Lyapunov-like transformations

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Motivation

$\mathcal{S}^n =$ Set of all $n \times n$ real symmetric matrices,

$K = \mathcal{S}^n_+ = \{X \in \mathcal{S}^n : X \succeq 0\}$ (Positive semidefinite cone).

Lyapunov's Theorem (1893)

For any $A \in R^{n \times n}$, TFAE:

- A is positive stable.
- There exists $X \succ 0$ with $AX + XA^T \succ 0$.
- $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable.

Connection to semidefinite LCPs

Gowda and Song 2000:
Let $L_A(X) = AX + XA^T$ $(X \in \mathcal{S}^n)$ — Lyapunov transformation.

The previous conditions are equivalent to:

For any $Q \in \mathcal{S}^n$, there exists $X \succeq 0$ such that

$Y := L_A(X) + Q \succeq 0$ and $\langle X, Y \rangle = 0.$

A key property of L_A is:

$$X, Y \succeq 0, \langle X, Y \rangle = 0 \Rightarrow \langle L_A(X), Y \rangle = 0.$$

Stein's Theorem (1952)

For any $A \in R^{n\times n}$, TFAE:

- A is Schur stable.
- There exists $X \succ 0$ with $X - AXA^T \succ 0$.
- $x(k+1) = Ax(k)$, $k = 1, 2, \ldots,$ is asymptotically stable.

Gowda and Parthasarathy 2000

Let $S_A(X) := X - AXA^T$ — Stein transformation.

The above conditions are equivalent to:

For any $Q \in \mathcal{S}^n$, there exists $X \succeq 0$ such that

$Y := S_A(X) + Q \succeq 0$ and $\langle X, Y\rangle = 0$.

A key property of S_A is:

$$X, Y \succeq 0, \langle X, Y \rangle = 0 \Rightarrow \langle S_A(X), Y \rangle \leq 0.$$

Can we extend these results to proper cones?

Let K be a proper cone in a real Hilbert space H.

A linear transformation $L : H \to H$ is a **Z**-transformation if

$$x \in K, y \in K^*, \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.$$

and *Lyapunov-like* on K if

$$x \in K, y \in K^*, \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$$

Here, $K^* = \{y \in H : \langle y, x \rangle \geq 0 \ \forall \ x \in K\}.$

Outline

- Proper cones and examples
- **Z** and Lyapunov-like transformations
- Lie algebraic characterization

Characterizations of Lyapunov-like transformations on

- Polyhedral cones
- Symmetric cones
- completely positive cones

- Bilinearity relations and rank
- A Schur type result
- Complementarity problems
- P and GUS properties on symmetric cones
- A conjecture
- A result for Lyapunov-like transformations
- A result for **Z**-transformations

Cones

Throughout, H is a real finite dimensional Hilbert space.

$K \subseteq H$ is

- *convex* if $0 \leq t \leq 1$ and $x, y \in K \Rightarrow (1-t)x + ty \in K$.
- *cone* if $0 \leq t$ and $x \in K \Rightarrow tx \in K$.

A closed convex cone K is a **proper cone** if

- K is *pointed*: $x, -x \in K \Rightarrow x = 0$ and
- K is *solid*: interior of K is nonempty.

Examples of proper cones

Example 1: $H = R^n$, $K = R^n_+$ (Nonnegative orthant)

Example 2: $H = \mathcal{S}^n$ (set of all real $n \times n$ symmetric matrices),

$K = \mathcal{S}^n_+ = \{X \in \mathcal{S}^n : X \succeq 0\}$ (Semidefinite cone)

Example 3: $H = R^n \ (n > 1)$, $K = \mathcal{L}^n_+$ (Ice-cream cone),

$K = \{x \in R^n : x_1 \geq \sqrt{\sum_2^n |x_i|^2}\}$

Example 4: H=Euclidean Jordan algebra,

$K = \{x \circ x : x \in H\}$ (symmetric cone)

Example 5: $C \subseteq R^n$ is a closed convex cone with interior.

$K = \{\sum uu^T : u \in C\}$ (completely positive cone of C)

Special cases:

$C = R^n \Rightarrow K = \mathcal{S}^n_+$.

$C = R^n_+ \Rightarrow K =$ (standard) completely positive cone.

Example 6: $||\cdot||$ is a norm on R^n,

$\phi : R^n \rightarrow R$ is a linear functional, $||\phi|| > 1$.

$K = \{x \in R^n : ||x|| \leq \phi(x)\}$ (Bishop-Phelps cone)

Lyapunov-like transformations

H is a Hilbert space, K is a proper cone.

$L : H \to H$ is a linear transformation.

L is a • **Z-transformation** on K if

$$x \in K, y \in K^*, \; \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.$$

• L is a **Lyapunov-like transformation** on K if

$$x \in K, y \in K^*, \; \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$$

Notation: $\mathbf{Z}(\mathbf{K})$ and $\mathbf{LL}(\mathbf{K}) = Z(K) \cap -Z(K)$ – lineality space of $Z(K)$.

