Some structural properties of copositive and completely positive cones

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Setting

Consider $\mathbb{R}^n$ with the usual inner product.

$\mathcal{S}^n$ is the space of all $n \times n$ real symmetric matrices with inner product $\langle X, Y \rangle := tr(XY)$.

Throughout, H denotes a (finite dimensional) real Hilbert space, such as $\mathbb{R}^n$ or $\mathcal{S}^n$.

Copositive and completely positive cones

$\mathcal{C}$ in $\mathbb{R}^n$ is a **closed cone** that is not necessarily convex.

(Cone: $x \in \mathcal{C},\ \lambda \geq 0 \Rightarrow \lambda x \in \mathcal{C}$.)

$A \in \mathcal{S}^n$ is copositive on $\mathcal{C}$ means: $x^T A x \geq 0$ for all $x \in \mathcal{C}$.

The **copositive cone of** $\mathcal{C}$ is:

$$\mathcal{E}_{\mathcal{C}} := \{A \in \mathcal{S}^n : A \text{ is copositive on } \mathcal{C}\}.$$

The **completely positive cone of** $\mathcal{C}$ is

$$\mathcal{K}_{\mathcal{C}} := \left\{ \sum_{u \in S} uu^T : S \subset \mathcal{C},\ S \text{ finite} \right\}.$$

Both $\mathcal{E}_{\mathcal{C}}$ and K are closed convex cones in $\mathcal{S}^n$.

• When $\mathcal{C} = \mathbb{R}^n$, $\mathcal{K}_{\mathcal{C}} = \mathcal{E}_{\mathcal{C}} = \mathcal{S}^n_+$ (semidefinite cone).

• When $\mathcal{C} = \mathbb{R}^n_+$, $\mathcal{K}_{\mathcal{C}}$ is the *standard* cone of completely positive matrices and $\mathcal{E}_{\mathcal{C}}$ is the *standard* cone of copositive matrices.

Various problems in optimization could be (re)formulated as linear programs over completely positive or copositive cones. For example,

Burer (2009):

Any nonconvex quadratic minimization problem over the nonnegative orthant with linear and binary constraints can be reformulated as a linear program over a completely positive cone.

Motivated by the good properties of the semidefinite cone, we look for structural properties of completely positive cones (copositive cones). In particular, we ask:

When is a completely positive cone $\mathcal{K}_{\mathcal{C}}$

- **self-dual**?
- **irreducible**?
- **homogeneous**?

- **What is the automorphism group of** $\mathcal{K}_{\mathcal{C}}$?
- **Is there a way to quantify 'goodness' of** $\mathcal{K}_{\mathcal{C}}$?

Some definitions

Let K be a closed cone in H.

• K is **pointed** if $K \cap -K = \{0\}$.

• K is a **proper cone** if K is a closed convex pointed cone with nonempty interior.

• $K^* := \{x \in H : \langle x, y \rangle \geq 0$ for all $y \in K\}$ is the **Dual of** K.

• K is **self-dual** if $K = K^*$.

• A is an **automorphism** of K if A is linear and invertible on H with $A(K) = K$.

• K is **homogeneous** if K is a proper cone and for any $x, y \in int(K)$, there is $A \in Aut(K)$ such that $Ax = y$.

- K is **reducible** if there exist nonzero closed cones K_1 and K_2 such that

$$K = K_1 + K_2 \quad \text{and} \quad \text{span}(K_1) \cap \text{span}(K_2) = \{0\}.$$

If K is not reducible, then it is *irreducible*.

Our results

Recall: For any closed cone $\mathcal{C}$ in $\mathbb{R}^n$,

$\mathcal{K}_{\mathcal{C}}$ is the completely positive cone of $\mathcal{C}$.

We show

- $\mathcal{K}_{\mathcal{C}}$ *is self-dual if and only if* $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.
- $\mathcal{S}^n_+$ *is the only self-dual completely positive cone.*
- *When* $\mathcal{C}$ *has nonempty interior,* $\mathcal{K}_{\mathcal{C}}$ *is irreducible.*
- *When* $\mathcal{C}$ *is a proper cone,* $\mathcal{K}_{\mathcal{C}}$ *is non-homogeneous.*

We relate Aut$(\mathcal{C})$ *and Aut*$(\mathcal{K}_{\mathcal{C}})$ *when* $\mathcal{C}$ *is a proper cone.*

We introduce 'Lypaunov rank' to quantify 'goodness'.

