $C[a,b]$ (of all real-valued continuous functions on the interval $[a,b]$ with sup-norm) such that $T_k(h) \to h$ for all $h \in \{1, \phi, \phi^2\}$, where $\phi(t) = t$ on $[a,b]$, then $T_k(h) \to h$ for all $h \in C[a,b]$. In particular, if T is a positive linear transformation on $C[a,b]$ such that $T(h) = h$ for all $h \in \{1, \phi, \phi^2\}$, then $T = I$ (the Identity transformation). There are numerous generalizations and analogs of Korovkin's theorem in various settings such as Banach function spaces, C^*-algebras, etc., see e.g, [2, 1, 16]. In many of these settings, associativity of the product (as in $C[a,b]$ and the space of $n \times n$ complex matrices) and an inequality of the form $T(h)^2 \leq T(h^2)$ (known as Kadison's inequality) are crucially used [17, 16]. In this paper, we focus on Euclidean Jordan algebras, where, generally, associativity is not available and a Kadison-type inequality is not (yet) known. Here, we formulate several Korovkin-type results and make an interesting connection to doubly stochastic transformations.

Let $\mathcal{V}$ be a Euclidean Jordan algebra with unit element e (see Section 2 for definitions and examples) and T be a positive linear transformation on it (so T keeps the symmetric cone of $\mathcal{V}$ invariant). Let p be an element of $\mathcal{V}$ with distinct eigenvalues. Our main result, Theorem 4.2, asserts that the condition $T(h) = h$ for all $h \in \{e, p, p^2\}$ is equivalent to $T = I$ and also to $T^* = I$ on the span of the Jordan frame corresponding to p, where I denotes the Identity transformation on $\mathcal{V}$ and T^* denotes the adjoint of T. An immediate consequence is that if a positive linear transformation coincides with the Identity transformation on a Jordan frame, then it is doubly stochastic (i.e., it is positive, unital, and trace-preserving). The sequential version – proved as a consequence of our main result – is as follows: Let p be as above and suppose $\{T_k\}$ is a sequence of positive linear transformations on $\mathcal{V}$ such that $T_k(h) \to h$ for all $h \in \{e, p, p^2\}$. Then $T_k(h) \to h$ and $T_k^*(h) \to h$ for all h in the span of the Jordan frame corresponding to p. Along with the above equality and sequential versions, we also discuss (weak) majorization formulations. We show that under certain conditions, a positive linear transformation T satisfying $T(e) \underset{w}{\prec} e$, $p \underset{w}{\prec} T(p)$, and $T(p^2) \underset{w}{\prec} p^2$ coincides with an (algebra) automorphism of $\mathcal{V}$ on the Jordan frame of p. We also formulate the problem of characterizing positive linear transformations T for which the conditions $T(e) \prec e$, $T(p) \prec p$, and $T(p^2) \prec p^2$ hold.

An outline of the paper is as follows. We cover some preliminary material in Section 2. Section 3 deals with a Korovkin-type result for matrices and its weak-majorization modification. In Section 4, we present our main result (Theorem 4.2), describe its connection to Priestley's generalization of Korovkin's theorem ([16], Theorem 1.3), and provide some examples. Section 5 deals with sequential and weak-majorization versions on Euclidean Jordan algebras.

2 Preliminaries

In $\mathcal{R}^n$, vectors are considered as either column vectors or row vectors depending on the context. For any $x = (x_1, x_2, \ldots, x_n) \in \mathcal{R}^n$, we write $x^2 := (x_1^2, x_2^2, \ldots, x_n^2)$. We write e for the vector of ones (reserving the same symbol for the unit element in a general Euclidean Jordan algebra, see below). We say that a real $n \times n$ matrix A is *nonnegative* if all its entries are nonnegative; it is *unital* if $Ae = e$ and *subunital* if $Ae \leq e$. A nonnegative unital matrix is said to be (row) *stochastic.* A nonnegative matrix A with both A and A^T unital is said to be *doubly stochastic.*

For a vector $x \in \mathcal{R}^n$ with entries/components $x_1, x_2, \ldots, x_n$, let $x^{\downarrow}$ denote the vector obtained by rearranging the of entries of x in a decreasing manner (so that $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$). Clearly $x^{\downarrow} = Px$ for some permutation matrix P and $(Ex)^{\downarrow} = x^{\downarrow}$ for every permutation matrix E. (Recall that a permutation matrix is obtained by permuting the rows/columns of the Identity matrix.) It is known ([3], page 29) that

$$x_1 + x_2 + \cdots + x_m \leq x_1^{\downarrow} + x_2^{\downarrow} + \cdots + x_m^{\downarrow}$$

for all $m = 1, 2, \ldots, n$. Given $x, y \in \mathcal{R}^n$, we say that x is weakly-majorized by y and write $x \underset{w}{\prec} y$ if $x_1^{\downarrow} + x_2^{\downarrow} + \cdots + x_k^{\downarrow} \leq y_1^{\downarrow} + y_2^{\downarrow} + \cdots + y_k^{\downarrow}$ for each $k \in \{1, 2, \ldots, n\}$. If, additionally, equality holds for $k = n$, we say that x is majorized by y and write $x \prec y$. By a well-known theorem of Hardy-Littlewood-Polya ([3], Theorem II.1.10), $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix D. Moreover, by Birkhoff's theorem ([3], Theorem II.2.3), every doubly stochastic matrix is a convex combination of permutation matrices. We note one useful property of majorization: $x \prec y \Rightarrow f(x) \leq f(y)$ for any real-valued convex function f on $\mathcal{R}^n$.

The standard material on Euclidean Jordan algebras given below can be found in [4, 7]. A *Euclidean Jordan algebra* is a finite dimensional real inner product space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ together with a bilinear product (called the Jordan product) $(x, y) \to x \circ y$ satisfying the following properties:

- $x \circ y = y \circ x$,
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$, and
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$.

In such an algebra, there is the 'unit element' e with the property $x \circ e = x$ for all x. In $\mathcal{V}$,

$$K = \{x \circ x : x \in \mathcal{V}\}$$

is called the *symmetric cone* of $\mathcal{V}$. It is a self-dual cone.

The space $\mathcal{R}^n$ is a Euclidean Jordan algebra under the componentwise product and the usual inner product. In this algebra, the symmetric cone is the nonnegative orthant. Any (nonzero) Euclidean

Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra $\mathcal{O}^3$ of 3×3 octonion Hermitian matrices and the Jordan spin algebra $\mathcal{L}^n$. In the algebras $\mathcal{S}^n$ (of all $n \times n$ real symmetric matrices) and $\mathcal{H}^n$ (of all $n \times n$ complex Hermitian matrices), the Jordan product and the inner product are given, respectively, by

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle := \operatorname{tr}(XY),$$

where the trace of a real/complex matrix is the sum of its diagonal entries.

