Z- transformations in complementarity theory and dynamical systems

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- Restatement of theorems of Lyapunov and Stein
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Continuous linear dynamical systems

Given $A \in R^{n \times n}$, consider

$\frac{dx}{dt} + Ax(t) = 0$, $x(0) = x_0 \in R^n$.

Lyapunov's theory is concerned with the stability of the above system.

When is the above system asymptotically stable?

That is, when will the trajectory $x(t) = e^{-tA}x(0) \to 0$ from any starting point $x(0)?$

Some definitions/notations:

- A is positive stable: Eigenvalues of A lie in the open right half-plane.
- $\mathcal{S}^n =$ Set of all $n \times n$ real symmetric matrices.

For $X \in \mathcal{S}^n$, $X \succeq 0$ means X is positive semidefinite and $X \succ 0$ means X is positive definite.

Lyapunov's Theorem (1893)

For $A \in R^{n\times n}$, the following are equivalent:

- A is positive stable.
- There exists $X \succ 0$ with $AX + XA^T \succ 0$.
- $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable.

Discrete linear dynamical systems

Given $A \in R^{n \times n}$, consider

$x(k+1) = Ax(k)$, $k = 0, 1, 2, \ldots,$ and $x(0) \in R^n$.

When is the above system asymptotically stable?

That is, when will the trajectory $x(k) = A^k\, x(0) \to 0$

from any starting point $x(0)$?

A definition:

• A is Schur stable: All eigenvalues of A lie in the open unit disk.

Stein's Theorem (1952)

For $A \in R^{n \times n}$, the following are equivalent:

- A is Schur stable.
- There exists $X \succ 0$ with $X - AXA^T \succ 0$.
- $x(k+1) = Ax(k)$, $k = 0, 1, 2, \ldots,$ is asymptotically stable.

Note that the results of Lyapunov and Stein are very similar.

Is there a unifying result?

Constrained linear systems

Given a closed set K, when will the continuous linear system evolve in K?

That is, $e^{-tA}(K) \subseteq K$ for all $t \geq 0$?

Viability theorems

Nagumo (1942), Bony (1969), Brezis (1970).

For our discussion, we consider a result of **Schneider-Vidyasagar (1970)** on proper cones.

Cones

Throughout,

H denotes a real finite dimensional Hilbert space.

$K \subseteq H$ is

- *convex* if $0 \le t \le 1$ and $x, y \in K \Rightarrow (1-t)x + ty \in K$.
- *cone* if $0 \le t$ and $x \in K \Rightarrow tx \in K$.

A closed convex cone K is a **proper cone** if

- K is *pointed*: $x, -x \in K \Rightarrow x = 0$ and
- K is *solid*: interior of K is nonempty.

Examples of proper cones

Example 1: $H = R^n$, $K = R^n_+$ (Nonnegative orthant)

Example 2: $H = \mathcal{S}^n$ (set of all real $n \times n$ symmetric matrices),

$K = \mathcal{S}^n_+ = \{X \in \mathcal{S}^n : X \succeq 0\}$ (Semidefinite cone)

Example 3: $H = R^n$ $(n > 1)$, $K = \mathcal{L}^n_+$ (Ice-cream cone),

$K = \{x \in R^n : x_1 \geq \sqrt{\sum_2^n |x_i|^2}\}$

Example 4: H=Euclidean Jordan algebra,

$K = \{x \circ x : x \in H\}$ (symmetric cone)

Example 5: $C \subseteq R^n$ is a closed convex cone with nonempty interior.

$K = \{\sum uu^T : u \in C\}$ (completely positive cone of C)

Special cases:

$C = R^n \Rightarrow K = \mathcal{S}^n_+$.

$C = R^n_+ \Rightarrow K =$ (standard) completely positive cone.

Example 6: $||\cdot||$ is a norm on R^n,

$\phi : R^n \rightarrow R$ is a linear functional, $||\phi|| > 1$.

