Z matrices, linear transformations, and tensors

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International Conference on
Tensors, Matrices, and their Applications
Tianjin, China

May 21-24, 2016

This is an expository talk on Z matrices, transformations on proper cones, and tensors. The objective is to show that these have very similar properties.

Outline

- The Z-property
- M and strong (nonsingular) M-properties
- The P-property
- Complementarity problems
- Zero-sum games
- Dynamical systems

Some notation

- R^n : The Euclidean n-space of column vectors.
- R^n_+: Nonnegative orthant, $x \in R^n_+ \Leftrightarrow x \geq 0$.
- R^n_{++} : The interior of R^n_+, $x \in^n_{++} \Leftrightarrow x > 0$.
- $\langle x, y \rangle$: Usual inner product between x and y.
- $R^{n \times n}$: The space of all $n \times n$ real matrices.
- $\sigma(A)$: The set of all eigenvalues of $A \in R^{n \times n}$.

The Z-property

$A = [a_{ij}]$ is an $n \times n$ real matrix

- A is a Z-*matrix* if $a_{ij} \leq 0$ for all $i \neq j$.

  (In economics literature, $-A$ is a Metzler matrix.)

- We can write $A = rI - B$, where $r \in R$ and $B \geq 0$.

  Let $\rho(B)$ denote the spectral radius of B.

- A is an M-*matrix* if $r \geq \rho(B)$,
- *nonsingular (strong)* M-*matrix* if $r > \rho(B)$.

The P-property

- A is a P*-matrix if all its principal minors are positive.*

Theorem: *The following are equivalent:*

- A is a P-matrix.
- $x * Ax \leq 0 \Rightarrow x = 0.$
- $\max_i\; x_i(Ax)_i > 0$ for all $x \neq 0$.

Here, $x * y$ denotes the componentwise product of vectors x and y.

Z together with P

Theorem: *The following are equivalent for a Z-matrix A:*

- A *is a P-matrix.*
- A *is a nonsingular M-matrix.*
- *There exists a vector $d > 0$ such that $Ad > 0$.*
- A *is invertible and A^{-1} is a nonnegative matrix.*

The book on Nonnegative matrices by Berman and Plemmons lists 52 equivalent conditions.

The linear complementarity problem

Given $A \in R^{n\times n}$ and $q \in R^n$, LCP(A,q) :

Find $x \in R^n$ such that

$$x \geq 0,\ y := Ax + q \geq 0,\ \text{ and } \langle x, y\rangle = 0.$$

Theorem:

- *A is a P-matrix iff LCP(A,q) has a unique solution for all q.*
- *Suppose A is a Z-matrix. If $\{x \geq 0, Ax + q \geq 0\}$ is nonempty, then its least element solves LCP(A,q).*

Zero-sum matrix game

Two players I and II start with a matrix A and

Strategy set $\Delta := \{x \in R^n_+ : \sum_1^n x_i = 1\}$.

I chooses $x \in \Delta$ and II chooses $y \in \Delta$.

Then, payoff for I is $\langle Ax, y\rangle$ and payoff for II is $-\langle Ax, y\rangle$.

Theorem: *There exist 'optimal strategies'* $p, q \in \Delta$ *such that*

$$\langle Ax, q\rangle \leq \langle Ap, q\rangle \leq \langle Ap, y\rangle \quad \textit{for all } x\, y \in \Delta.$$

The value of the game $v(A) := \langle Ap, q\rangle$.

$$v(A) := \max_{x\in\Delta} \min_{y\in\Delta} \langle Ax, y\rangle = \min_{y\in\Delta} \max_{x\in\Delta} \langle Ax, y\rangle.$$

Completely mixed games

The game is *completely mixed* if $p > 0$ and $q > 0$ for all optimal strategy pairs.

Theorem: (Kaplansky, 1945) *If the game is completely mixed, then the game has unique optimal strategies.*

Theorem: (Raghavan, 1978) *Suppose A is a Z-matrix. Then, $v(A) > 0$ iff A is a P-matrix. In this case, the game is completely mixed.*

Dynamical systems

Given A, consider the *continuous linear dynamical system*

$$\frac{dx}{dt} + Ax = 0,\ x(0) \in R^n.$$

Its trajectory in R^n: $x(t) = e^{-tA}x(0)$ for all $t \in R$.