Some basic properties

- $\mathcal{K}_{\mathcal{C}} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}_{\mathcal{C}}$.
- $\mathcal{K}_{\mathcal{C}}$ is pointed, that is, $\mathcal{K}_{\mathcal{C}} \cap -\mathcal{K}_{\mathcal{C}} = \{0\}$.
- $\mathcal{E}_{\mathcal{C}}$ is the dual of $\mathcal{K}_{\mathcal{C}}$ in $\mathcal{S}^n$.
- If int($\mathcal{C}$) is nonempty, then $\mathcal{K}_{\mathcal{C}}$ and $\mathcal{E}_{\mathcal{C}}$ are proper.
- Ext$(\mathcal{K}_{\mathcal{C}}) = \{uu^T : 0 \neq u \in \mathcal{C}\}$.
- int$(\mathcal{K}_{\mathcal{C}}) = \{\sum u_i u_i^T : \, u_i \in int(\mathcal{C}), \, span\{u_1, \ldots, u_n\} = \mathbb{R}^n\}$.

Self-duality

Theorem: *$\mathcal{K}_{\mathcal{C}}$ is self-dual if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.*

Proof. If $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$, then $\mathcal{K}_{\mathcal{C}} = \mathcal{S}^n_+$ is self-dual.

If $\mathcal{K}_{\mathcal{C}}$ is self-dual, then $\mathcal{K}_{\mathcal{C}} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}_{\mathcal{C}} \Rightarrow \mathcal{K}_{\mathcal{C}} = \mathcal{S}^n_+$.

Now, for any nonzero $x \in \mathbb{R}^n$, xx^T is an extreme vector of $\mathcal{S}^n_+ = \mathcal{K}_{\mathcal{C}}$.

By a known characterization, $xx^T = uu^T$ for some $0 \neq u \in \mathcal{C}$.

But then, $x = \pm u$. So, $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.

Irreducibility

Example. In $\mathbb{R}^n$, consider the standard unit vectors $e_1, e_2, \ldots, e_n$; let $\mathcal{C}$ be the cone generated by these. Then, the corresponding completely positive cone is the cone of nonnegative diagonal matrices in $\mathcal{S}^n$. This is isomorphic to $\mathbb{R}^n_+$, hence reducible.

Note that in this example, interior of $\mathcal{C}$ is empty.

Theorem: *If $\mathcal{C}$ has nonempty interior, then $\mathcal{K}_{\mathcal{C}}$ is irreducible.*

Proof. Suppose $\mathcal{C}$ has nonempty interior and $\mathcal{K}_{\mathcal{C}}$ is reducible in $\mathcal{S}^n$. Let $\mathcal{K}_{\mathcal{C}} = K_1 + K_2$, where K_1 and K_2 are nonzero closed cones with span$(K_1) \cap$ span$(K_2) = \{0\}$.

For any $0 \neq u \in \mathcal{C}$, uu^T is an extreme vector of $\mathcal{K}_{\mathcal{C}}$.

If $uu^T = x_1 + x_2$ with $x_i \in K_i \subseteq \mathcal{K}_{\mathcal{C}}$, we must have $x_1 = uu^T$ (say) and $x_2 = 0$.

Then, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 := \{u \in \mathcal{C} : uu^T \in K_1\} \quad \text{and} \quad \mathcal{C}_2 := \{u \in \mathcal{C} : uu^T \in K_2\}.$$

Baire category theorem implies $\mathcal{C}_1$ (say) has nonempty interior. Then the corresponding completely positive cone $\mathcal{K}_{\mathcal{C}_1}$ is proper and so $\mathcal{K}_{\mathcal{C}_1} - \mathcal{K}_{\mathcal{C}_1} = \mathcal{S}^n$. As $\mathcal{K}_{\mathcal{C}_1} \subseteq K_1$,
we must have $K_1 - K_1 = \mathcal{S}^n$, that is, span$(K_1) = \mathcal{S}^n$.
Then, span$(K_2) = \{0\}$, a contradiction.

