Weakly homogeneous variational inequalities and solvability of nonlinear equations over cones

M. Seetharama Gowda

Department of Mathematics and Statistics

University of Maryland, Baltimore County

Baltimore, Maryland

gowda@umbc.edu

****

(Joint work with David Sossa)

This talk is based on two articles:

- M.S. Gowda, *Polynomial complementarity problems,* Pacific Journal of Optimization, 13 (2017) 227-241.
- M.S. Gowda and D. Sossa, *Weakly homogeneous variational inequalities and solvability of nonlinear equations over cones*, To appear in Math. Programming.

The variational inequality problem

Setting: H is a finite dimensional real Hilbert space,

K is a closed convex set in H,

$\phi : K \to H$ is continuous, and $q \in H$.

Then, the *variational inequality problem* VI(ϕ, K, q)

is to find $x^* \in K$ such that

$$\langle \phi(x^*) + q, x - x^* \rangle \geq 0 \quad \forall\, x \in K.$$

The complementarity Problem

If K is a closed convex *cone*, then

VI(ϕ, K, q) becomes the *complementarity problem*

CP(ϕ, K, q): Find $x^* \in H$ such that

$$x^* \in K,\ \phi(x^*) + q \in K^*,\ \text{and } \langle x^*, \phi(x^*) + q\rangle = 0.$$

Here $K^* := \{x \in H : \langle x, y\rangle \geq 0\, \forall\, y \in K\}$ is the dual cone.

Standard complementarity problem: $H = R^n$, $K = R^n_+$.

Semidefinite complementarity problem: $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$.

When $K = H$, VI$(\phi, K, -q)$ becomes an equation $\phi(x) = q$.

Motivation

We begin with some examples.

Example 1 On R, let $f(x) := x^3 + \sin x$.

By the Intermediate Value Theorem, the equation $f(x) = q$ is solvable for all $q \in R$.

Example 2 On R, let $f(x) := x^2 + \sin x$.

The equation $f(x) = q$ is not solvable for all $q \in R$, but solvable when $q \geq 0$ (that is, in the cone R_+).

In these examples, f is a sum of a homogeneous term (call it a ‘leading term’) and a lower-order term. Solvability depends on the behavior of the leading term at zero.

Is there an analog for variational inequalities?

Example 3 In $H = R^2$, let $K := \{(x, y) : x > 0, y > 0,\ xy \geq 1\}$,

and $f(x, y) := (x, y) + (\sin x, \sqrt{y})$.

Then, f is a sum of homogeneous/leading term

$f^\infty(x, y) = (x, y) = I(x, y)$ and a ‘lower-order’ term.

It turns out that the solvability of VI(f, K, q)

is tied to CP$(f^\infty, K^\infty, 0)$, where

$K^\infty = R^2_+$ is the recession cone of K.

Motivated by the above examples, in this talk, we consider VI(f, K, q), where f is a sum of a homogeneous term (call it a 'leading term') and a lower-order term and tie the solvability of VI(f, K, q) to the behavior of VI$(f^\infty, K^\infty, 0)$, where f^∞ is the leading term of f and K^∞ is the recession cone of K.

Outline

- Weakly homogeneous maps
- The main result
- A copositivity result
- A generalization of Karamardian’s theorem
- Solvability of nonlinear equations over cones

Weakly homogeneous maps

Let H be a finite dimensional real Hilbert space and C be a closed convex cone in H.

- A continuous map $h : C \to H$ is

homogeneous of degree $\gamma\ (> 0)$ *on* C if

$h(\lambda x) = \lambda^\gamma h(x) \quad (\forall x \in C,\ \lambda \geq 0).$

• A map $f : C \to H$ is *weakly homogeneous of degree* γ (> 0) *on* C if $f = h + g$, where $h, g : C \to H$ are continuous, h is homogeneous of degree γ and $\frac{g(x)}{||x||^\gamma} \to 0$ as $||x|| \to \infty$ in C.

Since $h(0) = 0$, we assume that $g(0) = 0 = f(0)$.

We let $f^\infty(x) := \lim_{\lambda \to \infty} \frac{f(\lambda\, x)}{\lambda^\gamma} = h(x)$.

We say h is the ‘recession/leading’ term of f and g is ‘subordinate’ to h (or ‘follower’ of h).

Examples

- A polynomial map $f : R^n \to R^n$ is weakly homogeneous.