Examples

Example 1: $H = R^n$, $K = R^n_+$ (Nonnegative orthant)

$A = [a_{ij}]$ is a $\mathbf{Z}$-matrix iff $a_{ij} \leq 0$ for all $i \neq j$,

Lyapunov-like iff A is diagonal.

Example 2: $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$ (Positive semidefinite cone),

$K = \{X \in \mathcal{S}^n : X \succeq 0\}$.

For any $A \in R^{n \times n}$, $L_A(X) = AX + XA^T \ (X \in \mathcal{S}^n)$

is a Lyapunov transformation.

Then L_A is Lyapunov-like on $\mathcal{S}^n_+$.

Characterizations

H is a finite dimensional Hilbert space and K is a proper cone.

Schneider-Vidyasagar, 1970 TFAE:

- L is Lyapunov-like on K
- $e^{tL}(K) \subseteq K$ for all $t \in R$
- Every forward/backward trajectory of $\frac{dx}{dt} + L(x) = 0$ that starts in K stays in K.

Related to Nagumo's viability theorem 1942; Bony 1969; Brezis 1970)

Our observation: These are further equivalent to

- $e^{tL} \in Aut(K)$ for all $t \in R$
- $L \in Lie(Aut(K))$

Here, $Aut(K)$ is the automorphism group of K and $Lie(Aut(K))$ is the corresponding Lie algebra.

Specialized characterizations

Example 2: $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$ (Positive semidefinite cone),

$K = \{X \in \mathcal{S}^n : X \succeq 0\}$.

For any $A \in R^{n\times n}$, $L_A(X) = AX + XA^T \ (X \in \mathcal{S}^n)$.

Then L_A is Lyapunov-like on $\mathcal{S}^n_+$.

In fact, on $\mathcal{S}^n_+$, every Lyapunov-like transformation arises this way (Damm 2004).

Example 3: $H = \mathcal{L}^n$, $K = \mathcal{L}^n_+$ (Ice-cream cone),

$K = \{x \in R^n : x_1 \geq \sqrt{\sum_2^n |x_i|^2}\}$

$A \in R^{n \times n}$ is Lyapunov-like on $\mathcal{L}^n$ iff

$$A = \begin{bmatrix} a & b^T \\ b & D \end{bmatrix},$$

where $a \in R$, $D + D^T = 2aI$.

(Tao, 2006)

Euclidean Jordan algebras

$(V, \langle \cdot, \cdot \rangle, \circ)$ is a **Euclidean Jordan algebra** if
V is a finite dimensional real inner product space
and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

$K = \{x^2 : x \in V\}$ is the symmetric cone in V.

Any EJA is a product of the following:

- $\mathcal{S}^n = \text{Herm}(\mathcal{R}^{n\times n})$ - $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n\times n})$ - $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n\times n})$ - $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3\times 3})$ - 3×3 octonion Hermitian matrices.
- $\mathcal{L}^n$ - Jordan spin algebra.

Example 4: H=Euclidean Jordan algebra,

$K = \{x \circ x : x \in H\}$ (symmetric cone)

L is Lyapunov-like on K iff

$$L = L_a + D,$$

where $L_a(x) = a \circ x$ and D is a derivation,

that is, $D(x \circ y) = D(x) \circ y + x \circ D(y)$.

(Gowda-Tao-Ravindran, 2010)

Example 5: $C \subseteq R^n$ is a proper closed convex cone in R^n,

$K = \{\sum uu^T : u \in C\}$ (completely positive cone of C).

L is Lyapunov-like on K iff $L = L_A$, where

A is Lyapunov-like on C.

(Gowda-Sznajder-Tao, 2011)

Polyhedral cones

L is Lyapunov-like on a proper polyhedral cone iff every extreme vector of the cone is an eigenvector of L.

(Gowda-Tao, 2011)

Bilinearity relations

Let $H = R^n$ and K proper.

The optimality conditions for a primal-dual cone-linear program are of the form

$$Ax = b$$
$$A^T y + s = c$$
$$x \in K, s \in K^*, \langle x, s \rangle = 0.$$

To make the above system square, it is desirable to have n independent bilinear relations describing the set $C(K) := x \in K, s \in K^*, \langle x, s \rangle = 0$.

Bilinearity rank of a cone

Rudolf et al, 2011:

A matrix Q is called a *bilinearity relation* on K if

$(x, s) \in C(K) \Rightarrow x^T Q s = 0.$

The **bilinearity rank** of K is:

$\beta(K)$= Dimension of the set of all bilinearity relations.

Note that Q is a bilinearity relation iff Q^T is Lyapunov-like on K.

Thus, for a proper cone in H,

$$\beta(K) = dim\, Lie(Aut(K)).$$

Bilinearity rank of symmetric cones

(i) In $Herm(R^{n\times n})$, $\beta(K) = n^2$.

(ii) In $Herm(C^{n\times n})$, $\beta(K) = 2n^2 - 1$.

(iii) In $Herm(Q^{n\times n})$, $\beta(K) = 4n^2$.

(iv) In $Herm(O^{3\times 3})$, $\beta(K) = 79$.

(v) In $\mathcal{L}^n$, $\beta(K) = \frac{n^2-n+2}{2}$.

