On the P-property of Z and Lyapunov-like transformations on Euclidean Jordan algebras

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On the **P**-property of **Z** and Lyapunov-like transformations

on Euclidean Jordan algebras.

Outline

- Motivation and a conjecture
- Euclidean Jordan algebras
- **Z** and Lypaunov-like transformations
- Validity of the conjecture for Lyapunov-like transformations
- A result for **Z**-transformations

Motivation

Recall a result from complementarity problems:

The following are equivalent for $M \in \mathcal{R}^{n\times n}$:

- All principal minors of M are positive.
- $x * Mx \leq 0 \Rightarrow x = 0.$
- LCP(M, q) has a unique solution for all $q \in R^n$.

LCP(M, q): Find $x \in R^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad \langle Mx + q, x\rangle = 0.$$

When M is a **z**-matrix, i.e., when all off-diagonal entries of M are non-positive, the **above statements are further equivalent to**:

- LCP(M, q) has a solution for all q.
- There exists a $d > 0$ such that $Md > 0$.
- M is positive stable: Real part of any eigenvalue of M is positive.

$\mathcal{S}^n$ - All $n \times n$ real symmetric matrices.

$\mathcal{S}^n_+$ - All PSD matrices in $\mathcal{S}^n$.

Notation: $X \succeq 0$ if $X \in \mathcal{S}^n_+$.

$\langle X, Y \rangle := trace(XY)$.

$X \circ Y := \frac{XY+YX}{2}$ - Jordan product.

Semidefinite LCP:

$L : \mathcal{S}^n \to \mathcal{S}^n$ linear, $Q \in \mathcal{S}^n$.

SDLCP(L, Q): Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0,\ L(X) + Q \succeq 0,\ \text{ and } \langle X, L(X) + Q \rangle = 0.$$

For $A \in R^{n \times n}$,

$L_A(X) := AX + XA^T$ - Lyapunov transformation on $\mathcal{S}^n$.

$S_A(X) := X - AXA^T$ - Stein transformation on $\mathcal{S}^n$.

L denotes either L_A or S_A.

Gowda-Song (2000), Gowda-Parthasarathy (2000):

The following are equivalent:

- $[XL(X) = L(X)X,\ X \circ L(X) \preceq 0] \Rightarrow X = 0.$
- SDLCP(L, Q) has a solution for all Q.
- There exists $D \succ 0$ with $L(D) \succ 0$.
- L is positive stable.

The above result is very similar to the matrix theory result for **z**-matrices.

Why is this happening?

Do L_A and S_A have some sort of **z**-property?

Can the two results be unified and extended?

Note: Both $\mathcal{R}^n$ and $\mathcal{S}^n$ are Euclidean Jordan algebras!

$(V, \langle \cdot, \cdot \rangle, \circ)$ is a **Euclidean Jordan algebra** if
V is a finite dimensional real inner product space
and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

$K = \{x^2 : x \in V\}$ is the symmetric cone in V.
Notation: $x \geq 0$ if $x \in K$ and $x > 0$ if $x \in int(K)$.

Any EJA is a product of the following:

- $\mathcal{S}^n = \text{Herm}(\mathcal{R}^{n\times n})$ - $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n\times n})$ - $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n\times n})$ - $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3\times 3})$ - 3×3 octonion Hermitian matrices.
- $\mathcal{L}^n$ - Jordan spin algebra.

For $a \in V$, $L_a(x) := a \circ x.$

a and b *operator commute* if $L_a L_b = L_b L_a$.

Let L be linear on V and $q \in V$.

LCP$(L, K, q) : x \geq 0,\ L(x) + q \geq 0,\ \langle L(x) + q, x \rangle = 0.$

- **GUS**-property: Unique solution in all LCP(L, K, q).
- **P**-property:
  $[\, x$ and $L(x)$ operator commute, $x \circ L(x) \leq 0\,] \Rightarrow x = 0.$
- **Q**-property: For all $q \in V$, LCP(L, K, q) has a solution.
- **S**-property: There exists $d > 0$ such that $L(d) > 0$.

