A characterization of Q-property for rank-one linear transformations on Euclidean Jordan algebras

M. Seetharama Gowda
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21250, USA
gowda@umbc.edu

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Abstract

In a recent talk, Sushmitha Parameswaran showed that in the setting of standard LCP in $\mathcal{R}^n$, a rank-one matrix has the Q-property if and only if it is positive. Answering a question raised by Prof. K.C. Sivakumar, we extend this result from $\mathcal{R}^n$ to Euclidean Jordan algebras. Specifically, we show that in the setting of symmetric cone linear complementarity problem, a rank-one linear transformation $a \otimes b$ has the Q-property if and only if $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

Key Words: Euclidean Jordan algebra, complementarity problem, Q-property

AMS Subject Classification:

1 Introduction

Let K be a proper cone (= closed convex pointed cone with nonempty interior) in a finite dimensional real Hilbert space H. Given a linear transformation $L : H \to H$ and a $q \in H$, we consider the *linear complementarity problem*, LCP(L, K, q): Find $x \in H$ such that

$$x \in K,\ L(x) + q \in K^*,\ \text{and}\ \langle x, L(x) + q\rangle = 0,$$

where $K^* = \{y \in H : \langle y, x\rangle \geq 0 \text{ for all } x \in K\}$ is the dual of K. This is a special of a *variational inequality problem*, VI(f, C): Find $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0 \text{ for all } x \in C,$$

where C is a closed convex set in H and $f : H \to H$.

Examples:

1. When $H = \mathcal{R}^n$ and $K = \mathcal{R}^n_+$, LCP(L, K, q) = standard LCP.
2. When $H = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$, LCP(L, K, q) = semidefinite LCP.
3. When $H = \mathcal{V}$ (Euclidean Jordan algebra) and $K = \mathcal{V}_+$ (corresponding symmetric cone), LCP(L, K, q) = symmetric cone LCP.

Definition 1.1 *Let K be a proper cone in H and L be linear on H. We say that L has the Q-property on K if for every $q \in H$, LCP(L, K, q) has a solution.*

Problem: *Find necessary and/or sufficient conditions for L to have the Q-property on K.*

While there is no (meaningful) necessary condition, a well-known sufficient condition is given by Karamardian [7].

Theorem 1.2 (Karamardian's theorem) *Suppose there is a $d \in int(K^*)$ such that zero is the only solution of the problems LCP$(L, K, 0)$ and LCP(L, K, d). Then, L has the Q-property.*

There is an immediate corollary:

Corollary 1.3 *Suppose L is strictly copositive on K, that is, $\langle L(x), x\rangle > 0$ for all $0 \neq x \in K$ or, equivalently, there exists $\alpha > 0$ such that $\langle L(x), x\rangle \geq \alpha\, ||x||^2$ for all $x \in K$. Then L has the Q-property on K.*

The present talk is motivated by a simple result in the standard LCP theory:

A nonnegative matrix is a Q-matrix if and only if its diagonal entries are positive.

(If $A = [a_{ij}]$ is a nonnegative matrix with $a_{11} = 0$, then, with $q = (-1, 1, 1, \ldots, 1)$, LCP$(A, q)$ cannot have a solution. In the reverse direction, one can apply the above corollary.)

This raises an interesting **Question**: *Is there an analog for general cones?*

Since nonnegative matrices correspond to matrices M with $M(\mathcal{R}^n_+) \subseteq \mathcal{R}^n_+$, for a general cone K, we consider

$$\pi(K) := \{L \in \mathcal{B}(H) : L(K) \subseteq K\},$$

where $\mathcal{B}(H)$ is the space of all (bounded) linear transformations on H. While much is known about $\pi(K)$ (see references in Orlitzky [8]), characterizing it is notoriously hard. It is known (see Tam [9]) that $\pi(K)$ is a proper cone in $\mathcal{B}(H)$ and its dual (in the Hilbert space $\mathcal{B}(H)$ with inner product $\langle L_1, L_2\rangle := tr(L_1 L_2^*)$) is given by