Let $H = \mathcal{S}^n$, the space of all $n \times n$ real symmetric matrices. $C = \mathcal{S}^n_+$, the semidefinite cone (of positive semidefinite matrices). Let $A \in R^{n \times n}$ and $B \in \mathcal{S}^n$.

- Lyapunov transformation $f(X) := AX + XA^T$.
- Stein transformation $f(X) := X - AXA^T$.
- Riccati transformation $f(X) := XBX + AX + XA^T$.

More examples on $C = \mathcal{S}^n_+$

- $f(X) := X + \sin(X)$.
- $f(X) := XAXBXAX$ (where $A, B \in \mathcal{S}^n$).
- $f(X) := X^{r_m} A_m \cdots X^{r_2} A_2 X^{r_1} A_1 X^{r_1} A_2 X^{r_2} \cdots A_m X^{r_m}$.
- $f(X) := X - \sum_1^k A_i X^\delta A_i$ (where $0 < \delta < 1$, $A_i \in \mathcal{S}^n$).

The classical result of Hartman-Stampacchia:

If K is compact, then VI(ϕ, K, q) has a solution.

When K is not compact (e.g., K is a nonzero cone), coercive type conditions are imposed. In many settings (e.g., complementarity problems), these are too restrictive. *Our goal here is to study variational inequalities corresponding to weakly homogeneous maps by considering only the recession parts of the map and the closed convex set.*

Weakly homogeneous VIs

Setting: $K \subseteq C \subseteq H.$

Here K is a closed convex set, C is a closed convex cone, $f : C \to H$ is weakly homogeneous with $f = h + g$, where $h(= f^\infty)$ is the 'leading' part of f and g is 'subordinate' to h.

We extend f and f^∞ to all of H and use the same notation for the extensions. Let K^∞ denote the (closed convex) recession cone of K:

$K^\infty := \{u \in H : u + K \subseteq K\}.$

Reduction of VI to CP

Let

$$F(x) := x - \Pi_K\Big(x - [f(x) + q]\Big),$$

$$F^\infty(x) := x - \Pi_{K^\infty}\Big(x - f^\infty(x)\Big).$$

VI(f, K, q) is equivalent to solving $F(x) = 0$ and VI$(f^\infty, K^\infty, 0)$ =CP$(f^\infty, K^\infty, 0)$ is equivalent to solving $F^\infty(x) = 0$.

Goal: Study VI(f, K, q) via CP$(f^\infty, K^\infty, 0)$.

Our main result

Let $f : C \to H$ *be weakly homogeneous with leading term* f^∞ *and* $f(0) = 0$. *Let* K *be a closed convex subset of* C *with recession cone* K^∞.

Let $F^\infty(x) := x - \Pi_{K^\infty}\Big(x - f^\infty(x)\Big)$.

Suppose

- $F^\infty(x) = 0 \Leftrightarrow x = 0$ *and*
- $\mathrm{ind}\,(F^\infty, 0) \neq 0$.

Then, for all $q \in H$, *VI*(f, K, q) *and CP*(f, K^∞, q) *have nonempty compact solution sets.*

Copositive maps

We specialize our main result to copositive maps.

A map $\phi : E \to H$ is *copositive on* E if

$\langle \phi(x), x \rangle \geq 0 \quad$ for all $x \in E$.

If $\langle \phi(x), x \rangle > 0 \quad$ for all $0 \neq x \in E$,

we say that ϕ is *strictly copositive* on E.

Example On $H = \mathcal{S}^n$ and $E = \mathcal{S}^n_+$ with $A \in \mathcal{S}^n$,

$\phi(X) := XAX$ is copositive if A is positive semidefinite and

strictly copositive if A is positive definite.

A copositivity result

Let $f : C \to H$ *be*

weakly homogeneous, $K \subseteq C \subseteq H$*.*

Suppose one of the following holds:

- $F^\infty(x) = 0 \Leftrightarrow x = 0$ *and* f^∞ *is copositive on* K^∞*.*
- f^∞ *is strictly copositive on* K^∞*.*

Then, for all $q \in H$*, VI*(f, K, q) *and CP*(f, K^∞, q) *have*

nonempty compact solution sets.

Back to our simple example

In $H = R^2$, let $K := \{(x, y) : x > 0, y > 0,\ xy \geq 1\}$,

$C := R^2_+$, and $f(x, y) := (x, y) + (\sin x, \sqrt{y})$.