(Gowda-Tao 2011)

Schur type results

Conjecture: Suppose K is an irreducible proper cone in H, $L : H \to H$ is Lyapunov-like on K and $L(K) \subseteq K$. Then L is a multiple of the Identity transformation.

Conjecture is true for

- H is a simple Euclidean Jordan algebra, K is its symmetric cone.
- $H = \mathcal{S}^n$, K is the completely positive cone (of a proper cone in R^n).
- If every principle subtransformation of L is Lyapunov-like.

Gowda-Tao 2011

Complementarity problems

Let $M \in R^{n \times n}$ and $q \in R^n$.

$x \geq 0$ means $x \in R^n_+$.

(Standard) Linear complementarity Problem LCP(M, q):

Find $x \in R^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad \langle Mx + q, x \rangle = 0.$$

Primal-dual LPs and bimatrix game problems can be posed this way.

(LCP book by Cottle, Pang, and Stone 1992)

Semidefinite and cone LCPs

$L : \mathcal{S}^n \to \mathcal{S}^n$ linear, $Q \in \mathcal{S}^n$.

SDLCP(L, Q): Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0,\ L(X) + Q \succeq 0,\ \text{ and } \langle X, L(X) + Q \rangle = 0.$$

H is a real Hilbert space, K is a proper cone in H.

$L : H \to H$ is linear and $q \in H$.

cone-LCP(L, K, q):

Find $x \in H$ such that

$$x \in K,\ L(x) + q \in K^*,\ \text{ and } \langle x, L(x) + q \rangle = 0.$$

Back to Euclidean Jordan algebras

V is a EJA and K is its symmetric cone.

We write $x \geq 0$ when $x \in K$.

Say that $a, b \in V$ operator commute if $L_a L_b = L_b L_a$,

where $L_a(x) = a \circ x$.

Let L be linear on V and $q \in V$.

LCP$(L, K, q) : x \geq 0,\ L(x) + q \geq 0,\ \langle L(x) + q, x \rangle = 0.$

- **GUS**-property: Unique solution in all LCP(L, K, q).
- **P**-property:
  $[\, x$ and $L(x)$ operator commute, $x \circ L(x) \leq 0\,] \Rightarrow x = 0.$
- **Q**-property: For all $q \in V$, LCP(L, K, q) has a solution.
- **S**-property: There exists $d > 0$ such that $L(d) > 0$.

Gowda, Sznajder, Tao (2004):

GUS $\Rightarrow$ **P** $\Rightarrow$ **Q** $\Rightarrow$ **S.**

Recall

- **Z**-property: $[x, y \in K, \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle \leq 0.$
- Lyapunov-like: $[x, y \in K, \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle = 0.$

Example: L_A is Lyapunov-like and S_A has **Z**-property.

($L_A(X) = AX + XA^T$ and $S_A(X) = X - AXA^T$ on $\mathcal{S}^n$.)

Stern (1981), Gowda-Tao (2009):

For a **Z**-transformation, the following are equivalent:

- **S**-property
- Positive stable property
- $L^{-1}(K) \subseteq K$.
- **Q**-property

This extends the results of Lyapunov and Stein.

Conjecture: For a **Z**-transformation, $\mathbf{P} = \mathbf{Q}$

Conjecture holds for matrices on R^n, L_A and S_A on $\mathcal{S}^n$.

New Results on a EJA

Theorem A

For a Lyapunov-like transformation, **P=Q**.

A sketch of the Proof:

Assume L is Lyapunov-like and positive stable.

Suppose $x \neq 0$ operator commutes with $L(x)$ and $x \circ L(x) \leq 0$.

Write spectral decompositions

$x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$

with $x_i y_i \leq 0$ for all i and $x_i \neq 0$ for $i = 1, 2, \ldots k$.

Let $c := e_1 + e_2 + \cdots + e_k$ and $W := \{x : x \circ c = x\}$.

Then $L(W) \subseteq W$ and so restriction L' of L to W

is also positive stable.

Thus L' has positive trace.

But the Lyapunov-like property together with $x_i y_i \leq 0$

for all i implies that trace of L' is non-positive.

This is a contradiction.

Theorem B

Let L be a **Z**-transformation with $L(e) > 0$. Then **P=Q**.

Sketch of the proof:

Suppose $x \neq 0$ operator commutes with $L(x)$ and $x \circ L(x) \leq 0$.

Write $x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$

with $x_i y_i \leq 0$ for all i.

Define $A = [a_{ij}]$, where $a_{ij} := \langle L(e_i), e_j \rangle$.

Then A is a **Z**-matrix, $Au > 0$, where u is the vector of ones and $p * A^T p \leq 0$ in R^n, where p is the vector with components x_i.

By matrix theory results, A is a **P**-matrix.

Hence A^T is a **P**-matrix and $p * A^T p \leq 0 \Rightarrow p = 0$,

This implies that $x = 0$, leading to a contradiction.

Open problems

- **Conjecture**: For any $\mathbf{Z}$-transformation, $\mathbf{P} = \mathbf{Q}$.
- Characterize the **GUS**-property for $\mathbf{Z}$-transformations.
- When is S_A **GUS**?

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