Weakly homogeneous variational inequalities and solvability of nonlinear equations over cones

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(Joint work with David Sossa)

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- M.S. Gowda, *Polynomial complementarity problems,* Pacific Journal of Optimization, 13 (2017) 227-241.

- M.S. Gowda and D. Sossa, *Weakly homogeneous variational inequalities and solvability of nonlinear equations over cones*, Research Report, UMBC, December 2016. Available: *Optimization online*, April 2017.

Motivation

We begin with three examples.

Example 1 Let $A \in R^{n \times n}$, $f(x) := Ax$. Then, the equation $f(x) = q$ is solvable for all $q \in R^n$ if and only if $Ax = 0 \Leftrightarrow x = 0$ (which means A is invertible).

Example 2 On R, let $f(x) := x^2 + \sin x$.

Is the equation $f(x) = q$ solvable for all $q \in R$?

Answer: No. (Take $q = -2.$)

Example 3 On R, let $f(x) := x^3 + \sin x$.

Is the equation $f(x) = q$ solvable for all $q \in R$?

Answer: Yes. (Apply Intermediate Value Theorem.)

In these examples, f is a sum of a 'leading' homogeneous term and a 'subordinate' (lower order) term.

Thus, 'solvability ensues if the leading term behaves well'.

The goal of this talk is to explain this mathematically for (what we call) weakly homogeneous maps and variational inequalities.

A teaser: If $A \in R^{n\times n}$ is invertible, show that the equation

$Ax + \sin x = q$ has a solution for all $q \in R^n$.

Outline

- Weakly homogeneous maps
- Variational inequality problem
- Polynomial/tensor complementarity problem
- Equation formulation
- Degree theory
- The main result
- A copositivity result
- A generalization of Karamardian’s theorem
- Solvability of nonlinear equations over cones

Weakly homogeneous maps

Let H be a finite dimensional real Hilbert space

and C be a closed convex cone in H.

• A continuous map $h : C \to H$ is

homogeneous of degree $\gamma\ (> 0)$ *on* C if

$h(\lambda x) = \lambda^\gamma h(x) \quad (\forall x \in C,\ \lambda \geq 0)$.

Example: For a tensor $\mathcal{A} := [a_{i_1\, i_2\, \ldots\, i_m}]$ on R^n,

$h(x) := \mathcal{A}x^{m-1}$ is a homogeneous polynomial map, where

the ith component of h is given by

$(\mathcal{A}x^{m-1})_i := \sum_{i_2, i_3, \ldots, i_k = 1}^{n} a_{i\, i_2\, \ldots\, i_m} x_{i_2} x_{i_3} \cdots x_{i_m}$.

- A map $f : C \to H$ is *weakly homogeneous of degree* γ (> 0) *on* C if $f = h + g$, where $h, g : C \to H$ are continuous, h is homogeneous of degree γ and $\frac{g(x)}{||x||^\gamma} \to 0$ as $||x|| \to \infty$ in C.

Since $h(0) = 0$, we assume that $g(0) = 0 = f(0)$.

We let $f^\infty(x) := \lim_{\lambda \to \infty} \frac{f(\lambda\, x)}{\lambda^\gamma} = h(x)$.

We say h is the 'recession/leading' term of f and g is 'subordinate' to h (or 'follower' of h).

Examples

• $f : R^n \to R^n$ is a polynomial map (that is, each component of f is a polynomial function).

This is a sum of homogeneous polynomials, hence weakly homogeneous.

Small size example:

$$\begin{bmatrix} x_1^3 - x_1x_2 + x_2^2 \\ x_1^2 + 3x_2 \end{bmatrix} = \begin{bmatrix} x_1^3 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_1x_2 + x_2^2 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3x_2 \end{bmatrix}.$$

Examples continued

$H = \mathcal{S}^n$, the space of all $n \times n$ real symmetric matrices.

$C = \mathcal{S}^n_+$, the semidefinite cone (of positive semidefinite matrices). Let $A \in R^{n\times n}$ and $B \in \mathcal{S}^n$.