Then, $K^\infty = R^2_+$ and f is weakly homogeneous on C

with $f^\infty(x, y) = (x, y) = I(x, y)$.

Note that f^∞ is strictly copositive on R^n_+.

Hence, VI(f, K, q) has a nonempty compact

solution set for every q.

A surjectivity result

Let $K = C = H$ and f be weakly homogeneous on H.

Suppose $f^{\infty}(x) = 0 \Leftrightarrow x = 0$ and $\operatorname{ind}(f^{\infty}, 0) \neq 0$.

Then, f is surjective:

for every q, $f(x) = q$ has a solution.

This is especially true for polynomial maps on R^n.

Example: $f(x) = Ax + \sin x$ on R^n. If A is an invertible matrix, then f is surjective.

Karamardian's Theorem

A well-known result of Karamardian asserts that if

C is a proper cone in H, $h : C \to H$ is positively homogeneous with complementarity problems CP$(h, C, 0)$ and CP(h, C, d) having zero as the only solution for some $d \in int(C^*)$, then for all $q \in H$, CP(h, C, q) has a solution.

Our improvement: For any g that is 'subordinate' to h, CP$(h + g, C, q)$ has a nonempty compact solution set.

Nonlinear equations over cones

Given a closed convex cone C in H, $q \in C$, and a map $f : C \to H$, one looks for a solution of

$f(x) = q$ in C.

For example, in $H = \mathcal{S}^n$ with $C = \mathcal{S}^n_+$ and $Q \in \mathcal{S}^n_+$:

- Lyapunov equation $AX + XA^T = Q$ (where $A \in R^{n\times n}$).
- Stein equation $X - AXA^T = Q$ (where $A \in R^{n\times n}$).
- Riccati equation $XBX + AX + XA^T = Q$ (where $A \in R^{n\times n}, B \in \mathcal{S}^n$).
- Word equation $XAXBXAX = Q$ (where $A, B \in \mathcal{S}^n$).

Is there a unified way of proving solvability?

A simple result: *Suppose C is a closed convex cone in H, $f : C \to H$ and*

$$\Big[x \in C,\ y \in C^*,\ \textit{and } \langle x, y\rangle = 0\Big] \Rightarrow \langle f(x), y\rangle \leq 0.$$

(The above property is called the Z-property of f on C.)

Then,

- $\Big[q \in C \textit{ and } x^* \textit{ solves } CP(f, C, -q)\Big] \Rightarrow f(x^*) = q.$
- $\Big[x^* \in C \textit{ and } f(x^*) \in \textit{int}(C)\Big] \Rightarrow x^* \in \textit{int}(C).$

We now combine this with our main result.

Let C be a closed convex cone in H, $f : C \to H$ be weakly homogeneous with leading term f^∞ and $f(0) = 0$.

With $F^\infty(x) := x - \Pi_C\Big(x - f^\infty(x)\Big)$, suppose

- *f has the Z-property on C,*
- *Zero is the only solution of CP$(f^\infty, C, 0)$, and*
- $\mathrm{ind}\,(F^\infty, 0) \neq 0$.

Then, for all $q \in C$, the equation $f(x) = q$ has a solution in C. If $q \in \mathrm{int(C)}$, then the solution belongs to $\mathrm{int(C)}$.

Various solvability results in the (dynamical systems) literature (Lyapunov, Stein, Lim et al, etc.,) all follow from this result. A new one (Hillar and Johnson, 2004): Consider a *symmetric word equation* over $\mathcal{S}^n$:

$$X^{r_m}A_m\cdots X^{r_2}A_2X^{r_1}A_1X^{r_1}A_2X^{r_2}\cdots A_mX^{r_m}=Q.$$

Here, $A_1, A_2, \ldots, A_m$ are positive definite matrices, $r_1, r_2, \ldots, r_m$ are positive exponents, and Q is a positive (semi)definite matrix.

Then, there is a positive (semi)definite solution X.

Concluding Remarks

In this talk, we discussed some solvability issues for weakly homogeneous variational inequalities. We showed that under appropriate conditions, solvability can be reduced to that of recession map/cone complementarity problems. Many open issues/problems remain, for example,

- Uniqueness of solution,
- Applications to polynomial optimization,
- Specialized results for symmetric cones.