Let $\mathcal{V}$ be a Euclidean Jordan algebra. A nonzero element c in $\mathcal{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is not the sum of two other idempotents. *A Jordan frame* $\{e_1, e_2, \ldots, e_n\}$ *in* $\mathcal{V}$ *consists of primitive idempotents that are mutually orthogonal (equivalently,* $e_i \circ e_j = 0$ *when* $i \neq j$*) with sum equal to the unit element.* All Jordan frames in $\mathcal{V}$ have the same number of elements, called the rank of $\mathcal{V}$. *Let the rank of* $\mathcal{V}$ *be* n*.* According to the *spectral decomposition theorem* [4], any element $x \in \mathcal{V}$ has a decomposition

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n, \tag{1}$$

where the real numbers $x_1, x_2, \ldots, x_n$ are (called) the eigenvalues of x and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same. However, if all the eigenvalues are distinct, then, up to permutation, there is only one spectral decomposition, see [4], Theorem III.1.1.) For notational simplicity, we write the above spectral decomposition (1) in the form $x = r * \mathcal{E}$, where $r = (x_1, x_2, \ldots, x_n)$ and $\mathcal{E} := \{e_1, e_2, \ldots, e_n\}$.

For any $x \in \mathcal{V}$*, let* $\lambda(x)$ *denote the vector of eigenvalues of* x *written in the decreasing order.* Then, we can always write the spectral decomposition of any $x \in \mathcal{V}$ in the form $x = \lambda_1(x) f_1 + \lambda_2(x) f_2 + \cdots + \lambda_n(x) f_n = \lambda(x) * \mathcal{F}$ relative to a Jordan frame $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$.

Given $a \in \mathcal{V}$, we define linear transformations L_a and P_a (called the quadratic representation of a) on $\mathcal{V}$ by

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x \ (x \in \mathcal{V}).$$

We say that elements $a, b \in \mathcal{V}$ *operator commute* if the transformations L_a and L_b commute. It is known, see [4], Lemma X.2.2, that *a and b operator commute if and only if a and b have their spectral decompositions with respect to the same Jordan frame.*

For any $x \in \mathcal{V}$ with eigenvalues $x_1, x_2, \ldots, x_n$, the *trace* of x is defined by

$$\mathrm{tr}(x) := x_1 + x_2 + \cdots + x_n.$$

It is known that $(x, y) \mapsto \mathrm{tr}(x \circ y)$ defines another inner product on $\mathcal{V}$ that is compatible with the Jordan product. We let

$$\langle x, y \rangle_{tr} := \mathrm{tr}(x \circ y)$$

and call this, the *trace inner product.* When we replace the given inner product by the trace inner product, Jordan frames as well as the eigenvalues of an element remain the same. One advantage is: In the trace inner product, the norm of any primitive idempotent is one and so any Jordan frame in $\mathcal{V}$ is an orthonormal set. Additionally, $\mathrm{tr}(x) = \langle x, e \rangle_{tr} \quad \text{for all}\, x \in \mathcal{V}$.

We use the notation $x \geq 0$ ($x > 0$) when $x \in K$ (respectively, interior of K) or, equivalently, all the eigenvalues of x are nonnegative (respectively, positive); when $x > 0$, we say that x is a positive element. We also write $x \leq y$ in $\mathcal{V}$ when $y - x \geq 0$. *Since K is self-dual, we see that $x \geq 0$ if and only if $\langle x, y \rangle \geq 0$ for all $y \geq 0$.* For any $x \in \mathcal{V}$ with spectral decomposition $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$,, we define $x^+ := x_1^+ e_1 + x_2^+ e_2 + \cdots + x_n^+ e_n$, and $x^- := x^+ - x$ so that $x^+, x^- \in K$ and $x = x^+ - x^-$. (Here, for any real number λ, $\lambda^+ := \max\{\lambda, 0\}$.)

We record one useful consequence of the well-known Hirzebruch's min-max theorem [9]:

$$x \leq y \Rightarrow \lambda(x) \leq \lambda(y). \tag{2}$$

Given a Jordan frame $\{e_1, e_2, \ldots, e_n\}$, we have the *Peirce orthogonal decomposition* ([4], Theorem IV.2.1): $\mathcal{V} = \sum_{i \leq j} \mathcal{V}_{ij}$, where $\mathcal{V}_{ii} := \{x \in \mathcal{V} : x \circ e_i = x\} = \mathcal{R}\, e_i$ and for $i < j$, $\mathcal{V}_{ij} := \{x \in \mathcal{V} : x \circ e_i = \frac{1}{2} x = x \circ e_j\}$. Then, for any $x \in \mathcal{V}$, we have

$$x = \sum_{i \leq j} x_{ij} = \sum_{i=1}^{n} x_i e_i + \sum_{i<j} x_{ij} \quad \text{with} \quad x_i \in \mathcal{R} \text{ and } x_{ij} \in \mathcal{V}_{ij}. \tag{3}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be a (fixed) Jordan frame in $\mathcal{V}$. For arbitrary $x, y \in \mathcal{V}$, consider the corresponding Peirce decompositions $x = \sum_{i=1}^{n} x_i e_i + \sum_{i<j} x_{ij}$ and $y = \sum_{i=1}^{n} y_i e_i + \sum_{i<j} y_{ij}$. Define the real symmetric matrix

$$x \Delta y := \sum_{i=1}^{n} x_i y_i ||e_i||^2 E_{ii} + \frac{1}{2} \sum_{i<j} \langle x_{ij}, y_{ij} \rangle E_{ij},$$

where E_{ij} is the $n \times n$ matrix with 1s in the (i, j) and (j, i) slots and zeros elsewhere. It has been proved in [8], Theorem 8, that $x \Delta y$ is a (symmetric) positive semidefinite matrix when $x, y \geq 0$.

