On the non-homogeneity and the bilinearity rank of a completely positive cone

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Definition of completely positive cone

Consider $\mathbb{R}^n$ with the usual inner product.

$\mathcal{C}$ in $\mathbb{R}^n$ is a closed cone that is not necessarily convex.

$\mathcal{S}^n$ is the set of all $n \times n$ real symmetric matrices.

The **completely positive cone of** $\mathcal{C}$ is

$$\mathcal{K} := \left\{ \sum uu^T : u \in \mathcal{C} \right\}.$$

The **copositive cone of** $\mathcal{C}$ is:

$$\mathcal{E} := \{ A \in \mathcal{S}^n : A \text{ is copositive on } \mathcal{C} \}.$$

A copositive on $\mathcal{C}$ means: $x^T A x \geq 0$ for all $x \in \mathcal{C}$.

When $\mathcal{C} = \mathbb{R}^n$, $\mathcal{K} = \mathcal{E} = \mathcal{S}^n_+$ (semidefinite cone).

When $\mathcal{C} = \mathbb{R}^n_+$, $\mathcal{K}$ is the cone of completely positive matrices and $\mathcal{E}$ is the cone of copositive matrices.

Burer (2009):

Any nonconvex quadratic minimization problem over the nonnegative orthant with linear and binary constraints can be reformulated as a linear program over the cone of completely positive matrices.

Eichfelder and Povh (2011):

Any nonconvex quadratic minimization problem over a nonempty set with linear and binary constraints can be reformulated as a linear program over a completely positive cone.

$$\min \; x^T M x + 2c^T x$$

such that

$$Ax = b,$$

$$x_j \in \{0, 1\} \text{ for all } j \in J,$$

$$x \in S$$

Reformulation in $\mathcal{S}^{n+1}$:

$$\begin{aligned} \min \langle \widehat{M}, Y \rangle \\ L(Y) = B \\ Y \in \mathcal{K}, \end{aligned}$$

where $\widehat{M} = \begin{bmatrix} 0 & c^T \\ c & M \end{bmatrix}$ and

$$\mathcal{K} = closure \left\{ \sum_k \lambda_k \begin{pmatrix} 1 \\ x_k \end{pmatrix} \begin{pmatrix} 1 \\ x_k \end{pmatrix}^T : \lambda_k \geq 0\,,\ x_k \in S \right\}.$$

Since,

$$\mathcal{K} = \left\{ \sum uu^T : u \in \overline{cone(\{1\} \times S)} \right\},$$

this is a linear program over the completely positive cone

of $\mathcal{C} = \overline{cone(\{1\} \times S)}$.

Motivated by the good properties of the semidefinite cone,

we ask if a completely positive cone can be

- **self-dual**
- **irreducible**
- **homogeneous.**

Our results

For any closed cone $\mathcal{C}$ in $\mathbb{R}^n$,

$\mathcal{K} := \left\{ \sum uu^T : u \in \mathcal{C} \right\}$ denotes the

completely positive cone of $\mathcal{C}$.

We show

- $\mathcal{K}$ is self-dual if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.
- $\mathcal{S}^n_+$ is the only self-dual completely positive cone.
- When $\mathcal{C}$ has nonempty interior, $\mathcal{K}$ is irreducible.
- When $\mathcal{C}$ is a proper convex cone, $\mathcal{K}$ is non-homogeneous.

($\mathcal{C}$ proper means: $\mathcal{C}$ is convex, pointed and $int(\mathcal{C}) \neq \emptyset$.)

Some preliminary results

- $\mathcal{K} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}$.
- $\mathcal{K}$ is pointed, that is, $\mathcal{K} \cap -\mathcal{K} = \{0\}$.
- $\mathcal{E}$ is the dual of $\mathcal{K}$.
- If int($\mathcal{C}$) is nonempty, then $\mathcal{K}$ and $\mathcal{E}$ are proper.
- Ext$(\mathcal{K}) = \{uu^T : 0 \neq u \in \mathcal{C}\}$.
- int$(\mathcal{K}) = \{\sum u_i u_i^T : \ u_i \in int(\mathcal{C}), \ span\{u_1, \ldots, u_n\} = \mathbb{R}^n\}$.

Gowda-Sznajder-Tao 2012: Suppose $\mathcal{C}$ is a proper cone. Then every automorphism of $\mathcal{K}$ is of the form

$$L(X) = QXQ^T \quad (X \in \mathcal{S}^n)$$

for some automorphism Q of $\mathcal{C}$.

Self-duality

Theorem: *$\mathcal{K}$ is self-dual if and only if* $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.

Proof. If $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$, then $\mathcal{K} = \mathcal{S}^n_+$ is self-dual.