Gowda, Sznajder, Tao (2004):

GUS $\Rightarrow$ **P** $\Rightarrow$ **Q** $\Rightarrow$ **S**.

- **Z**-property: $[x, y \in K,\ x \perp y] \Rightarrow \langle L(x), y\rangle \leq 0.$
- Lyapunov-like: $[x, y \in K,\ x \perp y] \Rightarrow \langle L(x), y\rangle = 0.$

Example: L_A is Lyapunov-like and S_A has **Z**-property.
($L_A(X) = AX + XA^T$ and $S_A(X) = X - AXA^T$ on $\mathcal{S}^n$.)

Schneider-Vidyasagar (1970)
Z-property is equivalent to:

- $exp(-tL)(K) \subseteq K$ for all $t \geq 0$.
- $\dot{x} + L(x) = 0,\ x(0) \in K \Rightarrow x(t) \in K$ for all $t \geq 0$.

Stern (1981), Gowda-Tao (2009):

For a **Z**-transformation, the following are equivalent:

- **S**-property
- Positive stable property
- $L^{-1}(K) \subseteq K.$
- **Q**-property

Conjecture: For a **Z**-transformation, **P=Q**

Conjecture holds for matrices on $\mathcal{R}^n$, L_A and S_A on $\mathcal{S}^n$.

Will show:

Conjecture holds for all Lyapunov-like transformations and those **Z**-transformations with $L(e) > 0$, where e is the unit element in V.

New Results

A characterization of Lyapunov-like transformations:

The following are equivalent:

- L is Lyapunov-like.
- $e^{tL}(K) = K$ for all $t \in \mathcal{R}$.
- L belongs to the Lie algebra of $Aut(K)$.
- $L = L_a + D$, where $L_a(x) = a \circ x$ and D is a derivation.

Derivation:

$D(x \circ y) = D(x) \circ y + x \circ D(y)$ for all $x, y \in V$.

Theorem A

For a Lyapunov-like transformation, **P=Q**.

A sketch of the Proof:

Assume L is Lyapunov-like and positive stable.

Suppose $x \neq 0$ operator commutes with $L(x)$ and $x \circ L(x) \leq 0$.

Write spectral decompositions

$x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$

with $x_i y_i \leq 0$ for all i and $x_i \neq 0$ for $i = 1, 2, \ldots k$.

Let $c := e_1 + e_2 + \cdots + e_k$ and $W := \{x : x \circ c = x\}$.

Then $L(W) \subseteq W$ and so restriction L' of L to W

is also positive stable.

Thus L' has positive trace.

But the Lyapunov-like property together with $x_i y_i \leq 0$

for all i implies that trace of L' is non-positive.

This is a contradiction.

Theorem B

Let L be a **Z**-transformation with $L(e) > 0$. Then **P=Q**.

Sketch of the proof:

Suppose $x \neq 0$ operator commutes with $L(x)$ and $x \circ L(x) \leq 0$.

Write $x = \sum x_i e_i$ and $L(x) = \sum y_i e_i$

with $x_i y_i \leq 0$ for all i.

Define $A = [a_{ij}]$, where $a_{ij} := \langle L(e_i), e_j \rangle$.

Then A is a **Z**-matrix, $Au > 0$, where u is the vector of ones and $p * A^T p \leq 0$ in R^n, where p is the vector with components x_i.

By matrix theory results, A is a **P**-matrix.

Hence A^T is a **P**-matrix and $p * A^T p \leq 0 \Rightarrow p = 0$,

This implies that $x = 0$, leading to a contradiction.

Open problems

- **Conjecture**: For any $\mathbf{Z}$-transformation, $\mathbf{P} = \mathbf{Q}$.
- Characterize the **GUS**-property for $\mathbf{Z}$-transformations.
- When is S_A **GUS**?