$K = \{x \in R^n : ||x|| \leq \phi(x)\}$ (Bishop-Phelps cone)

Viability in a proper cone

H is a finite dimensional real Hilbert space,

K is a proper cone, $L : H \to H$ is linear, and

$K^* := \{x \in H : \langle x, y \rangle \geq 0,\ \forall\ y \in K\}$ denotes the dual of K.

Schneider-Vidyasagar (1970): The following are equivalent:

- Every forward trajectory of $\frac{dx}{dt} + L(x) = 0$ that starts in K stays in K.
- $e^{-tL}(K) \subseteq K$ for all $t \geq 0$ in R.
- $x \in K,\ y \in K^*,\ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0$.

Special case

$H = R^n$, $K = R^n_+$, and $A = [a_{ij}] \in R^{n \times n}$.

The following are equivalent:

- Every forward trajectory of $\frac{dx}{dt} + Ax = 0$ that starts in R^n_+ stays in R^n_+.
- $a_{ij} \leq 0$ for all $i \neq j$.

Some definitions:

- $A = [a_{ij}]$ is a **Z**-matrix if $a_{ij} \leq 0$ for all $i \neq j$.
- A is a Metzler matrix if $-A$ is a **Z**-matrix.

Z and Lyapunov-like transformations

K is a proper cone in H and

$L : H \to H$ is a linear transformation.

- L is a **Z-transformation** on K if

$$x \in K, y \in K^*, \ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

- L is a **Lyapunov-like transformation** on K if

$$x \in K, y \in K^*, \ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

Restatement of S-V result

Consider the following for any $L : H \to H$ with K proper:

(a) Every forward trajectory of $\frac{dx}{dt} + L(x) = 0$ that starts in K stays in K.

(b) L is a **Z-transformation** on K.

(c) Every forward/backward trajectory of $\frac{dx}{dt} + L(x) = 0$ that starts in K stays in K.

(d) L is a **Lyapunov-like transformation** on K.

Then $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$.

Examples

Example 1: $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$ (Positive semidefinite cone),

For any $A \in R^{n \times n}$, $L_A(X) = AX + XA^T \ (X \in \mathcal{S}^n)$

is the so-called Lyapunov transformation.

Then L_A is Lyapunov-like on $\mathcal{S}^n_+$.

Example 2: $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$ (Positive semidefinite cone),

For any $A \in R^{n \times n}$, $S_A(X) = X - AXA^T \ (X \in \mathcal{S}^n)$

is the so-called Stein transformation.

Then S_A is a **Z**-transformation on $\mathcal{S}^n_+$.

A connection to Lie algebras

For a proper cone K in H, let

$Aut(K)$ denote the automorphism group of K and

$Lie(Aut(K))$ denote the corresponding Lie algebra.

Then the following are equivalent:

- L is Lyapunov-like on K
- $e^{tL} \in Aut(K)$ for all $t \in R$
- $L \in Lie(Aut(K))$

In Rudolf et al., Math. Programming, 2011,

Lyapunov-like transformations are called

Bilinearity relations.

Also, bilinearity rank of K is defined as the

dimension of the space of all such relations.

Clearly,

bilinearity rank of K is the same as $dim(Lie(Aut(K)))$.

Complementarity problems

Let $M \in R^{n \times n}$ and $q \in R^n$.

In R^n, $x \geq 0$ means $x \in R^n_+$.

(Standard) Linear complementarity Problem LCP(M, q):

Find $x \in R^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad \langle Mx + q, x \rangle = 0.$$

Primal-dual linear programs and bimatrix game problems can be posed this way.

(Reference: Cottle, Pang, and Stone 1992)

Semidefinite and cone LCPs

$L : \mathcal{S}^n \to \mathcal{S}^n$ linear, $Q \in \mathcal{S}^n$.