Since

$$x\Delta x = \sum_{i=1}^{n} x_i^2 ||e_i||^2 E_{ii} + \frac{1}{2}\sum_{i<j} ||x_{ij}||^2 E_{ij},$$

and all (in particular, 2×2 and 1×1) principal minors of a real symmetric positive semidefinite matrix are nonnegative, we see that for $x \geq 0$ and $i < j$,

$$||x_{ij}||^2 \leq 2x_i x_j ||e_i||\,||e_j||.$$

(Note: When $\mathcal{V}$ carries the trace inner product, this inequality reduces to $||x_{ij}||^2 \leq 2x_i x_j$, see [4], Page 80.) We record a useful consequence:

Proposition 2.1. *Suppose $x \geq 0$ in $\mathcal{V}$ and let $x = \sum_{i=1}^{n} x_i e_i + \sum_{i<j} x_{ij}$ be its Peirce decomposition relative to a given Jordan frame $\{e_1, e_2, \ldots, e_n\}$. If $x_i = 0$ for some i, then $x_{il} = 0$ for $l > i$ and $x_{li} = 0$ for $l < i$.*

A linear transformation $T : \mathcal{V} \to \mathcal{V}$ is said to be positive if $x \geq 0 \Rightarrow T(x) \geq 0$ and unital if $T(e) = e$. T is said to be trace-preserving if $tr(T(x)) = tr(x)$ for all $x \in \mathcal{V}$. A positive unital trace-preserving transformation is said to be doubly stochastic.
Note that positivity means that $T(K) \subseteq K$, where K is the symmetric cone of $\mathcal{V}$. By the self-duality of K, if T is positive, then so is the adjoint T^* (defined by the condition $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in \mathcal{V}$). Writing T^*_{tr} for the adjoint of T relative to the trace inner product, we see that T is trace-preserving if and only if $T^*_{tr}(e) = e$.

A linear transformation $\phi : \mathcal{V} \to \mathcal{V}$ is an (algebra) automorphism if it is bijective and $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in \mathcal{V}$. An automorphism ϕ maps a Jordan frame to a Jordan frame and so $\lambda(\phi(x)) = \lambda(x)$ for all x. Moreover, for any $p \in \mathcal{V}$, $\phi(p)$ and $\phi(p^2)$ have their spectral decompositions with respect to the same Jordan frame. We observe that automorphisms are doubly stochastic. We mention two more useful results: On a simple Euclidean Jordan algebra, any Jordan frame can be mapped onto any another by an automorphism, see [4], Theorem IV.2.5. Also, every automorphism on $\mathcal{H}^n$ is of the form $\phi : X \mapsto UXU^*$ for some unitary matrix U.

We define majorization in $\mathcal{V}$ by: $x \prec y$ in $\mathcal{V}$ if $\lambda(x) \prec \lambda(y)$ in $\mathcal{R}^n$. Likewise, $x \underset{w}{\prec} y$ if $\lambda(x) \underset{w}{\prec} \lambda(y)$ in $\mathcal{R}^n$.

We have the following result from [5]:

Theorem 2.2. *For $x, y \in \mathcal{V}$, consider the following statements:*

(a) *$x = T(y)$, where T is a convex combination of automorphisms of $\mathcal{V}$.*

(b) *$x = T(y)$, where T is doubly stochastic on $\mathcal{V}$.*

(c) $x \prec y$ *in* $\mathcal{V}$.

Then, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *Furthermore, reverse implications hold when* $\mathcal{V}$ *is* $\mathcal{R}^n$ *or simple.*

Along with the above, we mention a result of Jeong and Gowda ([12], Lemma 2): *T is doubly stochastic if and only if $T(x) \prec x$ for all $x \in \mathcal{V}$.* For some results related to weak-majorization, we refer to [13].

Throughout this paper, depending on the context, I denotes either the Identity matrix or the Identity transformation (on a vector space).

3 Results over $\mathcal{R}^n$

Our first result is stated in the setting of the algebra $\mathcal{R}^n$. While it can be derived from known results such as Theorem 1.3 in [16], for completeness, we provide a simple and direct proof.

Theorem 3.1. *Suppose $A \in \mathcal{R}^{n\times n}$ is a nonnegative matrix such that $Ah = h$ for all $h \in \{e, p, p^2\}$, where $p \in \mathcal{R}^n$ is a vector with distinct entries. Then $A = I$.*

Proof. Let $A = [a_{ij}]$ and $p = (p_1, p_2, \ldots, p_n)$. For any fixed i, we claim that the ith row of A, namely, $(a_{i1}, a_{i2}, \ldots, a_{in})$ has 1 in the ith slot and zeros elsewhere. We observe that the entries of this row are nonnegative. The given conditions $Ae = e$, $Ap = p$, and $Ap^2 = p^2$ imply that $(Ae)_i = 1$, $(Ap)_i = p_i$, and $(Ap^2)_i = p_i^2$. Then, from the convexity of the function $t \mapsto t^2$ on $\mathcal{R}$,

$$p_i^2 = \Big(\sum_{k=1}^{n} a_{ik}p_k\Big)^2 \le \sum_{k=1}^{n} a_{ik}p_k^2 = p_i^2.$$

Consequently, since the entries of p are distinct, by the strict convexity of the function $t \mapsto t^2$, only one a_{ik} can be nonzero. From $\sum_{k=1}^{n} a_{ik} = 1$ and $\sum_{k=1}^{n} a_{ik}p_k = p_i$, we see that $k = i$ and $a_{ii} = 1$. This proves our claim. Thus, $A = I$.

□

In our next result, we replace the equality $Ah = h$ by an appropriate weak-majorization inequality.

Theorem 3.2. *Let $A \in \mathcal{R}^{n\times n}$ be a nonnegative matrix and $p \in \mathcal{R}^n$ be a positive vector with distinct entries. If $Ae \underset{w}{\prec} e$, $p \underset{w}{\prec} Ap$, and $Ap^2 \underset{w}{\prec} p^2$, then A is a permutation matrix; additionally, if the entries of p and Ap are decreasing, then $A = I$.*

Note: In this result we assume that p has positive entries. Without this assumption, the result may not hold. For example, over $\mathcal{R}^2$, let p be a vector with entries 1 and -1, and A be the matrix with rows $(1,0)$ and $(\frac{1}{2},\frac{1}{4})$.

Proof. Suppose $Ae \underset{w}{\prec} e$, $p \underset{w}{\prec} Ap$, and $Ap^2 \underset{w}{\prec} p^2$. From $Ae \underset{w}{\prec} e$, we see that $Ae \leq e$, that is, A is subunital. Let $q := p^{\downarrow}$ so that the *entries of q are positive and strictly decreasing.* Then, $q^2 = (p^2)^{\downarrow}$. Let $p = E_1 q$ and $Ap = E_2(Ap)^{\downarrow}$ for some permutation matrices E_1 and E_2. With $B := E_2^{-1}AE_1$, we verify that

$$B \textit{ is nonnegative, } Be \leq e, \; q \underset{w}{\prec} Bq, \textit{ and } Bq^2 \underset{w}{\prec} q^2.$$

Additionally, $Bq = (Ap)^{\downarrow}$ so Bq *has decreasing entries.* We now claim that $B = I$.