$$\pi(K)^* := \Big\{ \sum_{i=1}^{N} a_i \otimes b_i : \, a_i \in K^*, \, b_i \in K, \, i = 1, 2, \ldots, N \Big\},$$

where, for $a, b \in H$,

$$(a \otimes b)(x) := \langle b, x\rangle \, a \quad (x \in H).$$

So, the previous question reduces to: *Which linear transformations in $\pi(K)$ have the Q-property?* This turns out to be too broad. We restrict our attention to self-dual cones (those satisfying $K^* = K$) and, in particular, to symmetric cones in Euclidean Jordan algebras.

Here is a simple result.

Theorem 1.4 *Suppose K is self-dual. Then,*

1. *Every $L \in \pi(K)$ is copositive on K.*
2. *Every $L \in int\big(\pi(K)\big)$ is strictly copositive on K, hence has the Q-property.*

Proof. Let K be self-dual. If $L \in \pi(K)$ and $x \in K$, then $L(x) \in K = K^*$ and so $\langle L(x), x\rangle \geq 0$. Thus, L is copositive on K.
Suppose $L \in \text{int}\big(\pi(K)\big)$. Then, for some $\varepsilon > 0$, $L - \varepsilon I \in \pi(K)$, where I denotes the identity transformation on H. Then, for all nonzero $x \in K$, $\langle (L - \varepsilon I)x, x\rangle \geq 0$, that is, $\langle L(x), x\rangle \geq \varepsilon \,||x||^2 > 0$. Finally, an application of Karamardian's theorem gives the Q-property. $\square$

Example: Let $H = \mathcal{H}^n$ (the space of all $n \times n$ complex Hermitian matrices) with inner product $\langle X, Y\rangle := tr(XY)$ and $K = \mathcal{H}^n_+$, the semidefinite cone. A map (= linear transformation) L on $\mathcal{H}^n$ is said to be a *completely positive map* if there exist complex matrices $A_1, A_2, \ldots, A_N$ such that

$$L(X) = \sum_{i=1}^{N} A_i X A_i^* \text{ for all } X \in \mathcal{H}^n.$$

(Such maps appear in quantum information theory; a celebrated result of Choi is related to this. Some buzz words: L is unital if $L(I) = I$, trace-preserving if $L^*(I) = I$; doubly stochastic if it is unital and trace-preserving. A quantum channel is a completely positive map that is trace-preserving.)

We note that if $X \in \mathcal{H}^n_+$, that is, if X is positive semidefinite, then, AXA^* is so for any complex square matrix A. Thus, every completely positive map is in $\pi(\mathcal{H}^n_+)$. Working with $A \in \mathcal{R}^{n\times n}$ and $H = \mathcal{S}^n$ (the space of all $n \times n$ real symmetric matrices), various authors have investigated complementarity problems of the transformation $M_A(X) := AXA^T$. In particular, Bhimasankaram et al. [2] (see also, Chandrashekaran et al. [3]) have shown that M_A is strictly copositive on $\mathcal{S}^n_+$ if and only if A is either positive definite or negative definite and Balaji [1] has shown that M_A has the Q-property if and only if A is either positive definite or negative definite. Motivated by these, we now pose the following

Problem: *Characterize completely positive maps with the Q-property.*

Note: It is tempting to bring in the Z-property into this discussion. However, a result of Gowda-Song-Sivakumar [5] says that when K is irreducible, nonnegative multiples of the Identity map are the only ones in $Z(K) \cap \pi(K)$.

