On the game-theoretic value of a linear transformation on a symmetric cone

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Outline

- Zero-sum matrix games
- Dynamical systems
- Euclidean Jordan algebras
- **z** and Lyapunov-like transformations
- Value of a linear transformation on a EJA
- Completely mixed games
- Value of a **z**-transformation
- Value of a Lyapunov-like transformation
- Value inequalities
- Concluding remarks

Zero-sum game and value of a matrix

Consider two players I and II with payoff matrix $A \in R^{n\times n}$.

Player I chooses columns of A with probability/strategy x:

$x = (x_1, x_2, \ldots, x_n)^T \in R^n$, $x \geq 0,\ \sum_1^n x_i = 1;$

Player II chooses rows of A with probability/strategy y.

Then payoff for I is $\langle Ax, y\rangle$ and payoff for II is $-\langle Ax, y\rangle$.

Theorem of von Neumann: There exist optimal/equilibrium strategies $\bar{x}$ and $\bar{y}$ such that

$$\langle Ax, \bar{y}\rangle \leq \langle A\bar{x}, \bar{y}\rangle \leq \langle A\bar{x}, y\rangle$$

for all strategies x and y.

$v(A) := \langle A\bar{x}, \bar{y} \rangle$ is called the **value** of the game.

Min-max Theorem of von Neumann:

$$\max_{x \in \Delta} \min_{y \in \Delta} \langle Ax, y \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle Ax, y \rangle,$$

where $\Delta := \{x = (x_1, x_2, \ldots, x_n)^T \in R^n : x \geq 0, \sum_1^n x_i = 1\}$.

The common value is $v(A)$.

A zero-sum matrix game is a special case of

Bimatrix game.

A bimatrix game is a special case of

(standard) linear complementarity problem.

Uniqueness of optimal strategies

A game is **completely mixed** if $\bar{x} > 0$ and $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$.

Theorem of Kaplansky (1945):

(i) If $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$, then the game is completely mixed.

(ii) If the game is completely mixed, then the optimal pair is unique.

z-matrices

A square real matrix $A = [a_{ij}]$ is a **z**-matrix

if $a_{ij} \leq 0$ for all $i \neq j$.

(In Economics, $-A$ is called a Metzler matrix.)

Theorem of Raghavan (1978):

Let A be a **z**-matrix.

(i) If $v(A) > 0$, then the game is completely mixed.

(ii) $v(A) > 0$ iff there exists $d > 0$ in R^n such that $Ad > 0$.

Note: For a **z**-matrix, the condition $d > 0, Ad > 0$ can be described in more than 52 equivalent ways.

Dynamical systems

For an $n \times n$ real matrix A, the continuous dynamical system $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable on R^n (i.e., any trajectory starting from an arbitrary point in R^n converges to the origin) if and only if there exists a real symmetric matrix D such that

$$D \succ 0 \quad \text{and} \quad L_A(D) \succ 0,$$

where $D \succ 0$ means that D is positive definite, and

$$L_A(X) := AX + XA^T \quad (X \in \mathcal{S}^n).$$

Here, $\mathcal{S}^n$ denotes the space of all $n \times n$ real symmetric matrices and L_A is the so-called *Lyapunov transformation*.

Similarly, the discrete dynamical system $x(k+1) = Ax(k)$, $k = 0, 1, \ldots,$ is asymptotically stable on R^n if and only if there exists a real symmetric matrix D such that

$$D \succ 0 \quad \text{and} \quad S_A(D) \succ 0,$$

where S_A is the so-called *Stein transformation* on $\mathcal{S}^n$:

$$S_A(X) := X - AXA^T \quad (X \in \mathcal{S}^n).$$

Note similar inequalities:

- In R^n, $d > 0$, $Ad > 0$ for a **z**-matrix
- In $\mathcal{S}^n$, $D \succ 0 \quad \text{and} \quad L_A(D) \succ 0$
- In $\mathcal{S}^n$, $D \succ 0 \quad \text{and} \quad S_A(D) \succ 0$

Why is this happening? Is there a unifying result?

- R^n and $\mathcal{S}^n$ are both Euclidean Jordan algebras,
- In R^n, let $e = (1, 1, \ldots, 1)$. Then $x = (x_1, x_2, \ldots, x_n)$ is a probability vector iff all its eigenvalues $x_1, x_2, \ldots, x_n$ are nonnegative and $\langle x, e\rangle = \sum_1^n x_i = 1$.

In $\mathcal{S}^n$, let $e = I$ (Identity matrix). We could consider $X \in \mathcal{S}^n$ with all its eigenvalues nonnegative and $\langle X, I\rangle = trace(X) = \sum_1^n \lambda_i(X) = 1$.

z-transformations

- In R^n, let $K = R^n_+$. Then $A \in R^{n\times n}$ is a **z**-matrix iff

$$x \in K, y \in K^*(= K), \quad \text{and} \quad \langle x, y\rangle = 0 \Rightarrow \langle Ax, y\rangle \leq 0.$$

- In $\mathcal{S}^n$, let $K = \mathcal{S}^n_+$ (symmetric cone of $\mathcal{S}^n$).

Then, for any $A \in R^{n\times n}$,

$$X \in K, Y \in K^*(= K), \quad \text{and} \quad \langle X, Y\rangle = 0 \Rightarrow \langle L_A(X), Y\rangle = 0.$$

- In $\mathcal{S}^n$, let $K = \mathcal{S}^n_+$. Then, for any $A \in R^{n\times n}$,

$$X \in K, Y \in K^*(= K), \quad \text{and} \quad \langle X, Y\rangle = 0 \Rightarrow \langle S_A(X), Y\rangle \leq 0.$$

Euclidean Jordan algebras

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$, where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional real inner product space and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$, $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, and

(ii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$.