Consider the first row of B. As q and Bq have decreasing entries, from $q \underset{w}{\prec} Bq$ we have $q_1 \leq (Bq)_1$. As B is nonnegative, $Be \leq e$ and q_1 is the largest entry in q, by the convexity of the function $t \mapsto t^2$ on $\mathcal{R}$,

$$q_1^2 \leq \Big(\sum_{j=1}^{n} b_{1j}q_j\Big)^2 \leq \sum_{j=1}^{n} b_{1j}q_j^2 \leq \sum_{j=1}^{n} b_{1j}q_1^2 \leq q_1^2. \tag{4}$$

From the ensuing equality, we have $b_{1j}q_j^2 = b_{1j}q_1^2$ for all j. As the entries of q are positive and distinct, $b_{1j} = 0$ for all $j \neq 1$ and, from (4), $b_{11} = 1$. Thus, the first row of B is $(1,0,0,\ldots,0)$. In particular, $q_1 = (Bq)_1$. We use induction to show that the kth row of B is of the form $(0,\ldots,0,1,0,\ldots,0)$ with 1 in the kth slot. Assume that this statement holds for all indices in $\{1,2,\ldots,k\}$, where $k < n$. (From the above argument, this statement holds for $k = 1$.) We show that the statement holds of $k+1$. To simplify the notation, let $l = k+1$, $x := Bq^2$ and $y := q^2$. From the induction hypothesis (by the form of B), $q_i = (Bq)_i$ for all $i = 1,2,\ldots,k$. Then, from $q \underset{w}{\prec} Bq$ and the fact that the entries of q and Bq are decreasing, we have

$$q_l \leq (Bq)_l.$$

From $x = Bq^2 \underset{w}{\prec} q^2 = y$, we have, for all $m = 1,2\ldots,n$,

$$x_1 + x_2 + \cdots + x_m \leq x_1^{\downarrow} + x_2^{\downarrow} + \cdots + x_m^{\downarrow} \leq y_1 + y_2 + \cdots + y_m, \tag{5}$$

where the second inequality is due to the fact that the entries of y are decreasing. From the form of the first k rows of B, we have $x_i = y_i$ for all $i = 1,2\ldots,k$; by successively putting $m = 1,2,\ldots,k$ in (5), we get $x_i = x_i^{\downarrow} = y_i$ for all $i = 1,2\ldots,k$. By putting $m = l\,(= k+1)$ in (5), we get

$$x_l \leq x_l^{\downarrow} \leq y_l, \text{ that is, } (Bq^2)_l \leq (Bq^2)_l^{\downarrow} \leq q_l^2.$$

Since $q_l \leq (Bq)_l$ with B nonnegative and $Be \leq e$, by the convexity of the function $t \mapsto t^2$ on $\mathcal{R}$,

$$q_l^2 \leq \Big(\sum_{j=1}^{n} b_{lj}q_j\Big)^2 \leq \sum_{j=1}^{n} b_{lj}q_j^2 = (Bq^2)_l \leq (q^2)_l.$$

Then, by the ensuing equality and the strict convexity of function $t \mapsto t^2$, we get $b_{lj} = 0$ for all $j \neq l$ and $b_{ll} = 1$. This proves that our induction statement holds for l $(= k+1)$. We conclude that $B = I$. Now, $I = B = E_2^{-1} A E_1$ implies that A is a product of permutation matrices, hence a permutation matrix.

Finally, suppose p and Ap have decreasing entries. Since A is a permutation matrix, it follows that A must be the Identity matrix. (This can also be seen by letting $q = p$ and $E_1 = E_2 = I$ in the above proof so that $A = B = I$.) This completes the proof. $\square$

We state two immediate consequences.

Corollary 3.3. *Suppose $A \in \mathcal{R}^{n\times n}$ is nonnegative and p is a positive vector with distinct entries. If $(Ah)^{\downarrow} = h^{\downarrow}$ for all $h \in \{e, p, p^2\}$, then A is a permutation matrix; additionally if the entries of p and Ap are decreasing, then $A = I$.*

Corollary 3.4. *Suppose $A \in \mathcal{R}^{n\times n}$ is doubly stochastic (so that $Ax \prec x$ for all $x \in \mathcal{R}^n$). If $(Ap)^{\downarrow} = p^{\downarrow}$ for some $p \in \mathcal{R}^n$ with positive and distinct entries, then A is a permutation matrix; additionally, if the entries of p and Ap are decreasing, then $A = I$.*

We remark that the latter corollary can also be deduced from Birkhoff's theorem via the strict convexity of the Euclidean norm.

In reference to the above theorem, one may ask if the condition $p \underset{w}{\prec} Ap$ can be replaced by $Ap \underset{w}{\prec} p$ to make A (at least) doubly stochastic. A simple example (such as the 2×2 matrix with rows $(0,1)$ and $(0,1)$ with p having entries 1 and 2) shows that for a nonnegative matrix, the conditions $Ae \underset{w}{\prec} e$, $Ap \underset{w}{\prec} p$, and $Ap^2 \underset{w}{\prec} p^2$, need not imply that A a doubly stochastic. What if we replace weak-majorization inequalities by majorization ones? As the answer is unclear, we pose the following.

Problem: *Let $p \in \mathcal{R}^n$ be a positive vector with distinct entries. Consider the compact convex set*

$$\Omega_p := \{A \in \mathcal{R}^{n\times n} : \ A \text{ is nonnegative and } Ah \prec h \text{ for all } h \in \{e, p, p^2\}\}.$$

Is every matrix in this set doubly stochastic? If not, what are the extreme points of this set?

When $Ah \prec h$ for some h, we have $\langle Ah, e\rangle = \langle h, e\rangle$, that is, $\langle A^T e - e, h\rangle = 0$. Hence, if A is nonnegative and p is a positive vector with distinct entries, then the condition $Ah \prec h$ for all h in (the basis) $\{e, p, p^2, \ldots, p^{n-1}\}$ implies that $A^T e = e$, that is, A is doubly stochastic. In particular, if $A \in \mathcal{R}^{n\times n}$ with $n \leq 3$, the above problem has an affirmative answer, that is, every A in Ω_p is doubly stochastic. The answer for $n \geq 4$ is unclear.

4 Equality versions over general Euclidean Jordan algebras

Throughout this section, we assume that $\mathcal{V}$ is a Euclidean Jordan algebra of rank n with unit element e.

Before proving our equality/identity version of Korovkin's theorem over Euclidean Jordan algebras, we provide a simple example to show that the direct analog of Theorem 3.1 is false.

Example 1 Let $\mathcal{V} = \mathcal{H}^n$ and T be the transformation that takes a matrix $X \in \mathcal{V}$ to the corresponding diagonal matrix, that is,

$$T(X) := \mathrm{Diag}(X),$$

where $\mathrm{Diag}(X)$ is the diagonal matrix whose diagonal is that of X. Then, T is linear, positive, and unital. Moreover, $T(H) = H$ for all $H \in \{I, P, P^2\}$, where I is the Identity matrix, P is any diagonal matrix with distinct diagonal entries. Yet, T does not coincide with the Identity transformation on $\mathcal{H}^n$. It is interesting to observe that T is trace-preserving and, hence, doubly stochastic. It follows, for example, from Theorem 2.2, that the diagonal of a Hermitian matrix is majorized by the eigenvalue vector of that matrix – this is the well-known Schur's theorem in matrix theory.

