The eigenvalue map and spectral sets/functions/cones

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(Joint work with Juyoung Jeong)

Motivation

Let H be a finite dimensional real Hilbert space and K be a closed convex cone in H.

A *conic linear program* on K is to solve

$$\min\langle c, x\rangle$$
$$L(x) = b$$
$$x \in K$$

where, c and b are given and L is linear.

Many cones appear in applications:

The non-negative orthant, semidefinite cone, second-order cone, copositive cone, completely positive cone, etc.

Among these, there are ‘good’ cones admitting polynomial time algorithms. Symmetric cones (e.g., first three cones) belong to this class. Any symmetric cone K can be written as

$$K = \lambda^{-1}(R^n_+),$$

where λ is the so-called ‘eigenvalue map’.

While this map is nonlinear and its inverse is a multivalued map, it exhibits remarkable ‘linear’ behavior.

The purpose of this talk is to illustrate this linear behavior via spectral sets and functions.

While doing this, we touch upon automorphism groups, majorization, transfer principle, dimensionality results, and commutativity principle, etc.

Some basic ideas

Let $R^n =$ The Euclidean n-space over R.

For $x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n)$,

inner product and componentwise product are given by

$\langle x, y \rangle := \sum_1^n x_i y_i$ and

$x * y :== (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$.

Let $R^n_+ =$ the non-negative orthant.

Note: $R^n_+ = \{x * x : x \in R^n\}$.

Let $\lambda(x)$ denote the vector of entries of x written in the decreasing order.

We are interested in sets/functions/cones that are *permutation invariant*, meaning that they remain the same when the entries of x are permuted.

Examples: R^n_+ and (usual) 2-norm.

Let $\mathcal{S}^n =$ Space of all $n \times n$ real symmetric matrices.

We define the inner product and Jordan product:

$\langle X, Y \rangle := trace(XY)$,

$X \circ Y := \frac{XY+YX}{2}$.

Let $\mathcal{S}^n_+ =$ the set of all positive semidefinite matrices in $\mathcal{S}^n$.

Note: $\mathcal{S}^n_+ = \{X \circ X : X \in \mathcal{S}^n\}$.

For $X \in \mathcal{S}^n$, let $\lambda(X)$ denote the vector

of eigenvalues of X written in the decreasing order.

So we have the **'eigenvalue map'** $\lambda : \mathcal{S}^n \to R^n$.

Note that $X \in \mathcal{S}^n_+$ iff $\lambda(X) \in R^n_+$, so

$$\mathcal{S}^n_+ = \lambda^{-1}(R^n_+).$$

Observe that R^n_+ is a closed convex cone,

λ^{-1} is a multivalued map, yet $= \lambda^{-1}(R^n_+)$

is a closed convex cone!

In optimization, the cones R^n_+ and $\mathcal{S}^n_+$ are fundamental: R^n_+ is the underlying cone in linear programming and $\mathcal{S}^n_+$ is the basis of semidefinite programming. (These are problems where a linear objective function is minimized subject to linear constraints.)

Question: Instead of R^n_+, can we start with a permutation invariant closed convex cone Q in R^n and define $K := \lambda^{-1}(Q)$ so as to generate a closed convex cone that is interesting/useful from the optimization perspective?

Goal: Study the eigenvalue map and its inverse for their 'linearity' and 'differentiability' properties.

In this talk, we will concentrate on the 'linearity' aspects.

Outline

- Permutation invariant sets and cones in R^n
- Euclidean Jordan algebras
- Spectral sets and cones
- The transfer principle
- A dimensionality result
- Commutation principles
- Cross complementarity concepts

Permutation invariant set in R^n

- A permutation matrix is a zero-one matrix with exactly one 1 in each row and column.
- Σ_n is the set of all $n \times n$ permutation matrices.

Note that Σ_n is a group.

For $x \in R^n$ and $\sigma \in \Sigma_n$,

$\sigma(x)$ is just a permutation of entries of x.