- Lyapunov transformation $f(X) := AX + XA^T$.
- Stein transformation $f(X) := X - AXA^T$.
- Riccati transformation $f(X) := XBX + AX + XA^T$.

More examples

- $f(X) := X + \sin(X)$.
- $f(X) := XAXBXAX$ (where $A, B \in \mathcal{S}^n$).
- $f(X) := X^{r_m} A_m \cdots X^{r_2} A_2 X^{r_1} A_1 X^{r_1} A_2 X^{r_2} \cdots A_m X^{r_m}$.
- $f(X) := X - \sum_1^k A_i X^\delta A_i$ (where $0 < \delta < 1$, $A_i \in \mathcal{S}^n$).

The variational inequality problem

Let E be a closed convex set in H, $\phi : E \to H$ be continuous, and $q \in H$.

Then, the *variational inequality problem* $VI(\phi, E, q)$ is to find $x^* \in E$ such that

$$\langle \phi(x^*) + q, x - x^* \rangle \geq 0 \quad \forall\, x \in E.$$

This appears in PDEs, Engineering, Economics, Optimization, etc.

Best source: Book(s) by Facchinei and Pang, 'Finite dimensional variational inequalities'.

• Optimization origin: For (differentiable) $\theta : E \to R$,

$\min\{\theta(x) : x \in E\}$ leads to $VI(\nabla\theta, E, 0)$.

Special cases

• If E is a closed convex *cone*, then

$VI(\phi, E, q)$ becomes a *complementarity problem*

$CP(\phi, E, q)$: Find $x^* \in H$ such that

$$x^* \in E,\ \phi(x^*) + q \in E^*,\ \text{and}\ \langle x^*, \phi(x^*) + q\rangle = 0.$$

Here $E^* := \{x \in H : \langle x, y\rangle \geq 0\, \forall\, y \in E\}$ is the dual cone.

Special cases continued

- If $E = H$, then $VI(\phi, E, q)$ reduces to

finding a solution to $\phi(x) + q = 0.$

- Let $H = \mathcal{S}^n$ and $E = \mathcal{S}^n_+$. Then, $VI(\phi, E, q)$ is a

semidefinite complementarity problem.

More generally, if H is a Euclidean Jordan algebra and E is the corresponding symmetric cone, then $VI(\phi, E, q)$ is a *symmetric cone complementarity problem*.

Special cases continued

In the following, we let $H = R^n$ and $E = R^n_+$. Then, $CP(\phi, R^n_+, q)$ is a *nonlinear complementarity problem*, denoted by NCP(ϕ, q).

- If ϕ is a polynomial map, then NCP(ϕ, q) is a *polynomial complementarity problem* PCP(ϕ, q).
- Given a tensor $\mathcal{A}$ on R^n, consider the corresponding homogeneous polynomial $h(x) := \mathcal{A}x^{m-1}$. Then, NCP$(h, q)$ becomes the *tensor complementarity problem* TCP$(\mathcal{A}, q)$.

• If $\phi(x) = Ax$, where $A \in R^{n \times n}$, then NCP(ϕ, q) becomes the *linear complementarity problem* LCP(A, q).

LCPs include linear/quadratic programming and bimatrix game problems.

Book: Cottle, Pang, Stone,
'The Linear complementarity problem'.

The classical result of Hartman-Stampacchia:

If E is compact, then $VI(\phi, E, q)$ has a solution.

When E is not compact (e.g., E is a nonzero cone), coercive type conditions are imposed. In many settings (e.g., complementarity problems), these are too restrictive. Our goal here is to study variational inequalities corresponding to weakly homogeneous maps by considering only the recession parts of the map and the closed convex set.

Equation formulation

Consider $VI(\phi, E, q)$. Let $\Phi_E(x)$ denote the orthogonal projection of x onto (the closed convex set) E.

By considering $\phi \circ \Pi_E$, we may assume that ϕ is defined on all of H (instead of just on E).

Let $F(x) := x - \Pi_E\Big(x - \phi(x) - q\Big)$.

This is called the *Natural map*.

Then, solving $VI(\phi, E, q)$ *is equivalent to finding a solution of the equation* $F(x) = 0$.