In preparation for the main theorem, we present a lemma. Here, we let δ_{ij} denote Kronecker's delta function.

Lemma 4.1. *Suppose $T : \mathcal{V} \to \mathcal{V}$ is a positive linear transformation and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame such that*

$$\langle T(e_j), e_i \rangle = ||e_i||^2 \, \delta_{ij} \quad (1 \leq i, j \leq n). \tag{6}$$

Then, $T(e_k) = T^*(e_k) = e_k$ for all k; moreover, T is doubly stochastic.

Proof. Let I denote the Identity transformation on $\mathcal{V}$. Fix any $k \in \{1, 2, \ldots, n\}$ and let $b := T(e_k)$. Consider the Peirce decomposition of b relative to the Jordan frame $\{e_1, e_2, \ldots, e_n\}$:

$$b = \sum_{i=1}^{n} b_i e_i + \sum_{1 \leq i < j \leq n} b_{ij}.$$

Now fix any index i, $i \neq k$. By (6) and the orthogonality of the individual terms in the Peirce decomposition,

$$0 = \langle T(e_k), e_i \rangle = \langle b, e_i \rangle = b_i \, ||e_i||^2.$$

Since $b \geq 0$ (due to the positivity of T), from Proposition 2.1, we must have $b_{il} = 0$ for all $l > i$ and $b_{li} = 0$ for all $l < i$. So, in the Peirce decomposition of b, only one term survives. Hence, $b = b_k e_k$.

Since $||e_k||^2 = \langle T(e_k), e_k\rangle = \langle b, e_k\rangle = b_k||e_k||^2$, we must have $b_k = 1$. Thus, $b = e_k$, proving the equality $T(e_k) = e_k$. As k is arbitrary, $T = I$ on the Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and on the span of $\{e_1, e_2, \ldots, e_n\}$.

We now show that $T^* = I$ on this span. As T is positive, $T(K) \subseteq K$. Since K is self-dual, $T^*(K) \subseteq K$, so T^* is also positive. Since the condition $\langle T(e_j), e_i\rangle = ||e_i||^2\,\delta_{ij}$ is the same as $\langle T^*(e_i), e_j\rangle = ||e_j||^2\delta_{ij}$, from the above proof we see that $T^*(e_k) = e_k$ for all k; hence $T^* = I$ on the span of $\{e_1, e_2, \ldots, e_n\}$; in particular, $T^*(e) = e$. As T is positive and $T(e) = e$, to show that T is doubly stochastic, we need only show that T is trace-preserving, that is, $T^*_{tr}(e) = e$, where T^*_{tr} is the adjoint of T relative to the trace inner product. From $T(e_k) = e_k$ for all k, we see that

$$\langle T(e_j), e_i\rangle_{tr} = \langle e_j, e_i\rangle_{tr} = ||e_i||^2_{tr}\,\delta_{ij} \quad (1 \le i, j \le n).$$

By what has been proved earlier (applied to the trace inner product), $T^*_{tr}(e_k) = e_k$ for all k and so, $T^*_{tr}(e) = e$. Thus, T is doubly stochastic. □

We now state our main theorem.

Theorem 4.2. *Suppose $\mathcal{V}$ is a Euclidean Jordan algebra of rank n and $T : \mathcal{V} \to \mathcal{V}$ is a positive linear transformation. Let $\{e_1, e_2, \ldots, e_n\}$ be a Jordan frame and $p = p_1e_2 + p_2e_2 + \cdots + p_ne_n$, where p_is are distinct. Then, the following statements are equivalent:*

(a) *$T(h) = h$ for all $h \in \{e, p, p^2\}$.*

(b) *$T(h) = h$ for all $h \in span\{e_1, e_2, \ldots, e_n\}$.*

(c) *$T^*(h) = h$ for all $h \in span\{e_1, e_2, \ldots, e_n\}$.*

(d) *$T^*(h) = h$ for all $h \in \{e, p, p^2\}$.*

Moreover, under any of the above conditions, T is doubly stochastic; hence $T(x) \prec x$ for all $x \in \mathcal{V}$.

Proof. $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$: Suppose (a) holds. As $p = p_1e_1 + p_2e_2 + \cdots + p_ne_n$, we have $p^2 = p_1^2e_1 + p_2^2e_2 + \cdots + p_n^2e_n$. Consider the matrix $A = [a_{ij}]$, where $a_{ij} := \frac{1}{||e_i||^2}\langle T(e_j), e_i\rangle$. As T is positive and the symmetric cone K is self-dual, we see that A is a nonnegative matrix. Since the sum of all e_js is e with $T(e) = e$ and $\langle e, e_i\rangle = ||e_i||^2$, we see that A is stochastic. From $T(p) = p$ and $T(p^2) = p^2$, we have, for all i,

$$\sum_{j=1}^{n} a_{ij}p_j = p_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}p_j^2 = p_i^2.$$

Thus, A satisfies the conditions of Theorem 3.1. It follows that A is the Identity matrix. Hence, $\langle T(e_j), e_i\rangle = ||e_i||^2\,\delta_{ij}$, for all $1 \le i, j \le n$. From the above lemma, $T(e_k) = T^*(e_k) = e_k$ for all k. We see that $T = T^* = I$ on the span of $\{e_1, e_2, \ldots, e_n\}$. Thus we have (b) and (c).

$(b) \Rightarrow (a)$, $(c) \Rightarrow (d)$: These are obvious, as $p, p^2 \in \text{span}\{e_1, e_2, \ldots, e_n\}$.

$(d) \Rightarrow (c)$, $(d) \Rightarrow (b)$: Suppose (d) holds. Since T is positive and the symmetric cone K is self-dual, T^* is also positive. Then, applying the implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$ to T^*, we see that (c) and (b) hold.