- A set Q in R^n is *permutation invariant* if

$$\sigma(Q) = Q \quad \forall\, \sigma \in \Sigma_n.$$

Majorization

An $n \times n$ real matrix A is *doubly stochastic*

if it is nonnegative and each row/column sum is one.

Birkhoff's Theorem: A *is doubly stochastic iff*

it is a convex combination of permutation matrices.

For vectors x and y in R^n,

we say x is *majorized* by y and write $x \prec y$

if $x = Ay$ for some doubly stochastic matrix A.

Examples of permutation invariant sets

- R^n and R^n_+.
- $\{x \in R^n : x_1 + x_2 + \cdots + x_n = 0\}$.

For $p = 1, 2, \ldots, n$, let $s_p(x) =$ sum of smallest p entries of x.

- $Q_p := \{x \in R^n : s_p(x) \geq 0\}$.

Note: $R^n_+ = Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n$.

- For any nonempty set S in R^n, $Q := \Sigma_n(S)$

is permutation invariant.

Euclidean Jordan algebra

$(V, \langle \cdot, \cdot \rangle, \circ)$ is a **Euclidean Jordan algebra** if

V is a finite dimensional real inner product space

and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

$K = \{x \circ x : \; x \in V\}$ is a closed convex cone.

It is called a *symmetric cone*.

Examples of Euclidean Jordan algebras

Any EJA is a product of the following:

- $\mathcal{S}^n = \text{Herm}(\mathcal{R}^{n\times n})$ - $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n\times n})$ - $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n\times n})$ - $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3\times 3})$ - 3×3 octonion Hermitian matrices.
- $\mathcal{L}^n$ $(n \geq 3)$- Jordan spin algebra.

The above algebras are (the only) *simple* algebras.

The algebra R^n

Let $e_1, e_2, \ldots, e_n$ denote the standard basis vectors in R^n. Note that $e_i \circ e_i = e_i$ and $e_i \circ e_j = 0$ for $i \neq j$.

The set $\{e_1, e_2, \ldots, e_n\}$ is a *Jordan frame*:

It forms an orthogonal set of primitive idempotents.

Any $x = (x_1, x_2, \ldots, x_n) \in R^n$ can be written as $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$.

The algebra $\mathcal{S}^n$

For any $X \in \mathcal{S}^n$, there is an orthogonal matrix U such that $X = UDU^T$, where D is a diagonal matrix consisting of the eigenvalues of X.

Let $u_1, u_2, \ldots, u_n$ denote the column vectors of U.

Writing $e_i := u_i u_i^T$, we see that $\{e_1, e_2, \ldots, e_n\}$ is a *Jordan frame*:

It forms a an orthogonal set of primitive idempotents.

Also, $X = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$,

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the entries

of D (i.e., eigenvalues of X).

The above expression is the 'spectral decomposition' of X.

Note: The Jordan frame depends on X, but has the same

number of elements – called the 'rank' of the algebra.

The eigenvalue map

Spectral representation in an EJA of rank n:

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame,

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of x.

The *eigenvalue map* $\lambda : V \to R^n$ is defined by

$$\lambda(x) := (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Note: The eigenvalue map is continuous,

positive homogeneous, and nonlinear.

Spectral sets

Let V be a Euclidean Jordan algebra of rank n.

A set E in V is a **spectral set** if for some permutation invariant set Q in R^n,

$$E = \lambda^{-1}(Q).$$

Examples:

A symmetric cone in a EJA, its interior, its boundary are spectral sets.

Characterization of spectral sets

In V, let $x \sim y$ if $\lambda(x) = \lambda(y)$.

For a set Q in R^n, let $Q^{\diamondsuit} := \lambda^{-1}(Q)$.

For a set E in V, let $E^{\diamondsuit} := \Sigma_n(\lambda(E))$.