Example Let $H = R^n$, $E = R^n_+$. Then, $F(x) = \min\{x, \phi(x) + q\}$.

Weakly homogeneous VIs

Setting: $K \subseteq C \subseteq H,$

where K is a closed convex set, C is a closed convex cone, $f : C \to H$ is weakly homogeneous with $f = h + g$, where $h(= f^{\infty})$ is the 'leading' part of f and g is 'subordinate' to h.

We extend f and f^{∞} to all of H and use the same notation for the extensions. Let K^{∞} denote the (closed convex) recession cone of K:

$K^{\infty} := \{u \in H : u + K \subseteq K\}.$

Reduction: VI to CP

Let

$$F(x) := x - \Pi_K\Big(x - [f(x) + q]\Big),$$

$$F^\infty(x) := x - \Pi_{K^\infty}\Big(x - f^\infty(x)\Big).$$

$VI(f, K, q)$ is equivalent to solving $F(x) = 0$ and $VI(f^\infty, K^\infty, 0)$ is equivalent to solving $F^\infty(x) = 0$.

$VI(f^\infty, K^\infty, 0)$ *is a complementarity problem.*

We denote it by $CP(f^\infty, K^\infty, 0)$.

A simple example

In $H = R^2$, let $K := \{(x, y) : x > 0, y > 0,\ xy \geq 1\}$,

$C := R^2_+$, and $f(x, y) := (x, y) + (\sin x, \sqrt{y})$.

Then, $K^\infty = R^2_+$ and f is weakly homogeneous on C

with $f^\infty(x, y) = (x, y) = I(x, y)$.

$CP(f^\infty, K^\infty, 0)$ is just the linear complementarity problem

LCP$(I, 0)$ which has zero as the only solution.

Will show that $VI(f, K, q)$ has a nonempty compact

solution set for every q.

Topological degree theory

Consider topological degree $\deg(\phi, \Omega, 0)$.

Here: $\Omega \subseteq H$ is a bounded open set, $\phi : \overline{\Omega} \to H$ is continuous, and $0 \notin \phi(\partial\Omega)$.

Fact: *This degree is an integer. If it is nonzero, then the equation* $\phi(x) = 0$ *has a solution in* Ω*.*

Suppose $x^* \in \Omega$ and $\phi(x) = 0 \Leftrightarrow x = x^*$. Then, $\deg(\phi, \Omega', 0)$ is constant over (open set) Ω', where $x^* \in \Omega' \subseteq \Omega$. This common value is denoted by $\mathrm{ind}(\phi, x^*)$.

Homotopy invariance

Let $\mathcal{H}(x,t) : H \times [0,1] \to H$ be continuous.

($\mathcal{H}$ is called a homotopy). Suppose that for some bounded open set Ω in H, $0 \notin \mathcal{H}(\partial\Omega, t)$ for all $t \in [0,1]$. Then,

$$\deg\Big(\mathcal{H}(\cdot,1),\Omega,0\Big) = \deg\Big(\mathcal{H}(\cdot,0),\Omega,0\Big).$$

This is the *homotopy invariance property of degree.*

Our main result

Let $f : C \to H$ *be weakly homogeneous with leading term* f^∞ *and* $f(0) = 0$. *Let* K *be a closed convex subset of* C *with recession cone* K^∞.

Let $F^\infty(x) := x - \Pi_{K^\infty}\Big(x - f^\infty(x)\Big)$.

Suppose

- $F^\infty(x) = 0 \Leftrightarrow x = 0$ *and*
- $\mathrm{ind}\,(F^\infty, 0) \neq 0$.

Then, for all $q \in H$, $VI(f, K, q)$ *and* $CP(f, K^\infty, q)$ *have nonempty compact solution sets.*

Sketch of the proof

Fix a $q \in H$ and define, for all $x \in H$ and $t \in [0,1]$,

$$\mathcal{H}(x,t) := x - \Pi_{tK+K^\infty}\Big(x - \{(1-t)f^\infty(x) + t[f(x)+q]\}\Big).$$

Then, $\mathcal{H}(x,0) = x - \Pi_{K^\infty}\Big(x - f^\infty(x)\Big) = F^\infty(x)$ and

$$\mathcal{H}(x,1) := x - \Pi_K\Big(x - [f(x)+q]\Big) = F(x).$$

We verify that the maps $(x,t) \mapsto \Pi_{tK+K^\infty}(x)$ and

$\mathcal{H}(x,t)$ are jointly continuous, and

the set $Z := \{x \in H : \mathcal{H}(x,t) = 0 \text{ for some } t\}$

is bounded. Let Ω be a bounded open set containing Z.