From the above, we see that the stated conditions are all equivalent. Now suppose (b) holds. Then, condition (6) holds. From the above lemma, T is doubly stochastic. By Theorem 2.2, $T(x) \prec x$ for all $x \in \mathcal{V}$. □

Corollary 4.3. *If a positive linear transformation coincides with an automorphism on a Jordan frame, then it is doubly stochastic.*

Proof. Let $T = \phi$ on a Jordan frame, where T is a positive linear transformation and ϕ is an automorphism. Then, using the properties of ϕ and ϕ^{-1}, we see that $\phi^{-1} \circ T$ is a positive linear transformation satisfying the condition (b) of the above theorem. Hence $\phi^{-1} \circ T$ is doubly stochastic. As ϕ is doubly stochastic, we see that the composition $\phi \circ (\phi^{-1} \circ T)$ is also doubly stochastic. Hence, T is doubly stochastic. □

Remarks. In the setting of a C^*-algebra $\mathcal{A}$, Kadison's inequality [14] asserts that if a linear transformation $T : \mathcal{A} \to \mathcal{A}$ is positive and $T(I) \leq I$, then

$$T(X)^2 \leq T(X^2)$$

for all self-adjoint elements X in $\mathcal{A}$. (Uchiyama's elementary proof of Korovkin's theorem [17] uses this inequality in the setting of $C[a, b]$.) Based on this inequality, Priestley [16] has shown that when $\{T_k\}$ is a sequence of positive linear transformations on $\mathcal{A}$ with $T_k(I) \leq I$ for all k, the set

$$\mathcal{J} := \{X \in \mathcal{A} : X^* = X,\, T_k(X) \to X,\, T_k(X^2) \to X^2\}$$

is a norm-closed Jordan algebra of self-adjoint elements of $\mathcal{A}$, that is, a real linear subspace of $\mathcal{A}$ closed under the Jordan product $X \circ Y := \frac{XY+YX}{2}$. We specialize Priestley's result by letting $\mathcal{A} = \mathcal{M}_n$ (the space of all $n \times n$ matrices) with $T_k = T$ for all k, where $T : \mathcal{M}_n \to \mathcal{M}_n$ is a positive unital linear transformation. Then, $\mathcal{H}^n$ is the Jordan algebra of self-adjoint elements of $\mathcal{M}_n$ (under the Jordan product mentioned above). By the above result, the set $\mathcal{J} := \{X \in \mathcal{H}^n : T(X) = X,\ T(X^2) = X^2\}$ is a Jordan subalgebra of $\mathcal{H}^n$. Suppose there is a $P \in \mathcal{H}^n$ with distinct eigenvalues such that $T(P) = P$ and $T(P^2) = P^2$. In $\mathcal{H}^n$, let $\mathcal{E}$ be a Jordan frame with respect to which P has its spectral decomposition. Because P has distinct eigenvalues, by uniqueness of Jordan frame (see [4], Theorem III.1.1), this Jordan frame must be the Jordan frame of P in the Jordan subalgebra $\mathcal{J}$. This means that $\mathcal{E} \subset \mathcal{J}$, showing that T coincides with the Identity transformation on the span of $\mathcal{E}$. So, in the setting of $\mathcal{H}^n$, the implication $(a) \Rightarrow (b)$ in Theorem 4.2 can be deduced from Priestley's result. We note, however, that Priestley's result does not provide any information about T^*. Motivated by the above discussion, we raise the following questions:

(1) *Is there a Kadison-type inequality for Euclidean Jordan algebras? That is, if T is a positive linear transformation that is (sub)unital, can we assert that $T(x)^2 \leq T(x^2)$ for all x?*

(2) *Does Priestley's result have an analog in the setting of Euclidean Jordan algebras? That is, if T is a positive (sub)unital transformation on $\mathcal{V}$, can we say that the set*

$$\{x \in \mathcal{V} : T(x) = x,\ T(x^2) = x^2\}$$

is a subalgebra of $\mathcal{V}$?

We now describe some examples where condition (b) of Theorem 4.2 holds.

Example 2 Let $A = [a_{ij}]$ be an $n \times n$ real symmetric positive semidefinite matrix with every diagonal entry 1 (that is, A is a correlation matrix) and $\{e_1, e_2, \ldots, e_n\}$ be a Jordan frame in $\mathcal{V}$. Then, writing the Peirce decomposition of any $x \in \mathcal{V}$ as $x = \sum_{i \leq j} x_{ij}$, we define the transformation

$$T : x \mapsto A \bullet x := \sum_{i \leq j} a_{ij} x_{ij}.$$

This transformation is positive, unital, and self-adjoint (see Example 8 in [5]) and satisfies condition (b). T being doubly stochastic leads to some interesting consequences, see [6]. For example, by taking nonzero numbers $a_1, a_2, \ldots, a_n$ and letting $A = \left[\frac{2a_i a_j}{a_i^2 + a_j^2}\right]$, we get the pointwise majorization inequality $[a_i a_j] \bullet x \prec \left[\frac{a_i^2 + a_j^2}{2}\right] \bullet x$. Written in the familiar form, this becomes

$$P_a(x) \prec L_{a^2}(x) \quad (x \in \mathcal{V}),$$

where for any $a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \in \mathcal{V}$, $L_a(x) := a \circ x$ and $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$.

Note: In the case of $\mathcal{H}^n$, the real matrix A can be modified as follows. Let B be an $n \times n$ complex (Hermitian) positive semidefinite matrix with every diagonal entry 1. Writing $X = [x_{ij}]$ and $B = [b_{ij}]$, we define the (Schur/Hadamard product) transformation T on $\mathcal{H}^n$ by $T(X) := B \bullet X := [b_{ij} x_{ij}]$. Then, T is positive (by Schur product theorem, see [10], Theorem 5.2.1) and $T(E_i) = E_i$ for all i, where $E_i \in \mathcal{H}^n$ is the matrix with 1 in (i, i) slot and zeros elsewhere. As $T^*(X) = \overline{B} \bullet X$, where $\overline{B}$ is the matrix of conjugates of entries of B, we see that $T^*(E_i) = E_i$ for all i. Note that, generally, T need not be self-adjoint.

Example 3 Let $\mathcal{V} = \mathcal{H}^n$. We consider a *completely positive* linear transformation T on $\mathcal{H}^n$ which, by definition, is of the form

$$T(X) := A_1 X A_1^* + A_2 X A_2^* + \cdots + A_N X A_N^* \quad (X \in \mathcal{H}^n),$$

where $A_1, A_2, \ldots, A_N$ are $n \times n$ complex matrices. If this transformation satisfies condition (b) of the above theorem, then $T(I) = I = T^*(I)$ and so $A_1 A_1^* + A_2 A_2^* + \cdots + A_N A_N^* = I$ and

$A_1^*A_1+A_2^*A_2+\cdots+A_N^*A_N = I$. We now characterize completely positive transformations satisfying condition (b).