Theorem: For a set E in V, the following are equivalent:

1. E is a spectral set.
2. $x \sim y$, $y \in E \Rightarrow x \in E$.
3. $E^{\diamondsuit\diamondsuit} = E$.

Note: A product of spectral sets need not be spectral.

Example: $R \times \mathcal{S}^2_+$ in $R \times \mathcal{S}^2$.

Automorphism invariance

$A : V \to V$ is an *automorphism* if

A is linear, invertible, and $A(x \circ y) = Ax \circ Ay$.

- In $V = \mathcal{S}^n$, automorphisms are of the form

$$X \to UXU^T$$

for some orthogonal U.

Theorem: *Every spectral set is automorphism invariant.*

Converse holds if V is simple.

Note: Converse may not hold in a general V.

The transfer principle

Let Q be permutation invariant set in R^n and

$$E = \lambda^{-1}(Q).$$

The transfer principle describes properties of Q that get transferred to E. We will look at topological, convexity, and linearity properties.

Topological properties

Thm: Let Q be permutation invariant in R^n. Then,

1. $\overline{\lambda^{-1}(Q)} = \lambda^{-1}(\overline{Q})$. (closure)
2. $[\lambda^{-1}(Q)]^\circ = \lambda^{-1}(Q^\circ)$. (interior)
3. $\partial(\lambda^{-1}(Q)) = \lambda^{-1}(\partial(Q))$. (boundary)

Thm: Suppose Q is permutation invariant in R^n.

Then, Q is closed/open/compact if and only if

$\lambda^{-1}(Q)$ is closed/open/compact.

Convexity/conic properties

Thm: Let Q be permutation invariant in R^n.

Then,

1. Q is convex if and only if $\lambda^{-1}(Q)$ is convex.
2. $conv(\lambda^{-1}(Q)) = \lambda^{-1}(conv(Q))$. (convex hull)
3. Q is a cone if and only if $\lambda^{-1}(Q)$ is a cone.

The 'only if' part of the first item was proved by Baes (2007) using majorization techniques.

‘Linearity’ properties

Thm: Let Q be permutation invariant set in R^n. Then

1. $\alpha\lambda^{-1}(Q) = \lambda^{-1}(\alpha Q)$ $(\alpha \in R)$.
2. $\lambda^{-1}(Q_1 + Q_2) = \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2)$.
3. $[\lambda^{-1}(Q)]^* = \lambda^{-1}(Q^*)$ (when V is simple).

Proof of second item relies on: $\lambda(x+y) \prec \lambda(x) + \lambda(y)$.

Proof of third item relies on Schur type majorization: $diag(x) \prec x$.

Cor: If Q is a (closed) convex cone in R^n, then

$\lambda^{-1}(Q)$ is a (closed) convex cone in V.

Spectral cones

Let Q be a permutation invariant convex cone in R^n.

Then $K := \lambda^{-1}(Q)$ is called a **spectral cone.**

Symmetric cones in EJAs are spectral cones.

What is the dimension of a spectral cone?

Recall: If K is a convex cone, $dim(K) := dim(K - K)$.

Theorem: $dim(K) \in \{0, 1, m-1, m\}$,

where $m = dim(V)$.

What are spectral cones with these dimensions?

Dimensions of spectral cones

1. $K = \{0\}$: dimension zero.
2. K nonzero, $K \subseteq \{t\, e : t \in R\}$: dimension one.
3. $K = \{x \in V \,:\, trace(x) = 0\}$: dimension $m - 1$.
4. K is a spectral cone with nonempty interior: dimension m.

Commutation principle in EJA

Suppose E is a spectral set in V and

$f : V \to R$ is Fréchet differentiable.

Theorem (Ramirez, Seeger, and Sossa, 2013)

If a is a local minimizer of

$$\min_{x \in E} f(x),$$

then, a operator commutes with $f'(a)$.

Recently, Gowda-Juyoung extended this result to 'weakly spectral' sets - those in V that are automorphism invariant.