SDLCP(L, Q): Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0,\ L(X) + Q \succeq 0, \text{ and } \langle X, L(X) + Q \rangle = 0.$$

H is a real Hilbert space, K is a proper cone in H,

$L : H \to H$ is linear, and $q \in H$.

Cone-LCP(L, K, q): Find $x \in H$ such that

$$x \in K,\ L(x) + q \in K^*, \text{ and } \langle x, L(x) + q \rangle = 0.$$

Lyapunov's theorem and SDLCPs

Gowda-Song 2000:

For any $A \in R^{n \times n}$, let $L_A(X) = AX + XA^T$ $(X \in \mathcal{S}^n)$ denote the corresponding Lyapunov transformation.

The following are equivalent:

- A is positive stable.
- There exists $X \succ 0$ with $AX + XA^T \succ 0$.
- $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable.
- For any $Q \in \mathcal{S}^n$, SDLCP(L_A, Q) has a solution.

Stein's theorem and semidefinite LCPs

Gowda-Parthasarathy 2000

Let $S_A(X) := X - AXA^T$ — Stein transformation.

For any $A \in R^{n\times n}$, the following are equivalent:

- A is Schur stable.
- There exists $X \succ 0$ with $X - AXA^T \succ 0$.
- $x(k+1) = Ax(k)$, $k = 0, 1, 2, \ldots,$ is asymptotically stable.
- For all Q, SDLCP(S_A, Q) has a solution.

Generalization to Z-transformations

K is a proper cone in H, $L : H \to H$ linear.

Notation: In H, $d > 0$ means $d \in int(K)$.

Stern (1981), Gowda-Tao (2009):

For a **Z**-transformation, the following are equivalent:

- There exists $d > 0$ such that $L(d) > 0$.
- All eigenvalues of L lie in the open right half-plane.
- $L^{-1}(K) \subseteq K.$
- For all $q \in H$, Cone-LCP(L, K, q) has a solution.

To get Lyapunov's result:

Take $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$, and $L_A(X) = AX + XA^T$.

To get Stein's result:

Take $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$, and $S_A(X) = X - AXA^T$.

Connections to the P-property

Setting: $H = R^n$, $K = R^n_+$, $M \in R^{n \times n}$.

Notation: $x \leq 0$ means all components of x are nonpositive.

$x * y$ is the componentwise product (Hadamard product).

Fiedler-Ptak (1962): The following are equivalent:

- All principal minors of M are positive.
- M has the **P**-property: $x * (Mx) \leq 0 \Rightarrow x = 0$.

Moreover, when M is a **Z**-matrix, the above

are equivalent to:

- There exists $d > 0$ such that $Md > 0$.

Murty (1966): The following are equivalent:

- $x * (Mx) \leq 0 \Rightarrow x = 0$
- For all $q \in R^n$, LCP(M, q) has a unique solution.

Question: Is it possible to introduce the **P**-property for general proper cones?

Satisfactory answer in Euclidean Jordan algebras.

Euclidean Jordan algebras

$(V, \langle \cdot, \cdot \rangle, \circ)$ is a **Euclidean Jordan algebra** if
V is a finite dimensional real Hilbert space
and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

$K = \{x^2 : x \in V\}$ is a closed convex self-dual homogeneous cone called the **symmetric cone** of V.

Jordan, Neumann, Wigner (1934):

Any EJA is a product of the following:

- $\mathcal{S}^n = \text{Herm}(\mathcal{R}^{n\times n})$ - $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n\times n})$ - $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n\times n})$ - $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3\times 3})$ - 3×3 octonion Hermitian matrices.
- $\mathcal{L}^n$ - Jordan spin algebra.

V is a EJA and K is its symmetric cone.

We write $x \geq 0$ when $x \in K$.

Say that $a, b \in V$ **operator commute** if $L_a L_b = L_b L_a$,

where $L_a(x) = a \circ x$.