Let T be as above and let $\{e_1, e_2, \ldots, e_n\}$ be a Jordan frame in $\mathcal{H}^n$ with $T(e_i) = e_i$ for all i. Let $\{E_1, E_2, \ldots, E_n\}$ denote the canonical Jordan frame in $\mathcal{H}^n$, where E_i is a diagonal matrix with 1 in the (i,i) slot and zeros elsewhere. As $\mathcal{H}^n$ is simple, the Jordan frame $\{E_1, E_2, \ldots, E_n\}$ can be mapped into the Jordan frame $\{e_1, e_2, \ldots, e_n\}$ by an automorphism. Hence, there is a unitary matrix U such that $e_i = UE_iU^*$ for all i. Define the transformation S on $\mathcal{H}^n$ by

$$S(X) := U^*T(UXU^*)U \quad (X \in \mathcal{H}^n).$$

Then, S is positive and $S(E_i) = E_i$ for all i. We see that $S(X) = B_1XB_1^*+B_2XB_2^*+\cdots+B_NXB_N^*$, where $B_k = U^*A_kU$ for all k. By considering the block form of each E_i, we deduce from $S(E_i) = E_i$ that B_k is a diagonal matrix. Let b_k denote the diagonal of B_k (viewed as a column vector). Then, $B_kXB_k^*$ can be written as $(b_kb_k^*) \bullet X$. Letting $C := \sum_{k=1}^N b_kb_k^*$, we see that

$$S(X) = C \bullet X \quad (X \in \mathcal{H}^n).$$

We observe that C, being a sum of rank-one matrices, is positive semidefinite; from $S(E_i) = E_i$ we see that each diagonal entry of C is one. Finally,

$$T(X) = U\Big(C \bullet U^*XU\Big)U^* \quad (X \in \mathcal{H}^n). \tag{7}$$

Clearly, the above arguments can be reversed to see that a transformation of the form (7) is completely positive and satisfies condition (b) of the theorem.

We now specialize by letting $k = 1$. Let T (defined by $T(X) = A_1XA_1^*$) coincide with the Identity transformation on the Jordan frame $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{H}^n$. Then, by the above, B_1 is a diagonal matrix where every diagonal entry has absolute value 1 and A_1 is unitarily similar to B_1.

5 Sequential and weak-majorization versions

Our next result deals with the sequential version of Theorem 4.2. First, some preliminary material. On the Euclidean Jordan algebra $\mathcal{V}$, for any $x \in \mathcal{V}$ with eigenvalues $x_1, x_2, \ldots, x_n$, the ∞-norm is defined by

$$||x||_\infty = \max_{1\le k\le n} |x_k|.$$

(It is known that $||\cdot||_\infty$ is a norm on $\mathcal{V}$, see e.g., [5].) For any linear transformation S on $\mathcal{V}$, let $||S||_\infty$ denote the operator norm relative to $||\cdot||_\infty$. Now assume that S is positive. For any $x \ge 0$, we have $0 \le x \le ||x||_\infty\, e$ and so

$$0 \le S(x) \le ||x||_\infty\, S(e).$$

We now apply (2) to get the inequality $||S(x)||_\infty \leq ||x||_\infty \, ||S(e)||_\infty$ for all $x \geq 0$.
Now let $x \in \mathcal{V}$. By considering the spectral decomposition of x, we can write $x = a - b$, where $a = x^+$ and $b = x^-$ (see Section 2 for definitions). Then, $||a||_\infty \leq ||x||_\infty$ and $||b||_\infty \leq ||x||_\infty$. Hence,

$$||S(x)||_\infty = ||S(a) - S(b)||_\infty \leq ||S(a)||_\infty + ||S(b)||_\infty \leq ||a||_\infty \, ||S(e)||_\infty + ||b||_\infty \, ||S(e)||_\infty$$

and so

$$||S(x)||_\infty \leq 2||x||_\infty \, ||S(e)||_\infty.$$

Hence, for a positive linear transformation S on $\mathcal{V}$,

$$||S||_\infty \leq 2||S(e)||_\infty.$$

Now, let

$$\mathcal{B}(\mathcal{V}, \mathcal{V}) := \text{Space of all linear transformations from } \mathcal{V} \text{ to } \mathcal{V}.$$

Since $\mathcal{V}$ is finite dimensional, the norm induced by the given inner product on $\mathcal{V}$ is equivalent to the ∞-norm. Correspondingly, the operator norms induced by these on $\mathcal{B}(\mathcal{V}, \mathcal{V})$ are also equivalent. Hence, there is a positive constant C (depending only on the dimension of $\mathcal{V}$) such that for any positive linear transformation S on $\mathcal{V}$,

$$||S|| \leq C||S(e)||, \tag{8}$$

where $||S||$ is the operator norm of S and $||S(e)||$ is the norm of $S(e)$ relative to the norm induced by the given inner product on $\mathcal{V}$.

Theorem 5.1. *Suppose $\mathcal{V}$ is a Euclidean Jordan algebra of rank n and $\{T_k\}$ is a sequence of positive linear transformations on $\mathcal{V}$ such that $T_k(h) \to h$ for all $h \in \{e, p, p^2\}$, where $p \in \mathcal{V}$ with distinct eigenvalues. Let $p = p_1 e_1 + p_2 e_2 + \cdots + p_n e_n$ be the spectral decomposition of p. Then,*

$$T_k(h) \to h \quad \textit{and} \quad T_k^*(h) \to h$$

for all $h \in span\{e_1, e_2, \ldots, e_n\}$.

Proof. For any k, T_k is positive; hence, from (8),

$$||T_k|| \leq C||T_k(e)||.$$

As $T_k(e) \to e$, the sequence $||T_k(e)||$ is bounded. Hence, from the above, the sequence $\{T_k\}$ is bounded in $\mathcal{B}(\mathcal{V}, \mathcal{V})$.
We now claim that $T_k(h) \to h$ for all $h \in \text{span}\{e_1, e_2, \ldots, e_n\}$. Since T_ks are linear, it is enough to show that $T_k(e_i) \to e_i$ for all $i = 1, 2, \ldots, n$. Suppose this is false; assume, without loss of generality, that $T_k(e_1) \not\to e_1$. Then there is a subsequence $\{T_{k_l}\}$ of $\{T_k\}$ and a positive number ε such that

$$||T_{k_l}(e_1) - e_1|| \geq \varepsilon \quad \text{for all } l. \tag{9}$$

On the other hand, T_{k_l} is a bounded sequence (in the finite dimensional space $\mathcal{B}(\mathcal{V},\mathcal{V})$), hence has a subsequence – continue to call this T_{k_l} – that converges to a linear transformation, say, T. We see that T is positive and (by the imposed conditions on $\{T_k\}$) satisfies the conditions

$$T(e) = e, T(p) = p, \text{ and } T(p^2) = p^2.$$

Now, by Theorem 4.2, $T(e_i) = e_i$ for all $i = 1, 2, \ldots, n$. In particular, $T(e_1) = e_1$. But this means that $T_{k_l}(e_1) \to e_1$ contradicting (9). Hence, $T_k(e_i) \to e_i$ for all $i = 1, 2, \ldots, n$.
We now claim that $T_k^*(e_i) \to e_i$ for all i. Suppose, without loss of generality, $T_k^*(e_1) \not\to e_1$. Since $||T_k^*|| = ||T_k||$ for all k, the sequence $\{T_k^*\}$ is bounded in $\mathcal{B}(\mathcal{V},\mathcal{V})$. Then, as argued before, there is a subsequence $T_{k_l}^*$ such that $T_{k_l}^*(e_1) \not\to e_1$ and $T_{k_l}^*$ converges to, say, S. As the adjoint operation is continuous, we have $T_{k_l} \to S^*$. As $T_{k_l}(e_i) \to e_i$ for all i, we have $S^*(e_i) = e_i$ for all i. Since S^* is positive, from our previous result, $S(e_i) = e_i$ for all i. But then, $T_{k_l}^*(e_i) \to e_i$ for all i, contradicting our assumption that $T_{k_l}^*(e_1) \not\to e_1$. Thus, we have our claim.