Theorem: A *is a* Z*-matrix iff* $e^{-tA}(R^n_+) \subseteq R^n_+$ for all $t \geq 0$.

Theorem: *Suppose A is a Z-matrix. Then, the following are equivalent:*

- *There exists $d > 0$ such that $Ad > 0$.*
- *A is a P-matrix.*
- *A is positive stable: All eigenvalues of A have positive real parts.*
- *For any $x(0) \in R^n$, $x(t) \to 0$ as $t \to \infty$.*

Proper cones

- H: Finite dimensional real inner product space.
- $\langle x, y\rangle$: inner product between $x, y \in H$.
- K : *Proper cone* in H.

  (closed convex pointed cone with nonempty interior)
- The *dual cone* $K^* := \{x \in H : \langle x, y\rangle \geq 0 \; \forall \; y \in K\}$.

Z and Lyapunov-like transformations

K is a proper cone in H, $L : H \to H$ is linear.

- L is a Z-*transformation* on K if

  $$x \in K,\ \ y \in K^*,\ \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.$$

  We write $L \in Z(K)$.

(Schneider-Vidyasagar: L is cross-positive if $-L \in Z(K)$.)

- L is a *Lyapunov-like transformation* on K if

  $$x \in K,\ \ y \in K^*,\ \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$$

  We write $L \in LL(K)$.

Examples

- $H = \mathcal{S}^n$: Space of all $n \times n$ real symmetric matrices.
- $\langle X, Y \rangle := trace(XY)$.
- $K = \mathcal{S}^n_+$: cone of positive semidefinite matrices in $\mathcal{S}^n$.
- For any $A \in R^{n \times n}$,

  $S_A(X) := X - AXA^T$ and $L_A(X) := AX + XA^T$.

  (S_A-Stein transformation, L_A-Lyapunov transformation)
- $S_A \in Z(\mathcal{S}^n_+)$ and $L_A \in LL(\mathcal{S}^n_+)$.

Properties equivalent to the Z-property on a proper cone K:

(Schneider-Vidyasagar, 1970)

- $e^{-tL}(K) \subseteq K$ for all $t \geq 0$.
- $L = \lim_{n\to\infty}(t_n I - S_n)$ where $S_n(K) \subseteq K$ and $t_n \in R$.

Properties equivalent to the LL-property on a proper cone K:

- $e^{tL}(K) \subseteq K$ for all $t \in R$.
- $L \in Lie(Aut(K))$ (Lie algebra characterization).

Lyapunov rank of K:= dimension of $Lie(Aut(K))$.

Useful in conic optimization.

Lyapunov rank

If the Lyapunov rank of K is more than the dimension n of the ambient space, then the complementarity system

$$x \in K, \; y \in K^*, \; \langle x, y \rangle = 0$$

can be expressed in terms of n linearly independent bilinear relations.

For example, in R^n, when $x \geq 0$, $y \geq 0$,

$$\langle x, y \rangle = 0 \Leftrightarrow x_i y_i = 0 \; \forall \; i.$$

This idea is useful in linear programs over cones.

Theorem: (Stern 1981, Gowda-Tao 2006)

The following are equivalent for a Z-transformation:

- *There exists $d \in int(K)$ such that $L(d) \in int(K)$.*
- *L^{-1} exists and $L^{-1}(K) \subseteq K$.*
- *L is positive stable.*
- *All real eigenvalues of L are positive.*
- *For any $q \in H$, there exists x such that*
  $x \in K, L(x) + q \in K^*, \langle x, L(x) + q \rangle = 0.$

Recall: $K^* := \{y \in H : \langle y, x \rangle \geq 0 \; \forall \, x \in K\}.$

Euclidean Jordan algebras

For a matrix $A \in R^{n\times n}$, the following are equivalent:

- A is a P-matrix: $x * Ax \leq 0 \Rightarrow x = 0.$
- All principal minors of A are positive.