Note: In $\mathcal{S}^n$, X and Y operator commute iff $XY = YX$.

For L linear on V and $q \in V$, **symmetric cone LCP** is:

LCP(L, K, q) : Find $x \in V$ such that

$x \geq 0,\ L(x) + q \geq 0,$ and $\langle L(x) + q, x \rangle = 0.$

Some definitions for a linear L on V:

- **GUS**-property: Unique solution in all LCP(L, K, q).
- **P**-property:
  $[\, x$ and $L(x)$ operator commute, $x \circ L(x) \leq 0 \,] \Rightarrow x = 0.$
- **Q**-property: For all $q \in V$, LCP(L, K, q) has a solution.
- **S**-property: There exists $d > 0$ such that $L(d) > 0$.

Gowda, Sznajder, Tao (2004): For any L,

GUS $\Rightarrow$ **P** $\Rightarrow$ **Q** $\Rightarrow$ **S**.

Conjecture: For a **Z**-transformation, $\mathbf{P} = \mathbf{S}$

Conjecture holds for matrices on R^n, L_A and S_A on $\mathcal{S}^n$.

New Results on a EJA

Gowda, Tao, Ravindran (2012):

Theorem A

For a Lyapunov-like transformation on a EJA, **P=S.**

Theorem B

Let e denote the unit element in V.

For a **Z**-transformation L with $L(e) > 0$, **P=S.**

The conjecture is still open for a general ***Z****-transformation.*

Lyapunov-like transformations

Characterizing **Z**-transformations on a proper cone K is a difficult problem, since $-L$ is a **Z**-transformation whenever $L(K) \subseteq K$. Even the problem of finding the automorphism group of K is difficult. So, we turn to describing Lyapunov-like transformations on a proper cone.

On the nonnegative orthant, a matrix is Lyapunov-like if and only if it is a diagonal matrix.

In the next several slides, we present some characterization results.

Damm (2004):

On $\mathcal{S}^n_+$, every Lyapunov-like transformation is of the form L_A, where $L_A(X) = AX + XA^T \ (X \in \mathcal{S}^n)$.

Tao (2006):

$A \in R^{n\times n}$ is Lyapunov-like on $\mathcal{L}^n_+$ iff

$$A = \begin{bmatrix} a & b^T \\ b & D \end{bmatrix},$$

where $a \in R$, $D + D^T = 2aI$.

Symmetric cone

Gowda-Tao-Ravindran (2010):

On a Euclidean Jordan algebra, L is Lyapunov-like on K iff

$$L = L_a + D,$$

where $L_a(x) = a \circ x$ and D is a derivation,

that is, $D(x \circ y) = D(x) \circ y + x \circ D(y)$.

Recall: For a proper cone C in R^n,

$K = \{\sum uu^T : u \in C\}$ is the completely positive cone of C.

Gowda-Sznajder-Tao (2012):

L is Lyapunov-like on K iff $L = L_A$, where

A is Lyapunov-like on C.

Gowda-Tao (2011):

L is Lyapunov-like on a proper polyhedral cone

iff every extreme vector of the cone is an eigenvector

of L.

There are a number of important proper cones in the literature.

Problem:

Describe Lyapunov-like transformations on them and find

the dimension of the space of all such transformations.

References

(1) Gowda-Parthasarathy, On the complementarity ...., LAA 2000.

(2) Gowda-Song, Semidefinite LCPs, Math Programming, 2000.

(3) Gowda-Sznajder-Tao, On the automorphism group ...., LAA 2012.

(4) Gowda-Tao-Ravindran, On complementarity properties ...., LAA, 2012.

(5) Gowda-Tao, Bilinearity rank of a proper cone..., Tech Report, Dec. 2011.

(6) Rudolf et al, Bilinearity optimality conditions..., Math Programming, 2011.

(7) Schneider-Vidyasagar, Cross-positive matrices, SIAM Numer. Anal., 1970.