Since we do not know how to describe the Q-property of transformations that are on the boundary of $\pi(K)$, we turn out attention to those in $\pi(K)^*$. In the case of a self-dual cone K, any element of $\pi(K)^*$ is a sum of rank-one transformations of the form $a \otimes b$, where $a, b \in K$. We consider just one transformation $L = a \otimes b$. In her UMBC online seminar talk (of July 16, 2020), Sushmitha Parameswaran shows that *in the setting of standard LCP, a rank-one matrix ab^T is a Q-matrix if and only if it is positive.* In a private communication, Prof. K.C. Sivakumar raises the following

Question: *When does a rank-one transformation on a Euclidean Jordan algebra have the Q-property?*

In this talk, we provide an answer. First, we review some necessary material.

Given a self-dual cone K in H, we write $x \geq 0$ when $x \in K$ and $x > 0$ when $x \in int(K)$. We also write $x < 0$ when $-x > 0$ etc. We note:

- $y \geq 0$ iff $\langle y, x\rangle \geq 0$ for all $x \geq 0$.
- $y > 0$ iff $\langle y, x\rangle > 0$ for all $0 \neq x \geq 0$.

(The second item can be seen by an application of the standard separation theorem.)

A Euclidean Jordan algebra is a finite dimensional real inner product space $\mathcal{V}$ together with a Jordan product $x \circ y$ satisfying certain properties, see the Appendix. In such an algebra, the cone of squares $K := \{x \circ x : x \in \mathcal{V}\}$ – called the symmetric cone of $\mathcal{V}$ – is a self-dual cone whose interior

is transitive. *We assume that $\mathcal{V}$ has rank n and carries the trace inner product.*
The result below is stated in the setting of a general Euclidean Jordan algebra. If unfamiliar with this general setting, one may let $\mathcal{V} = \mathcal{S}^n$, $K = \mathcal{S}^n_+$ (the semidefinite cone), with

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle := tr(XY).$$

In this case, for any $X \in \mathcal{S}^n$, we have the *spectral decomposition*

$$X = UDU^T = \lambda_1(u_1 u_1^T) + \lambda_2(u_2 u_2^T) + \cdots + \lambda_n(u_n u_n^T),$$

where U is an orthogonal matrix with column vectors $u_1, u_2, \ldots, u_n$ and D is a diagonal matrix with entries (eigenvalues) $\lambda_1, \lambda_2, \ldots, \lambda_n$. Writing $e_i := u_i u_i^T$, we see that $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{S}^n$. For $C \in \mathcal{S}^n$, the transformation M_C (defined by $M_C(X) = CXC$) is called a *quadratic representation* of C.

Recall that a rank-one transformation on H is given by $L = a \otimes b$, where $a, b \in H$; In the case of $H = \mathcal{R}^n$, this is just ab^T. Our result is the following:

Theorem 1.5 *Suppose K is self-dual in H and $L = a \otimes b$ for some $a, b \in H$. Consider the following statements:*

(i) $a > 0, b > 0$ *or* $a < 0, b < 0$.

(ii) The implication $0 \neq x \geq 0 \Rightarrow L(x) > 0$ *holds.*

(iii) L has the Q-property.

Then, $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.
Moreover, if $H = \mathcal{V}$ is a Euclidean Jordan algebra and K is the corresponding symmetric cone, then above statements are all equivalent.

Proof. $(i) \Rightarrow (ii)$: This is obvious as $L(x) = \langle b, x \rangle\, a$.
$(ii) \Rightarrow (i)$: We assume (ii) so that for any $0 \neq x \geq 0$, $\langle b, x \rangle\, a > 0$. Suppose $a > 0$, in which case, $\langle b, x \rangle > 0$ for all $0 \neq x \geq 0$. Then, by our previous observation, $b > 0$. Similarly, $b < 0$ when $a < 0$. Thus we have (i).
$(ii) \Rightarrow (iii)$: When (ii) holds, L becomes strictly copositive on K, that is, $0 \neq x \geq 0 \Rightarrow \langle L(x), x \rangle > 0$. As noted previously, L has the Q-property.
$(iii) \Rightarrow (i)$: We now assume that our Hilbert space is a Euclidean Jordan algebra and K is the corresponding symmetric cone. Suppose $L = a \otimes b$ has the Q-property. With e denoting the unit element of $\mathcal{V}$, LCP$(L, K, -e)$ has a solution, say, u. Then, $L(u) - e \geq 0$ implies that $0 \neq u \geq 0$ and $L(u) \geq e > 0$; so $\langle b, u \rangle\, a > 0$. Then, either $a > 0$ or $a < 0$. *Supposing $a > 0$, we will show that* $b > 0$. (Similarly, when $a < 0$ we can show that $b < 0$).
We will employ a *standard trick* to drive a to e and look at the induced/simplified transformation.