Cor: *If K is a spectral cone in V and*

$a \in K,\ b \in K^*,$ *and* $\langle a, b \rangle = 0,$

then, a and b operator commute.

This means that there is a (common) Jordan frame

$\{e_1, e_2, \ldots, e_n\}$ such that

$a = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n$ and

$b = b_1 e_1 + b_2 e_2 + \ldots + b_n e_n$, where

$a_1, a_2, \ldots, a_n$ are eigenvalues of a, etc.

When $K = \lambda^{-1}(Q)$ and V is simple, above Corollary implies that $\bar{a} := (a_1, a_2, \ldots, a_n) \in Q$, $\bar{b} := (b_1, b_2, \ldots, b_n) \in Q^*$, and $\langle \bar{a}, \bar{b} \rangle = 0$.

This means that if (a, b) is a complementary pair of a spectral cone $K = \lambda^{-1}(Q)$ in V, then some eigenvalue vectors $\bar{a}$, $\bar{b}$ form a complementarity pair of the cone Q in R^n. This result can be sharpened.

Notation: For $x \in R^n$, $x^{\downarrow}$ ($x^{\uparrow}$) is the decreasing (increasing) rearrangement of x.

Based on Lewis, *Normal decomposition systems*, 1996,

we state a specialized commutation principle.

Suppose V *is* R^n *or a simple EJA.*

Let $K = \lambda^{-1}(Q)$ *be a spectral cone in* V. *If*

$a \in K,\ b \in K^*,$ *and* $\langle a, b\rangle = 0$, *then, there exists*

a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ *such that*

$a = \lambda_1^{\downarrow}(a)e_1 + \lambda_2^{\downarrow}(a)e_2 + \ldots + \lambda_n^{\downarrow}(a)e_n$ *and*

$b = \lambda_1^{\uparrow}(b)e_1 + \lambda_2^{\uparrow}(b)e_2 + \ldots + \lambda_n^{\uparrow}(b)e_n$.

Hence, $\lambda^{\downarrow}(a) \in Q,\ \lambda^{\uparrow}(b) \in Q^*,\ \langle \lambda^{\downarrow}(a), \lambda^{\uparrow}(b)\rangle = 0.$

Summarizing:

If V is R^n or a simple EJA, $K = \lambda^{-1}(Q)$ is a spectral cone in V, and (a, b) is a complementary pair with respect to K, then, $(\lambda^{\downarrow}(a), \lambda^{\uparrow}(b))$ is a complementary pair with respect to Q.

This may lead to a new type of problem:

Cross complementarity problem:

Given permutation invariant Q in R^n, $f : R^n \to R^n$,

Find $x \in R^n$ such that

$x \in Q,\ y = f(x) \in Q^*,\ \langle x^{\downarrow}, y^{\uparrow} \rangle = 0.$

Spectral functions

$f : R^n \to R$ is a permutation invariant function if

$f(\sigma(x)) = f(x)$ for all $x \in R^n$ and $\sigma \in \Sigma_n$.

- $F : V \to R$ is a **spectral function** if

there exists a permutation invariant f such that

$$F = f \circ \lambda.$$

Example: A set E in V is spectral iff its

characteristic function is a spectral function.

Many properties of f get transferred to F.

Transfer of convexity property

Theorem:

Suppose Q *is convex and permutation invariant in* R^n*,*

$f : Q \to R$ *is convex and permutation invariant.*

Then, $F := f \circ \lambda$ *is convex. Moreover,*

$x \prec y$ *in* V *implies* $F(x) \leq F(y)$*.*

Illustration: For $1 \leq p \leq \infty$,

$||x||_{sp,p} := ||\lambda(x)||_p$ is a norm on V.

Concluding remarks

In this talk, we introduced spectral sets and cones in Euclidean Jordan algebras. We showed that many properties of underlying permutation invariant sets transfer to spectral sets/cones.

It may be interesting and useful to discover more such properties. It is certainly worthwhile looking for spectral cones that are useful in optimization.

References

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