Non-homogeneity

Recall: A proper cone K in H is

homogeneous if for any $x, y \in int(K)$,

there is an $L \in Aut(K)$ such that $L(x) = y$.

- A self-dual homogeneous cone is a **symmetric cone.**
- Every such cone arises as the cone of squares in a

Euclidean Jordan algebra.

$\mathbb{R}^n_+$, $\mathcal{S}^n_+$, and second order cone are examples of

symmetric cones.

The automorphism group of $\mathcal{K}_{\mathcal{C}}$

Gowda-Sznajder-Tao (2013): *Suppose $\mathcal{C}$ is a proper cone.*

Then every automorphism of $\mathcal{K}_{\mathcal{C}}$ is of the form

$$L(X) = QXQ^T \quad (X \in \mathcal{S}^n)$$

for some automorphism Q of $\mathcal{C}$.

Example: Let $\mathcal{C} = \mathbb{R}^n_+$. Then, every automorphism of $\mathbb{R}^n_+$ is given by $Q = PD$, where P is a permutation matrix and D is a positive diagonal matrix.

Theorem: *If $\mathcal{C}$ is a proper cone in $\mathbb{R}^n$ $(n > 1)$, then $\mathcal{K}_{\mathcal{C}}$ cannot be homogeneous.*

We need two lemmas. Here $L \in \mathbb{R}^{n \times n}$.

Lemma 1: Suppose $L(\mathcal{C}) \subseteq \mathcal{C}$ with $L(u) = 0$ for some $u \in int(\mathcal{C})$. Then $L = 0$.

Lemma 2: Suppose $L \in \overline{Aut(\mathcal{C})}$ and $L(d) \in int(\mathcal{C})$ for some $d \in int(\mathcal{C})$. Then, $L \in Aut(\mathcal{C})$.

Sketch of the proof of the theorem:

We know $\mathcal{K}_{\mathcal{C}}$ is proper. Suppose $\mathcal{K}_{\mathcal{C}}$ is homogeneous.

Pick two bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v, u_2, \ldots, u_n\}$.

in $int(\mathcal{C})$. Define $X := u_1 u_1^T + u_2 u_2^T + \cdots + u_n u_n^T$ and

$Y_k := vv^T + \frac{1}{k}(u_2 u_2^T + \cdots + u_n u_n^T)$. These are in $int(\mathcal{K}_{\mathcal{C}})$.

There exist $L_k \in Aut(\mathcal{K}_{\mathcal{C}})$ of the form $L_k(X) = Q_k X Q_k^T$

with $Q_k \in Aut(\mathcal{C})$ such that $Q_k X Q_k = L_k(X) = Y_k$.

So, for all k,

$$Q_k(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q_k^T = vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T).$$

Case 1: Q_k unbounded. Without loss of generality, let $Q := \lim \frac{Q_k}{||Q_k||}$. A normalization argument leads to

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = 0$$

and to $Qu_i = 0$ for all i. This implies $Q = 0$.

Case 2: Q_k **bounded.** Let $Q := \lim Q_k \in \overline{Aut(\mathcal{C})}$.
Taking limits,

$$Q(u_1 u_1^T + u_2 u_2^T + \cdots + u_n u_n^T)Q^T = vv^T.$$

As $vv^T \in Ext(\mathcal{K}_{\mathcal{C}})$, we must have $Qu_i = \lambda_i v$ for all i.

Then Q has rank one. Now, by Lemma 1, $\lambda_1 \neq 0$.

As $\mathcal{C}$ is proper, λ_1 cannot be negative.

When λ_1 is positive, by Lemma 2, $Q \in Aut(\mathcal{C})$. As Q has rank one, we get a contradiction.

Bilinearity relations

Let K be a proper cone in $\mathbb{R}^n$.

The optimality conditions for a primal-dual cone-linear program on K are of the form

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ x \in K, s \in K^*, \langle x, s \rangle &= 0. \end{aligned}$$

To make the above system square, it is desirable to have n or more independent bilinear relations describing the complementarity condition.

Bilinearity rank of a cone

Let

$C(K) := \{(x, s) : x \in K, s \in K^*, \langle x, s \rangle = 0\}$.