Now, given that $a > 0$, let $c := \sqrt{a^{-1}}$ and consider the quadratic representation P_c defined by

$$P_c(x) := 2c \circ (c \circ x) - c^2 \circ x.$$

(In the case of $\mathcal{V} = \mathcal{S}^n$, $P_C(X) = CXC = M_C(X)$.)
Because c is invertible, from the well-known properties of quadratic representations, we see that P_c is self-adjoint and invertible, $(P_c)^{-1} = P_{c^{-1}}$, $P_c(K) = K$, $P_c(K^\circ) = K^\circ$, and $P_c(a) = e$. (Here, K° is the interior of K.) We define a new linear transformation T on $\mathcal{V}$ by

$$T := P_c L P_c.$$

Claim: T has the Q-property.
To see this, take any $p \in \mathcal{V}$, let $q \in \mathcal{V}$ be such that $P_c(q) = p$. Let x be a solution of LCP(L, K, q) so that $x \geq 0$, $L(x) + q \geq 0$ and $\langle x, L(x) + q\rangle = 0$. Writing $x = P_c(y)$, we see that $y \geq 0$ and $T(y) + p = P_c(LP_c(y) + q) = P_c(L(x) + q) \geq 0$. We also see that

$$0 = \langle x, L(x) + q\rangle = \langle P_c(y), L(P_c(y)) + q\rangle = \langle y, P_c(L(P_c(y) + q)\rangle = \langle y, T(y) + p\rangle.$$

Thus, LCP(T, K, p) has a solution, namely, y. This proves the claim.
Now, define $d := P_c(b)$ so that

$$T(x) = P_c L P_c(x) = P_c\big(\langle P_c(x), b\rangle\, a\big) = \langle x, P_c(b)\rangle\, P_c(a) = \langle x, d\rangle\, e \quad \text{for all } x \in \mathcal{V}.$$

(Note that T looks very much like L, except that we have e in place of a.)

Claim: $d > 0$.
We will prove the claim by writing the spectral decomposition

$$d = d_1 e_1 + d_2 e_2 + \cdots + d_n e_n$$

and showing that each $d_i > 0$. We show that $d_1 > 0$ (with a similar proof for other d_i). Corresponding to the Jordan frame coming from d, define

$$q := -1\, e_1 + 0\, e_2 + \cdots + 0\, e_n$$

and let x be a solution of LCP(T, K, q) whose Peirce decomposition with respect to $\{e_1, e_2, \ldots, e_n\}$ be given by

$$x = \sum_{i=1}^n x_i e_i + \sum_{i<j} x_{ij}.$$

Define vectors $\bar{x}$ and $\bar{d}$ in $\mathcal{R}^n$ by $\bar{x} := (x_1, x_2, \ldots, x_n)$ and $\bar{d} := (d_1, d_2, \ldots, d_n)$. As $x \geq 0$, we have $0 \leq \langle x, e_i\rangle = x_i$ for all i and so $\bar{x} \geq 0$ in $\mathcal{R}^n$. Also, since the above Peirce decomposition is an orthogonal decomposition and $\mathcal{V}$ is assumed to carry the trace inner product (so that $||e_i|| = 1$ for all i),

$$\langle x, d\rangle = \langle \bar{x}, \bar{d}\rangle.$$

As $e = e_1 + e_2 + \cdots + e_n$, we have

$$0 \leq T(x) + q = \langle x, d\rangle\, e + q = \big[\langle \bar{x}, \bar{d}\rangle - 1\big] e_1 + \langle \bar{x}, \bar{d}\rangle\, e_2 + \cdots + \langle \bar{x}, \bar{d}\rangle\, e_n.$$