How to generalize the P-property?

$(V, \langle\cdot,\cdot\rangle, \circ)$ is a **Euclidean Jordan algebra** if
V is a finite dimensional real inner product space
and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$

Examples

Any EJA is a product of the following *simple* algebras:

- $\mathcal{S}^n = \text{Herm}(\mathcal{R}^{n\times n})$: All $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n\times n})$: $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n\times n})$: $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3\times 3})$: 3×3 octonion Hermitian matrices.
- $\mathcal{L}^n$ $(n \geq 3)$: The Jordan spin algebra.

$K = \{x^2 : x \in V\}$ is the *symmetric cone* of V.

It is a self-dual, homogeneous cone.

Characterization of LL transformations on a EJA:

Theorem: (Tao-Gowda, 2013) *The following are equivalent:*

- L *is Lyapunov-like on the symmetric cone of* V*,*
- $L = L_a + D$*, where* $a \in V$ *and* $D : V \to V$ *is a derivation.*

Here,

$L_a(x) := a \circ x$ and $D(x \circ y) = D(x) \circ y + x \circ D(y)$.

Corollary: (Damm, 2004) On $\mathcal{S}^n$, L is Lyapunov-like iff it is of the form L_A for some $A \in R^{n \times n}$.

The P-property

On an EJA V, a and b *operator commute* if

$$L_a L_b = L_b L_a.$$

A linear $L : V \to V$ has the ***P****-property if*

$[\, x$ *and* $L(x)$ *operator commute,* $x \circ L(x) \leq 0\,] \Rightarrow x = 0$.

Theorem: (Gowda, Tao, and Ravindran, 2012)

Suppose L is Lyapunov-like on the symmetric cone of V.

- *L has P-property if and only if*
- *There exists $d \in int(K)$ such that $L(d) \in int(K)$.*

Conjecture: This result holds for Z-transformations.

Lyapunov's theorem revisited

Recall: $L_A(X) := AX + XA^T$ on $\mathcal{S}^n$.

Corollary: *For any $A \in R^{n\times n}$, the following are equivalent:*

- L_A *has P-property on $\mathcal{S}^n_+$.*
- *There exists $X \succ 0$ such that $AX + XA^T \succ 0$.*
- A *is positive stable.*

Here, $X \succ 0$ means that X is symmetric and positive definite.

Linear games on self-dual cones

Let K be *self-dual* that is, $K = K^*$ in H. For $e \in int(K)$,

define $\quad \Delta := \{x \in K : \langle x, e\rangle = 1\}$.

(We think of Δ as the strategy set.)

Given a linear transformation $L : H \to H$*,*

by von Neumann's min-max theorem, there exist

'optimal strategies' p *and* q *such that*

$$\langle L(x), q\rangle \leq \langle L(p), q\rangle \leq \langle L(p), y\rangle$$

for all strategies $x, y \in \Delta$*.*

$v(L) := \langle L(p), q\rangle$ is called the **value** of L.

A generalization of Raghavan's result:

Theorem: (Gowda-Ravindran, 2015) *Let K be self-dual and $L \in Z(K)$. Then the following are equivalent:*

- $v(L) > 0$.
- *There exists $d \in int(K)$ such that $L(d) \in int(K)$.*
- *L is positive stable.*

In this case, L is completely mixed.

Moreover, if L is Lyapunov-like and $v(L) \neq 0$, then L is completely mixed.

Z-tensor

- *A tensor is a multidimensional analog of a matrix.*
- Let $\mathcal{A} := [a_{i_1 i_2 \cdots i_m}]$ be an mth order, n-dimensional tensor.
- The entries $a_{iii\cdots i}$, $1 \leq i \leq n$, are the diagonal entries of $\mathcal{A}$ and the rest are 'off-diagonal' entries.
- A *nonnegative tensor* has all entries nonnegative.
- *A tensor is a Z-tensor if all its off-diagonal entries are non-positive.* Such a tensor can be written as $\mathcal{A} = r\mathcal{I} - \mathcal{B}$, where $r \in R$, $\mathcal{I}$ is the identity tensor, and $\mathcal{B}$ is a nonnegative tensor.