Rudolf, Noyan, Papp, and Alizadeh (2011):

An $n \times n$ matrix Q is called a *bilinearity relation* on K if

$(x, s) \in C(K) \Rightarrow x^T Q s = 0.$

The **bilinearity rank** of K is:

$\beta(K)$= Dimension of the space of all bilinearity relations.

This notion can be extended to a proper cone in our (finite dimensional) real Hilbert space H.

Lyapunov-like transformations

Let K be a proper cone in H.

Gowda-Sznajder (2007):

A linear transformation L on H is said to be **Lyapunov-like** *on K if*

$$x \in K, y \in K^*, \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$$

Thus, L is Lyapunov-like on K iff L^T is a bilinearity relation on K and

$\beta(K)$=Dimension of the space of all Lyapunov-like transformations on K.

K is a **good cone** if $\beta(K) \geq \dim(H)$.

Examples

Example 1: On $\mathbb{R}^n_+$, a matrix is Lyapunov-like iff it is a diagonal matrix.

Example 2: On $\mathcal{S}^n_+$, L is Lyapunov-like iff it is of the form $L_A(X) = AX + XA^T \; (X \in \mathcal{S}^n)$ for some $A \in R^{n \times n}$.

Example 3: On a Euclidean Jordan algebra, L is Lyapunov-like if and only if $L = L_a + D$, where $L_a(x) := a \circ x$ and D is a derivation.

Thanks to a result of **Schneider-Vidyasagar** (1970),

The following are equivalent:

- L is Lyapunov-like on K.
- $e^{tL} \in Aut(K)$ for all $t \in \mathbb{R}$.
- L belongs to the Lie algebra of the group $Aut(K)$.

Thus, for any proper cone K,

$$\beta(K) = dim(Lie(Aut(K)).$$

Simple symmetric cones

Gowda-Tao (2014): V – Euclidean Jordan algebra, $Herm(V)$ – Hermitian matrices in (the matrix algebra) V, and K – corresponding symmetric cone.

(i) In $Herm(R^{n\times n})$, $\beta(K) = n^2$.

(ii) In $Herm(C^{n\times n})$, $\beta(K) = 2n^2 - 1$.

(iii) In $Herm(Q^{n\times n})$, $\beta(K) = 4n^2$.

(iv) In $Herm(O^{3\times 3})$, $\beta(K) = 79$.

(v) In $\mathcal{L}^n$, $\beta(K) = \frac{n^2-n+2}{2}$.

Note: In each case, $\beta(K) \geq \dim(V)$.

Gowda-Sznajder-Tao (2013):

Let $\mathcal{C}$ be a proper cone. Then, every Lyapunov-like transformation on $\mathcal{K}_{\mathcal{C}}$ is of the form L_A, where $L_A(X) := AX + XA^T$ and A is Lyapunov-like on $\mathcal{C}$. Since $A \mapsto L_A$ is an isomorphism,

$$\beta(\mathcal{K}_{\mathcal{C}}) = \beta(\mathcal{C}).$$

Example: Let $\mathcal{C} = \mathbb{R}^n_+$.

Then $\mathcal{K}_\mathcal{C}$ is the cone of completely positive matrices.

Since a matrix is Lyapunov-like on $\mathbb{R}^n_+$ if and only if it is a diagonal matrix, it follows that $\beta(\mathbb{R}^n_+) = n$.

Thus, the bilinearity rank of the cone of completely positive matrices is n. This is less than the dimension of the ambient space $\mathcal{S}^n$. *Hence, the cone of completely positive matrices is 'bad' in the sense that the primal-dual LP system cannot be made square by means of bilinearity relations alone.*

Results for the copositive cone

Recall: $\mathcal{E}_{\mathcal{C}}$ is the copositive cone of $\mathcal{C}$.

(i) *$\mathcal{E}_{\mathcal{C}}$ is self-dual if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.*

(ii) *If $\mathcal{C}$ has nonempty interior, then $\mathcal{E}_{\mathcal{C}}$ is irreducible.*

(iii) *If $\mathcal{C}$ is a proper cone in $\mathbb{R}^n$ $(n > 1)$,*

then $\mathcal{E}_{\mathcal{C}}$ is not homogeneous.

Beyond copositive and completely positive cones –

Bishop-Phelps cones, spectral cones, ...

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