This implies that $\langle \bar{x}, \bar{d}\rangle - 1 \geq 0$ and $\langle \bar{x}, \bar{d}\rangle \geq 0$. Then, $\langle x, T(x) + q\rangle = 0$ leads to

$$x_1\big[\langle \bar{x}, \bar{d}\rangle - 1\big] + x_2\langle \bar{x}, \bar{d}\rangle + x_3\langle \bar{x}, \bar{d}\rangle + \cdots + x_n\langle \bar{x}, \bar{d}\rangle = 0$$

and to the complementarity relations

$$0 = x_1[\langle \bar{x}, \bar{d}\rangle - 1] = x_2\langle \bar{x}, \bar{d}\rangle = x_3\langle \bar{x}, \bar{d}\rangle = \cdots = x_n\langle \bar{x}, \bar{d}\rangle.$$

Since $\langle \bar{x}, \bar{d}\rangle \geq 1$, we see that $x_2 = x_3 = \cdots = x_n = 0$. From $\langle \bar{x}, \bar{d}\rangle - 1 \geq 0$ we get $x_1 d_1 \geq 1$; As $x_1 \geq 0$, we have $x_1 > 0$ and $d_1 > 0$. Likewise, we see that each $d_i > 0$. Thus, $d > 0$ in $\mathcal{V}$. Now, $d = P_c(b)$ implies that $b = P_{c^{-1}}(d) > 0$ as $P_c(K^\circ) = K^\circ$. We conclude that $b > 0$. Hence $(iii) \Rightarrow (i)$. This completes the proof of the theorem. □

Question: *Does the implication* $(iii) \Rightarrow (i)$ *hold for general self-dual cones?*

2 Appendix: Euclidean Jordan algebras

The material given below can be found in [4, 6]. A Euclidean Jordan algebra is a finite dimensional real inner product space $(\mathcal{V}, \langle \cdot, \cdot\rangle)$ together with a bilinear product (called the Jordan product) $(x, y) \to x \circ y$ satisfying the following properties:

- $x \circ y = y \circ x$,
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$, and
- $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$.

In such an algebra, there is 'unit element' e such that $x \circ e = x$ for all x. In $\mathcal{V}$, $K = \{x \circ x :\in \mathcal{V}\}$ is called the symmetric cone of $\mathcal{V}$. It is a self-dual cone.

Any nonzero Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra. In the algebras $\mathcal{S}^n$ (of all $n \times n$ real symmetric matrices) and $\mathcal{H}^n$ (of all $n \times n$ complex Hermitian matrices), the Jordan product and the inner product are given, respectively, by

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y\rangle := \mathrm{tr}(XY),$$

where the trace of a real/complex matrix is the sum of its diagonal entries.
A nonzero element c in $\mathcal{V}$ is an idempotent if $c^2 = c$; it is a primitive idempotent if it is not the sum of two other idempotents. A Jordan frame $\{e_1, e_2, \ldots, e_n\}$ consists of primitive idempotents that are mutually orthogonal and with sum equal to the unit element. All Jordan frames in $\mathcal{V}$ have the same number of elements, called the rank of $\mathcal{V}$. Let the rank of $\mathcal{V}$ be n. According to the *spectral decomposition theorem* [4], any element $x \in \mathcal{V}$ has a decomposition