Given a tensor $\mathcal{A} := [a_{i_1 i_2 \cdots i_m}]$, we define the function $F : R^n \to R^n$ whose ith component is given by

$$F_i(x) = \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}.$$

Commonly used notation: $\mathcal{A}x^{m-1} := F(x)$.

Each component of $F(x)$ is a homogeneous polynomial of degree $m-1$.

Given a tensor $\mathcal{A}$,

- $\lambda \in C$ is an *eigenvalue* of $\mathcal{A}$ if $\exists$ nonzero $x \in C^n$

such that $\mathcal{A}x^{m-1} = \lambda\, x^{m-1}$.

$\sigma(\mathcal{A}) :=$ Set of all eigenvalues of $\mathcal{A}$.

- *Spectral radius* $\rho(A) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$.

Theorem:(Yang-Yang, 2010)

For a nonnegative tensor, its spectral radius is an eigenvalue with a nonnegative eigenvector.

Tensor complementarity problems

Given a tensor $\mathcal{A}$ and $q \in R^n$, the

- *Tensor complementarity problem* TCP$(\mathcal{A}, q)$ is to

  find $x \in R^n$ such that

  $x \geq 0,\ y = F(x) + q \geq 0,$ and $\langle x, y \rangle = 0.$

  Note: $F(x) = \mathcal{A}x^{m-1}$ is a polynomial map.

  Thus, TCP is a *nonlinear complementarity problem*.

  Equation formulation: $\min\{x, F(x) + q\} = 0.$

Theorem: (Luo, Qi, Xiu, 2015)

Suppose $\mathcal{A}$ is a Z-tensor. If TCP$(\mathcal{A}, q)$ is feasible, then it is solvable.

A Mega theorem

Theorem: $\mathcal{A} = r\mathcal{I} - \mathcal{B}$ with $\mathcal{B}$ nonnegative, $F(x) = \mathcal{A}x^{m-1}$.

Then the following are equivalent:

- There exists $d > 0$ such that $F(d) > 0$.
- $\mathcal{A}$ is positive stable: $Re(\lambda) > 0 \,\forall\, \lambda \in \sigma(\mathcal{A})$.
- $\mathcal{A}$ is a strong M-tensor: $r > \rho(\mathcal{B})$.
- $\mathcal{A}$ is a P-tensor: $\forall\, x \neq 0$, $\max_i \; x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0$.
- TCP$(\mathcal{A}, q)$ is solvable for all q.
- TCP$(\mathcal{A}, q)$ has trivial solution for all $q \geq 0$.
- $R^n_+ \subseteq F(R^n_+)$.
- For all $x \geq 0$, $\max_i \; x_i(\mathcal{A}x^{m-1})_i > 0$.

Mega theorem, continued

If $\mathcal{A}$ is of even order, we have further equivalence:

- F is surjective.
- For all $x \neq 0$, $\max_i \, x_i(\mathcal{A}x^{m-1})_i > 0$.

Open problem: Characterize Z-tensors $\mathcal{A}$ for which TCP$(\mathcal{A}, q)$ has a unique solution for all q.

A viability result

Suppose $\mathcal{A}$ is a Z-tensor.

Then, $-F(x)$ is 'cooperative' (that is, its derivative is a Metzler matrix at all points).

Theorem: *Let $x(t)$ solve the dynamical system*

$$\frac{dx}{dt} + F(x) = 0, \ \ x(0) = x_0.$$

If $x(0) \in R^n_+$, then $x(t) \in R^n_+$ for all t.

This comes from an application of Nagumo's theorem on viability (also called Bony-Brezis theorem).

Asymptotic convergence

Theorem: *Suppose* $\mathcal{A}$ *is a* Z*-tensor.*

Then, $\frac{dx}{dt} + F(x) = 0$ *is exponentially stable iff*

$\mathcal{A}$ *is a strong* M*-tensor.*

Follows from a result in H.R. Feyzmahdavian et al,

“Exponential stability of positive homogeneous systems”,

IEEE Trans Auto Control, 2013.

Tensor zero-sum games

No results yet......