Using homotopy invariance property of degree,

$$\deg\Big(F,\Omega,0\Big)=\deg\Big(F^\infty,\Omega,0\Big)=\mathrm{ind}\,(F^\infty,0)\neq 0.$$

Hence, the equation $F(x)=0$ has a solution in Ω.

(In fact, all solutions lie in Ω.)

So, $VI(f,K,q)$ has a nonempty compact solution set.

We now replace K by K^∞ to see that

$VI(f,K^\infty,q)$ has a nonempty compact solution set.

Copositive maps

We specialize our main result to copositive maps.

A map $\phi : E \to H$ is *copositive on* E if

$\langle \phi(x), x \rangle \geq 0 \quad$ for all $x \in E$.

If $\langle \phi(x), x \rangle > 0 \quad$ for all $0 \neq x \in E$,

we say that ϕ is *strictly copositive* on E.

Example On $H = \mathcal{S}^n$ and $E = \mathcal{S}^n_+$ with $A \in \mathcal{S}^n$,

$\phi(X) := XAX$ is copositive if A is positive semidefinite and

strictly copositive if A is positive definite.

A copositivity result

Let $f : C \to H$ *be*

weakly homogeneous, $K \subseteq C \subseteq H$*.*

Suppose one of the following holds:

- $F^{\infty}(x) = 0 \Leftrightarrow x = 0$ *and* f^{∞} *is copositive on* K^{∞}*.*
- f^{∞} *is strictly copositive on* K^{∞}*.*

Then, for all $q \in H$*,* $VI(f, K, q)$ *and* $CP(f, K^{\infty}, q)$ *have*

nonempty compact solution sets.

Back to our simple example

In $H = R^2$, let $K := \{(x,y) : x > 0, y > 0,\ xy \geq 1\}$,

$C := R^2_+$, and $f(x,y) := (x,y) + (\sin x, \sqrt{y})$.

Then, $K^\infty = R^2_+$ and f is weakly homogeneous on C

with $f^\infty(x,y) = (x,y) = I(x,y)$.

Note that f^∞ is strictly copositive on R^n_+.

Hence,

$VI(f,K,q)$ has a nonempty compact solution set

for every q.

Polynomial complementarity problems

Let $H = R^n$ and $K = C = R^n_+$.

Let $f : R^n \rightarrow R^n$ be a polynomial map:

$f(x) = \mathcal{A}_m x^{m-1} + \mathcal{A}_{m-1} x^{m-2} + \cdots + \mathcal{A}_2 x + \mathcal{A}_1$,

where $\mathcal{A}_k$ is a tensor of order k. We assume

that the leading tensor $\mathcal{A}_m$ is nonzero.

Note: $f^\infty(x) = \mathcal{A}_m x^{m-1}$ and

$F^\infty(x) := \min\{x, f^\infty(x)\}$.

We now impose conditions on the leading tensor $\mathcal{A}_m$

to get the solvability of PCP(f, q).

Theorem: *Suppose* f *is a polynomial map on* R^n *such that*

• *Zero is the only solution of TCP*$(\mathcal{A}_m, 0)$ *and*

• $\mathrm{ind}\Big(F^\infty, 0\Big) \neq 0$.

Then, for all $q \in R^n$, *PCP*(f, q) *has a nonempty compact solution set.*

So, a condition (such as strict copositivity, P, nonsingular M, etc.,) on the leading tensor will yield the solvability of the polynomial complementarity problem!

Affine variational inequality

Let $H = C = R^n$, K be polyhedral, and f be linear.

Then $VI(f, K, q)$ is called an *affine variational inequality problem* (Gowda-Pang, 1994).