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n,$$

where the real numbers $x_1, x_2, \ldots, x_n$ are (called) the eigenvalues of x and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same.)
We use the notation $x \geq 0$ ($x > 0$) when $x \in K$ (interior of K) or, equivalently, all the eigenvalues of x are nonnegative (respectively, positive). When $x \geq 0$ with spectral decomposition $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$, we define $\sqrt{x} := \sqrt{x_1} e_1 + \sqrt{x_2} e_2 + \cdots + \sqrt{x_n} e_n$. If $x > 0$, then $x^{-1} := (x_1)^{-1} e_1 + \cdots + (x_n)^{-1} e_n$. If $x_1, x_2, \ldots, x_n$ are the eigenvalues of $x \in \mathcal{V}$, we define the *trace* of x by

$$\mathrm{tr}(x) := x_1 + x_2 + \cdots + x_n.$$

It is known that $(x, y) \mapsto \mathrm{tr}(x \circ y)$ defines another inner product on $\mathcal{V}$ that is compatible with the Jordan product. *Throughout this talk, we assume that the inner product on $\mathcal{V}$ is this trace inner product, that is,* $\langle x, y \rangle = \mathrm{tr}(x \circ y)$. In this inner product, the norm of any primitive element is one and so any Jordan frame in $\mathcal{V}$ is an orthonormal set. Additionally, $\mathrm{tr}(x) = \langle x, e \rangle \quad$ for all $x \in \mathcal{V}$.

Given a Jordan frame $\{e_1, e_2, \ldots, e_n\}$, we have the Peirce orthogonal decomposition ([4], Theorem IV.2.1): $\mathcal{V} = \sum_{i \leq j} \mathcal{V}_{ij}$, where $\mathcal{V}_{ii} := \{x \in \mathcal{V} : x \circ e_i = x\} = \mathcal{R}\, e_i$ and for $i < j$, $\mathcal{V}_{ij} := \{x \in \mathcal{V} : x \circ e_i = \frac{1}{2} x = x \circ e_j\}$. Then, for any $x \in \mathcal{V}$ we have

$$x = \sum_{i \leq j} x_{ij} = \sum_{i=1}^{n} x_i e_i + \sum_{i<j} x_{ij} \quad \text{with} \quad x_{ij} \in \mathcal{V}_{ij}. \tag{1}$$

3 Stepan Karamardian, 1933-1994

Armenian, studied in Syria; MS from University of Illinois (1962), PhD from University of California, Berkeley (1966, under George Dantzig); faculty at University of California, Irvine, Dean of School of Management (1982-1991); Co-founded American University of Armenia.

References

[1] R. Balaji, *Characterization of Q-property for multiplicative transformations in semidefinite linear complementarity problems*, Linear Algebra Appl. 432 (2010) 2754-2763.

[2] P. Bhimasankaram, A.L.N. Murthy, G.S.R. Murthy, and T. Parthasarathy, *Complementarity problems and positive definite matrices*, Algorithmic Operations Research, 7 (2013) 94-102. (ISI Technical Report, 2000)

[3] A. Chandrashekaran, T. Parthasarathy, and V. Vetrivel, *on the* P_2' *and* P_2 *properties in the semidefinite linear complementarity problems*, Linear Alg. Appl., 432 (2010) 134-143.

[4] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford University Press, Oxford, 1994.

[5] M.S. Gowda, Y. Song, and K.C. Sivakumar, *Some characterizations of cone preserving Z-transformations*, Ann. Oper. Research, 287 (2020) 727-736.

[6] M.S. Gowda, R. Sznajder, and J. Tao, *Some P-properties for linear transformations on Euclidean Jordan algebras,* Linear Algebra Appl., 393 (2004) 203-232.

[7] S. Karamardian, *An existence theorem for the complementarity problem*, Jour. Optimization Theory, 19 (1976) 227-232.

[8] M. Orlitzky, *Positive and* Z*-operators on closed convex cones*, Electron. J. Linear Alg., 34 (2018) 444-458.

[9] B.-S. Tam, *Some results on polyhedral cones and simplicial cones*, Linear Multi. Alg., 4 (1977) 281-284.