If $\mathcal{K}$ is self-dual, then $\mathcal{K} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E} \Rightarrow \mathcal{K} = \mathcal{S}^n_+$.

Now, for any nonzero $x \in \mathbb{R}^n$, xx^T is an extreme direction of $\mathcal{S}^n_+ = \mathcal{K}$.

By a known characterization, $xx^T = uu^T$ for some $u \in \mathcal{C}$.

But then, $x = \pm u$. So, $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.

Irreducibility

A closed cone K in $\mathcal{S}^n$ is **reducible** if there exist nonzero closed cones K_1 and K_2 and subspaces H_1 and H_2

such that $K_1 \subseteq H_1$, $K_2 \subseteq H_2$, with

$$K = K_1 + K_2, \quad \mathcal{S}^n = H_1 + H_2, \quad \text{and} \quad H_1 \cap H_2 = \{0\}.$$

If K is not reducible, we say that it is *irreducible*.

Example. In $\mathbb{R}^2$, consider the standard unit vectors e_1 and e_2 and let $\mathcal{C} = \{\lambda\, e_1, \mu\, e_2 : \lambda, \mu \geq 0\}$.
The corresponding completely positive cone is
$\mathcal{K} = \{\lambda\, e_1 e_1^T + \mu\, e_2 e_2^T : \lambda, \mu \geq 0\}$. This is reducible.

Theorem: *If $\mathcal{C}$ has nonempty interior, then $\mathcal{K}$ is irreducible.*

Proof. Suppose $\mathcal{C}$ has nonempty interior and $\mathcal{K}$ is reducible; let K_i and H_i be as above.

For any $0 \neq u \in \mathcal{C}$, uu^T is an extreme vector of $\mathcal{K}$.

If $uu^T = x_1 + x_2$ with $x_i \in K_i \subseteq \mathcal{K}$, we must have $x_1 = uu^T$ (say) and $x_2 = 0$.

Then, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 := \{u \in \mathcal{C} : uu^T \in K_1\} \quad \text{and} \quad \mathcal{C}_2 := \{u \in \mathcal{C} : uu^T \in K_2\}.$$

Baire category theorem implies $\mathcal{C}_1$ (say) has interior.

Then the corresponding completely positive cone $\mathcal{K}_1$ is proper and so $\mathcal{K}_1 - \mathcal{K}_1 = \mathcal{S}^n$. As $\mathcal{K}_1 \subseteq K_1$, we must have $K_1 - K_1 = \mathcal{S}^n$.

Then, $H_1 = \mathcal{S}^n$ and $H_2 = \{0\}$, a contradiction.

Non-homogeneity

A cone K (with interior) in $\mathbb{R}^n$ or $\mathcal{S}^n$ is said to be **homogeneous** if for any $x, y \in int(\mathcal{K})$, there is an $L \in Aut(K)$ such that $L(x) = y$.

- A self-dual homogeneous cone is a **symmetric cone.**
- Every such cone arises as the cone of squares in a Euclidean Jordan algebra.

$\mathbb{R}^n_+$, $\mathcal{S}^n_+$, second order cone are examples of symmetric cones.

Theorem: *If $\mathcal{C}$ is a proper cone in $\mathbb{R}^n$ $(n > 1)$, then*

$\mathcal{K}$ cannot be homogeneous.

Sketch of the proof. Suppose $\mathcal{K}$ is homogeneous.

Pick two bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v, u_2, \ldots, u_n\}$.

in $int(\mathcal{C})$. Define $X := u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T$ and

$Y_k := vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T)$. These are in $int(\mathcal{K})$.

There exist $L_k \in Aut(\mathcal{K})$ of the form $L_k(X) = Q_kXQ_k^T$

with $Q_k \in Aut(\mathcal{C})$ such that $Q_kXQ_k = L_k(X) = Y_k$. So, for all k,

$$Q_k(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q_k^T = vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T).$$

Case 1: Q_k unbounded. A normalization argument leads to

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = 0$$

and to a contradiction.

Case 2: Q_k bounded. Taking appropriate limits,

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = vv^T.$$

As $vv^T \in Ext(\mathcal{K})$, we must have $Qu_i = \lambda_i v$ for all i.

Then Q has rank one,....

Bilinearity relations

Let K be a proper cone in $\mathbb{R}^n$.

The optimality conditions for a primal-dual cone-linear program on K are of the form

$$\begin{gathered} Ax = b \\ A^T y + s = c \\ x \in K, s \in K^*, \langle x, s \rangle = 0. \end{gathered}$$

To make the above system square, it is desirable to have n or more independent bilinear relations describing the complementarity condition.