□

We now state a result that is analogous to Theorem 3.2 on a *simple* algebra. Recall that in $\mathcal{V}$, by definition, $u \underset{w}{\prec} v$ if $\lambda(u) \underset{w}{\prec} \lambda(v)$ in $\mathcal{R}^n$.

Theorem 5.2. *Let $\mathcal{V}$ be a simple Euclidean Jordan algebra of rank n and $T : \mathcal{V} \to \mathcal{V}$ be a positive linear transformation. Let $p \in \mathcal{V}$ with spectral decomposition $p = p_1e_1 + p_2e_2 + \cdots + p_ne_n$, where p_is are positive and distinct. Suppose the following conditions hold:*

(i) $T(e) \underset{w}{\prec} e$, $p \underset{w}{\prec} T(p)$, and $T(p^2) \underset{w}{\prec} p^2$, and

(ii) $T(p)$ and $T(p^2)$ operator commute.

Then

(a) T coincides with an automorphism on span$\{e_1, e_2, \ldots, e_n\}$,

(b) $\lambda(T(x)) = \lambda(x)$ for all $x \in$ span$\{e_1, e_2, \ldots, e_n\}$, and

(c) T is doubly stochastic.

Proof. We assume that all the assumptions are in place. By permuting $e_1, e_2, \ldots, e_n$, we may assume that $p_1 > p_2 > \cdots > p_n$. Then, $p = \lambda(p) * \mathcal{E}$, where $\mathcal{E} := \{e_1, e_2, \ldots, e_n\}$. Since $T(p)$ and $T(p^2)$ operator commute, they have their spectral representations with respect to the same Jordan frame, say, $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$. We write $T(p) = r * \mathcal{F} = r_1f_2 + r_2f_2 + \cdots + r_nf_n$ and $T(p^2) = s * \mathcal{F} := s_1f_1 + s_2f_2 + \cdots + s_nf_n$, where $r = (r_1, r_2, \ldots, r_n)$ and $s = (s_1, s_2, \ldots, s_n)$; we assume, without loss of generality, that the entries of r are decreasing. Note that $r = r^{\downarrow} = \lambda(T(p))$ and $s^{\downarrow} = \lambda(T(p^2))$. Now, since $\mathcal{V}$ is simple, there is an automorphism ϕ which takes $\mathcal{F}$ to $\mathcal{E}$, so

$\phi(f_i) = e_i$ for all i, see [4], Theorem IV.2.5. Then,

$$\phi(T(p)) = \phi(r * \mathcal{F}) = r * \mathcal{E} \text{ and } \phi(T(p^2)) = \phi(s * \mathcal{F}) = s * \mathcal{E}.$$

Let $S := \phi \circ T$ and $\overline{p} := \lambda(p)$. Then, S is positive and

$$T(e) \underset{w}{\prec} e \Rightarrow \lambda(T(e)) \underset{w}{\prec} \lambda(e) \Rightarrow \lambda(T(e)) \leq \lambda(e) \Rightarrow T(e) \leq e \Rightarrow S(e) \leq e,$$

where the second implication is due to the fact that $\lambda(e)$ is the vector of 1s in $\mathcal{R}^n$. Now consider the matrix B defined by

$$B = [b_{ij}], \quad b_{ij} := \frac{1}{||e_i||^2}\langle S(e_j), e_i\rangle.$$

Since S is positive and $S(e) \leq e$, we see that B is nonnegative and $\sum_{j=1}^n b_{ij} \leq 1$ for all i. From the relations

$$S\Big(\sum_{j=1}^n \bar{p}_j e_j\Big) = S(p) = r * \mathcal{E} = \sum_{i=1}^n r_i e_i \quad \text{and} \quad S\Big(\sum_{j=1}^n \bar{p}_j^2 e_j\Big) = S(p^2) = s * \mathcal{E} = \sum_{i=1}^n s_i e_i$$

we verify that $B\bar{p} = r$ and $B\bar{p}^2 = s$. Moreover, from condition (i), as ϕ preserves eigenvalues,

$$\lambda(p) \underset{w}{\prec} \lambda(T(p)) = r = \lambda(S(p)) \text{ and } \lambda(S(p^2)) = s^{\downarrow} \underset{w}{\prec} \lambda(p^2).$$

In summary: B is nonnegative, subunital, $\overline{p} \underset{w}{\prec} B\overline{p}$, and $B\overline{p}^2 \underset{w}{\prec} \overline{p}^2$; additionally, the entries of $\overline{p}$ are strictly decreasing and those of $B\overline{p}$ are decreasing.
From Theorem 3.2, we see that B is the Identity matrix. So, for all i, j,

$$\langle S(e_j), e_i\rangle = ||e_i||^2 \delta_{ij}.$$

From Lemma 4.1, $S = I$ on $\mathcal{W} := \text{span}\{e_1, e_2, \ldots, e_n\}$. So, $\phi(T(x)) = x$ for all $x \in \mathcal{W}$, that is,

$$T(x) = \phi^{-1}(x) \text{ for all } x \in \mathcal{W}.$$

As ϕ^{-1} is an automorphism on $\mathcal{V}$, we have Item (a). Since automorphisms preserve eigenvalues, T preserves eigenvalues of every element in $\mathcal{W}$. This gives (b). Finally, (c) comes from Corollary 4.3. □

Remarks. We note that conditions (i) and (ii) in the above result are necessary and sufficient for T to coincide with an automorphism on $\{e_1, e_2, \ldots, e_n\}$. Moreover, in the presence of (ii), (i) is equivalent to each of the following:

(1) $\lambda(T(h)) = \lambda(h)$ for all $h \in \{e, p, p^2\}$.

(2) $T(e) \underset{w}{\prec} e$, $\lambda(T(p)) = \lambda(p)$, and $T(p^2) \underset{w}{\prec} p^2$.

It is not clear if the assumption that $\mathcal{V}$ is simple can be dispensed with.

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