The Lyapunov rank of a proper cone

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(1) On the bilinearity rank of a proper cone and Lyapunov-like transformations, to appear in Math Prog.,

(2) On the irreducibility, Lyapunov rank, and automorphisms of special Bishop-Phelps cones, Journal of Math Analysis, 2014.

What is Lyapunov rank?

Let K be a proper cone in a finite dimensional real Hilbert space. Then, the *Lyapunov rank* of K is the dimension of the Lie algebra of the automorphism group of K:

$$\beta(K) = \dim\left(Lie(Aut(K))\right),$$

where

$Aut(K) := \{A \in \mathcal{L}(H, H) : A(K) = K\}.$

Why study this?

Motivation

For $f : R^n \to R^n$, consider the *nonlinear complementarity problem* NCP(f): Find $x, s \in R^n$ with

$$x \geq 0, \;\; s \geq 0, \;\; s = f(x), \;\; \text{and} \;\; \langle x, s \rangle = 0.$$

Apart from cone constraints, there are 2n variables and $n + 1$ equations. To make this into a square system, we replace $\langle x, s \rangle = 0$ by n equations $x_i s_i = 0$, $i = 1, 2, \ldots, n$.

Can such a thing be done for primal-dual conic LPs?

From Rudolf et al., 2011

Consider a proper cone K in R^n. Call a square matrix Q, a *bilinearity relation* on K if

$$x \in K,\ s \in K^*,\ \langle x, s \rangle = 0 \Rightarrow \langle x, Qs \rangle = 0.$$

Then, the *bilinearity rank of* K, $\beta(K)$, is the dimension of the space of all bilinearity relations on K. So this number measures the maximal number of linearly independent bilinearity relations on K.

Desirable situation $\beta(K) \geq n$:

In this case, the complementarity system

$$x \in K,\ s \in K^*,\ \langle x, s\rangle = 0 \Rightarrow \langle x, s\rangle = 0.$$

can be rewritten as a square system.

In many cases, Identity can be written as a linear combination of n linearly independent bilinear relations.

If $\beta(K) < n$: difficulty in problem reformulation and/or in finding a solution.

Lyapunov rank

Let H be a finite dimensional real Hilbert space,

K be a proper cone in H (that is, K is a closed pointed convex cone with nonempty interior).

A linear transformation L on H is *Lyapunov-like on* K (Gowda-Sznajder 2007) if

$$x \in K,\ s \in K^*,\ \langle x, s \rangle = 0 \Rightarrow \langle L(x), s \rangle = 0,$$

where K^* is the dual of K.

Thus, Q is a bilinearity relation on K iff Q^T is Lyapunov-like.

It is known (via a result of Schneider-Vidyasagar 1970) that L is Lyapunov-like iff $e^{tL}(K) \subseteq K$ for all $t \in R$, or equivalently, $e^{tL} \in Aut(K)$ for all $t \in R$.

Hence, L is Lyapunov-like iff L is an element of the Lie algebra of the automorphism group of L. We redefine

$$\beta(K) := \dim \left(Lie(Aut(K)) \right)$$

as the *Lyapunov rank of* K.

Why Lyapunov?

On the semidefinite cone $\mathcal{S}^n_+$, every Lyapunov-like transformation is of the form L_A for some matrix $A \in R^{n\times n}$ (Damm 2004): $L_A(X) := AX + XA^T$.

Such transformations appear in Lyapunov's theory of continuous linear dynamical systems.

Symmetric cone (or Euclidean Jordan algebra) characterization of Lyapunov-like transformations:

$$L = L_a + D,$$

where $L_a(x) = a \circ x$, and D is a derivation.

Some elementary properties of rank

- A proper cone and its dual have the same rank.
- Isomorphic cones have the same rank.
- Rank is additive on a product cone.
- If $\beta(K) = 1$, then K is irreducible.
- $1 \leq \beta(K) \leq n^2 - n$ for any K in R^n, $n \geq 2$.

Proper polyhedral cones

Theorem (Gowda-Tao, 2012)

The Lyapunov rank of a polyhedral cone in R^n can be any number between 1 and n, except $n-1$.

It is n iff the cone is isomorphic to R^n_+.

Theorem

A polyhedral cone is irreducible iff its Lyapunov rank is one.

Lyapunov rank of symmetric cones

- $\beta(R^n_+) = n$.
- $\beta(\mathcal{S}^n_+) = n^2$.
- $\beta(\mathcal{C}^n_+) = 2n^2 - 1$.
- $\beta(\mathcal{Q}^n_+) = 4n^2$.
- $\beta(\mathcal{O}^{3\times 3}_+) = 79$.
- $\beta(\mathcal{L}^n_+) = \frac{n^2-n+2}{2}$.

Moment cone

Cone of nonnegative polynomials in R^{2n+1}:

$$P_{2n+1} = \{(p_0, p_1, p_2, \ldots, p_{2n+1}) : \sum_0^{2n} p_k t^k \geq 0 \ \forall t \in R\}.$$

Moment cone:

$$M_{2n+1} = P^*_{2n+1}.$$

Theorem (Rudolf et al., 2011)

- $\beta(M_{2n+1}) = 4.$

Is there a simpler proof?

Completely positive cones

C is a closed cone (not necessarily convex) in R^n.

$$K_C := \{\sum uu^T : u \in C\}$$

is called the completely positive cone of C.

(Its dual is the cone of matrices copositive on C.)

Theorem (Gowda-Sznajder-Tao 2013)

When C is a proper cone, $Aut(K_C)$ is isomorphic to $Aut(C)$.

Corollary: $\beta(K_C) = \beta(C)$.

Corollary: $\beta(K_{R^n_+}) = \beta(R^n_+) = n$.

Special Bishop-Phelps cones

For $n > 1$, let $||\cdot||$ be a norm on R^{n-1}. Then

$$K = \{(t, x) \in R \times R^{n-1} : t \geq ||x||\}$$

is called a special Bishop-Phelps cone.

For $1 \leq p \leq \infty$, l_p-cone is:

$$l^n_{p,+} := \{(t, x) \in R \times R^{n-1} : t \geq ||x|||_p\}$$

- $\beta(l^n_{2,+}) = \frac{n^2-n+2}{2}$.

When $n \geq 3$, $\beta(l^n_{1,+}) = \beta(l^n_{\infty,+}) = 1$.

Theorem (Gowda-Trott, 2014)

- For $n \geq 3$, every special BP cone is irreducible.
- For $n \geq 3$, every polyhedral special BP cone has rank one.
- For $n \geq 3$, $p \neq 2$, $\quad \beta(l_{p,+}^n) = 1.$

Automorphism groups

Given a proper cone K, how to describe $Aut(K)$?

- R^n_+: Permutation matrix times positive diagonal matrix,
- Simple symmetric cones: Algebra automorphism times a quadratic representation,
- $l^n_{1,+}$ $(n \geq 3)$: Generalized permutations.
- Special BP cones: Some automorphisms induced by isometries.
- $l^n_{p,+}$ $(n \geq 3,\ p \neq 1, 2, \infty)$: Generalized permutations??

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