Our main result can be specialized to get a result on AVIs, which has been observed before via piecewise affine maps (Gowda, 1996).

A surjectivity result

Let $K = C = H$ and f be weakly homogeneous on H.

Suppose $f^{\infty}(x) = 0 \Leftrightarrow x = 0$ and $\mathrm{ind}(f^{\infty}, 0) \neq 0$.

Then, f is surjective:

for every q, $f(x) = q$ has a solution.

This is especially true for polynomial maps on R^n.

Example: $f(x) = Ax + \sin x$ on R^n. If A is an invertible matrix, then f is surjective.

Karamardian's Theorem

A well-known result of Karamardian asserts that if

C is a proper cone in H, $h : C \to H$ is positively homogeneous with complementarity problems CP$(h, C, 0)$ and CP(h, C, d) having zero as the only solution for some $d \in int(C^*)$, then for all $q \in H$, CP(h, C, q) has a solution.

Our improvement: For any g that is 'subordinate' to h, CP$(h + g, C, q)$ has a nonempty compact solution set.

Nonlinear equations over cones

Given a closed convex cone C in H, $q \in C$, and a map $f : C \to H$, one looks for a solution of

$f(x) = q$ in C.

For example, in $H = \mathcal{S}^n$ with $C = \mathcal{S}^n_+$ and $Q \in \mathcal{S}^n_+$:

- Lyapunov equation $AX + XA^T = Q$ (where $A \in R^{n \times n}$).
- Stein equation $X - AXA^T = Q$ (where $A \in R^{n \times n}$).
- Riccati equation $XBX + AX + XA^T = Q$ (where $A \in R^{n \times n}, B \in \mathcal{S}^n$).
- Word equation $XAXBXAX = Q$ (where $A, B \in \mathcal{S}^n$).

Is there a unified way of proving solvability?

A simple result: *Suppose* C *is a closed convex cone in* H*,* $f : C \to H$ *and*

$$\Big[x \in C,\ y \in C^*,\ \textit{and } \langle x, y\rangle = 0\Big] \Rightarrow \langle f(x), y\rangle \leq 0.$$

(The above property is called the Z*-property of* f *on* C*.)*

Then,

- $\Big[q \in C \textit{ and } x^* \textit{ solves } CP(f, C, -q)\Big] \Rightarrow f(x^*) = q.$
- $\Big[x^* \in C \textit{ and } f(x^*) \in \textit{int}(C)\Big] \Rightarrow x^* \in \textit{int}(C).$

We now combine this with our main result.

Let C be a closed convex cone in H, $f : C \to H$ be weakly homogeneous with leading term f^∞ and $f(0) = 0$.

With $F^\infty(x) := x - \Pi_C\Big(x - f^\infty(x)\Big)$, suppose

- *f has the Z-property on C,*
- *Zero is the only solution of $CP(f^\infty, C, 0)$, and*
- $\mathrm{ind}\,(F^\infty, 0) \neq 0$.

Then, for all $q \in C$, the equation $f(x) = q$ has a solution in C. If $q \in \mathrm{int(C)}$, then the solution belongs to $\mathrm{int(C)}$.

Various solvability results in the (dynamical systems) literature (Lyapunov, Stein, Lim et al, etc.,) all follow from this result. A new one (Hillar and Johnson, 2004): Consider a *symmetric word equation* over $\mathcal{S}^n$:

$$X^{r_m}A_m \cdots X^{r_2}A_2X^{r_1}A_1X^{r_1}A_2X^{r_2} \cdots A_mX^{r_m} = Q.$$

Here, $A_1, A_2, \ldots, A_m$ are positive definite matrices, $r_1, r_2, \ldots, r_m$ are positive exponents, and Q is a positive (semi)definite matrix.

Then, there is a positive (semi)definite solution X.

Concluding Remarks

In this talk, we discussed some solvability issues for weakly homogeneous variational inequalities. We showed that under appropriate conditions, solvability can be reduced to that of recession map/cone complementarity problems. Many open issues/problems remain, for example,

- Uniqueness (even for PCPs).
- Polynomial optimization.
- Weakly homogeneous VIs over symmetric cones.