Bilinearity rank of a cone

Let

$$C(K) := \{(x,s) : x \in K, s \in K^*, \langle x, s\rangle = 0\}.$$

Rudolf, Noyan, Papp, and Alizadeh, 2011:

An $n \times n$ matrix Q is called a *bilinearity relation* on K if

$$(x,s) \in C(K) \Rightarrow x^T Q s = 0.$$

The **bilinearity rank** of K is:

$\beta(K)$= Dimension of the space of all bilinearity relations.

This notion can be extended to a proper cone in a real Hilbert space.

Lyapunov-like transformations

Let H be a finite dimensional real Hilbert space, K be a proper cone in H.

Gowda-Sznajder, 2007:

A linear transformation L on H is **Lyapunov-like** on K

if $x \in K, y \in K^*, \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$

Thus, L is Lyapunov-like on K iff L^T is a bilinearity relation on K and

$\beta(K)$=Dimension of the space of all Lyapunov-like transformations on K.

Examples

Example 1: On $\mathbb{R}^n_+$, a matrix is Lyapunov-like iff it is a diagonal matrix.

Example 2: On $\mathcal{S}^n_+$, L is Lyapunov-like iff it is of the form $L_A(X) = AX + XA^T \ (X \in \mathcal{S}^n)$ for some $A \in R^{n\times n}$.

Example 3: On a Euclidean Jordan algebra, L is Lyapunov-like if and only if $L = L_a + D$, where $L_a(x) := a \circ x$ and D is a derivation.

Thanks to a result of **Schneider-Vidyasagar, 1970**,

The following are equivalent:

- L is Lyapunov-like on K.
- $e^{tL} \in Aut(K)$ for all $t \in \mathbb{R}$.
- L belongs to the Lie algebra of the group $Aut(K)$.

Thus, for any proper cone K,

$$\beta(K) = dim(Lie(Aut(K))).$$

simple symmetric cones

Gowda-Tao 2011:

$Herm(V)$ – Hermitian matrices in V and

K —- corresponding symmetric cone.

(i) In $Herm(R^{n\times n})$, $\beta(K) = n^2$.

(ii) In $Herm(C^{n\times n})$, $\beta(K) = 2n^2 - 1.$

(iii) In $Herm(Q^{n\times n})$, $\beta(K) = 4n^2.$

(iv) In $Herm(O^{3\times 3})$, $\beta(K) = 79.$

(v) In $\mathcal{L}^n$, $\beta(K) = \frac{n^2-n+2}{2}$.

completely positive cone

For a proper cone $\mathcal{C}$ in $\mathbb{R}^n$, let $\mathcal{K}$ be the corresponding completely positive cone in $\mathcal{S}^n$.

Gowda-Sznajder-Tao 2012: Every Lyapunov-like transformation on $\mathcal{K}$ is of the form L_A, where $L_A(X) := AX + XA^T$ and A is Lyapunov-like on $\mathcal{C}$.

Since $A \mapsto L_A$ is an isomorphism,

$$\beta(\mathcal{K}) = \beta(\mathcal{C}).$$

Example: Let $C = \mathbb{R}^n_+$.

Then $\mathcal{K}$ is the cone of completely positive matrices.

Since a matrix is Lyapunov-like on $\mathbb{R}^n_+$ if and only if it is a diagonal matrix, it follows that $\beta(\mathbb{R}^n_+) = n$.

Thus, the bilinearity rank of the cone of completely positive matrices is n.

Note that the dimension of $\mathcal{S}^n$ is $\frac{n(n+1)}{2}$.

Results for the copositive cone

Recall: $\mathcal{E}$ is the copositive cone of $\mathcal{C}$.

(i) $\mathcal{E}$ is self-dual if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.

(ii) If $\mathcal{C}$ has nonempty interior, then $\mathcal{E}$ is irreducible.

(iii) If $\mathcal{C}$ is a proper cone in $\mathbb{R}^n$ $(n > 1)$,
then $\mathcal{E}$ is not homogeneous.

References:

(1) S. Burer, On the copositive representation of binary and continuous nonconvex quadratic programs, Math. Prog. A, 2009.

(2) G. Eichfelder and J. Povh, On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets, Tech Report, 2010.

(3) Gowda-Sznajder, Some global uniqueness and solvability results for LCPs over symmetric cones, SIAM Opt. 2007.

(4) Gowda-Sznajder-Tao, The automorphism group of a completely positive cone, LAA, 2012.

(5) Gowda-Tao, Bilinearity rank of a proper cone..., Tech Report, Dec. 2011.

(6) Rudolf, Noyan, Papp, and Alizadeh, Bilinearity optimality conditions..., Math Prog., 2011.

(7) Schneider-Vidyasagar, Cross-positive matrices, SIAM Numer. Anal., 1970.