Examples: The Jordan spin algebra $\mathcal{L}^n$, the algebra(s) of $n \times n$ real/complex/quaternion Hermitian matrices, and the algebra of 3×3 octonion Hermitian matrices.

Any nonzero EJA is a product of these.

Let V be a EJA and K be its symmetric cone. A linear transformation $L : V \to V$ is

- a **Z**-transformation if

$$x \in K, y \in K^*(= K), \quad \text{and} \quad \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0,$$

- *Lyapunov-like* if

$$x \in K, y \in K^*(= K), \quad \text{and} \quad \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

Theorem: On a EJA V, L is Lyapunov-like iff

$$L = L_a + D,$$

where $a \in V$, $L_a(x) := a \circ x$ and D is a derivation.

How to define the value on a EJA?

Can we extend the results of Kaplansky and Raghavan?

From now on, V is an EJA with its symmetric cone K.

e is the unit element in V and

$\Delta := \{x \in K : \langle x, e \rangle = 1\}$.

We think of Δ as the strategy set. Also, instead of the unit element, we could take any $e \in int(K)$.)

Given a linear transformation $L : V \to V$,

as Δ is compact convex, by a Theorem of von Neumann,

there exist optimal strategies $\bar{x}$ and $\bar{y}$ such that

$$\langle L(x), \bar{y} \rangle \leq \langle L(\bar{x}), \bar{y} \rangle \leq \langle L(\bar{x}), y \rangle$$

for all strategies $x, y \in \Delta$.

$v(L) := \langle L(\bar{x}), \bar{y} \rangle$ is called the **value** of L.

We have

$$v(L) = \max_{x \in \Delta} \min_{y \in \Delta} \langle L(x), y \rangle = \min_{y \in \Delta} \max_{x \in \Delta} \langle L(x), y \rangle.$$

We write $x \geq 0$ when $x \in K$ and $x > 0$ when $x \in int(K)$.

Let L be linear on V.

Theorem:

(i) $(\bar{x}, \bar{y})$ is an optimal pair iff $L^T(\bar{y}) \leq v\, e \leq L(\bar{x})$.

(ii) If $(\bar{x}, \bar{y})$ is an optimal pair, then

$$0 \leq \bar{x} \perp v\, e - L^T(\bar{y}) \geq 0 \quad \text{and} \quad 0 \leq \bar{y} \perp L(\bar{x}) - v\, e \geq 0,$$

In addition, $\bar{x}$ and $L^T(\bar{y})$ operator commute and $\bar{y}$ and $L(\bar{x})$ operator commute.

Corollary:

(i) $v(-L^T) = -v(L)$.

(ii) $v(L + \lambda ee^T) = v(L) + \lambda$.

(iii) For any $A \in Aut(K)$, and $e \in int(K)$,

$$v(ALA^T, Ae) = v(L, e).$$

Note that as K is homogeneous, one could go from one interior point of K to another. Thus, for many properties, chosen interior point e is unimportant.

We say that L is *completely mixed* if

$\bar{x} > 0$ and $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$.

An EJA generalization of Kaplansky's Theorem:

(i) If $\bar{y} > 0$ for every optimal pair $(\bar{x}, \bar{y})$, then L is completely mixed.

(ii) If L is completely mixed, then the optimal pair is unique.

(iii) If L is completely mixed, then so is L^T and

$$v(L) = v(L^T).$$

An EJA generalization of Raghavan's Theorem:

The following are equivalent when L is a **Z**-transformation:

(i) $v(L) > 0$.

(ii) L is positive stable (real parts of eigenvalues of L are positive).

(ii) There exists $d > 0$ such that $L(d) > 0$.

In addition, L is completely mixed when $v(L) > 0$.

What happens when $v(L) < 0$?

Easy examples show that a **z**-transformation L may not be completely mixed when $v(L) < 0$. However, we have

Theorem:

When L is Lyapunov-like (or Stein-like*) and $v(L) \neq 0$, L is completely mixed.

L is said to be *Stein-like* if $L = I - \Lambda$, where $\Lambda \in \overline{Aut(K)}$.

Example: The Stein transformation S_A on $\mathcal{S}^n$ given by $S_A(X) = X - AXA^T$.

Value inequalities

Let $V = V_1 \times V_2$ with $K = K_1 \times K_2$.

Let L be linear on V. We write

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A : V_1 \to V_1$ is linear, etc. If A is invertible, we define the Schur complement

$$L/A := D - CA^{-1}B.$$

Theorem: Suppose L is a **Z**-transformation.

Then A is **Z** on V_1 and D is **Z** on V_2.

(i) If $v(L) > 0$, then $v(A) > 0$, $v(D) > 0$ and

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(D)}.$$

Reverse implications and inequalities hold if

L is Lypaunov-like.

(ii) If $v(L) > 0$, then $v(A) > 0$, $v(L/A) > 0$ and

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(L/A)}.$$

Concluding remarks

(i) The value of a linear transformation on a EJA can be computed by a (symmetric) cone linear program in polynomial time.

(ii) Let L be completely mixed. If L is invertible, then, $v(L) = \frac{1}{\langle L^{-1}(e), e\rangle}$.
Also, the unique optimal pair is given by
$\bar{x} = v(L)\, L^{-1}(e)$ and $\bar{y} = v(L)\, (L^T)^{-1}(e)$.

(iii) Many of the results presented here carry over to self-dual cones.

Some references

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(4) Karlin, Mathematical methods and theory in games.

(5) Parthasarathy and Raghavan, Some topics in two-person games.

(6) Raghavan, Completely mixed games and M-matrices.