Semi-FTvN systems

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The main objective of this talk is to introduce semi-FTvN systems and describe some examples and results.

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A **semi-FTvN system** is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces, and $\lambda : \mathcal{V} \to \mathcal{W}$ is a map satisfying the following *'sharpened Cauchy-Schwarz inequality'*:

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \leq ||x||\,||y|| \quad (\forall\ x, y \in \mathcal{V}).$$

The above condition is equivalent to: For all $x, y \in \mathcal{V}$,

- $||\lambda(x)|| = ||x||$ (norm preserving property) and
- $\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle$ (inner product expanding property).

It is known that above λ is Lipschitz-continuous.

FTvN $\equiv$ **Fan-Theobald-von Neumann**

Two simple examples

Example 1 $(\mathcal{V}, \mathcal{R}, \lambda)$, where $\mathcal{V}$ is a real inner product space and $\lambda(x) := ||x||$.

Example 2 $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where for $x \in \mathcal{R}^n$, $\lambda(x) := x^{\downarrow}$ is the decreasing rearrangement of entries of x.

Hardy-Littlewood-Polya: For $x, y \in \mathcal{R}^n$,

(i) $||x|| = ||x^{\downarrow}||$,

(ii) $\langle x, y \rangle \leq \langle x^{\downarrow}, y^{\downarrow} \rangle$, and

(iii) Equality holds in (ii) if and only if $x = P(x^{\downarrow})$ and $y = P(y^{\downarrow})$ for some permutation matrix P.

FTvN systems

A **FTvN system** is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces, and $\lambda : \mathcal{V} \to \mathcal{W}$ is a map satisfying the conditions

(1) $||\lambda(x)|| = ||x||$ for all $x \in \mathcal{V}$,

(2) $\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle$ for all $x, y \in \mathcal{V}$, and

(3) $\max\{\langle c, x \rangle : x \in [a]\} = \langle \lambda(c), \lambda(a) \rangle$ for all $c, a \in \mathcal{V}$.

Here, $[a] := \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\}$ is called the λ-orbit of a.

- It is known that $(3) \Rightarrow (1) + (2)$.

- *Every FTvN system is a semi-FTvN system.*

FTvN $\equiv$ **Fan-Theobald-von Neumann**
Why these names?

Let $\mathcal{S}^n$ = Set of all $n \times n$ real symmetric matrices with inner product $\langle X, Y \rangle = tr(XY)$.

- For $X \in \mathcal{S}^n$, $\lambda(X)$ = (decreasing) vector of eigenvalues of X.

  Note: $X = U\, Diag(\lambda(X))\, U^T$ for some orthogonal matrix U.

- For all $X, Y \in \mathcal{S}^n$,
  (i) $||X|| = ||\lambda(X)||$,
  (ii) (**Fan**, 1949) $\langle X, Y \rangle \leq \langle \lambda(X), \lambda(Y) \rangle$,
  (iii) (**Theobald**, 1975) Equality holds in (ii) if and only if for some orthogonal U,

$$X = U\, Diag(\lambda(X))\, U^T \quad \text{and} \quad Y = U\, Diag(\lambda(Y))\, U^T.$$

Considering the space of all $n \times n$ complex matrices with $\lambda(X)$ denoting the (decreasing) vector of singular values, **von Neumann** proved analogous results.

Examples of FTvN systems include

- $(\mathcal{V}, \mathcal{R}, \lambda)$, where $\mathcal{V}$ is a real IPS and $\lambda(x) = ||x||$,
- $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, $\lambda(x) = x^{\downarrow}$,
- **Euclidean Jordan algebras**,
- Systems induced by certain hyperbolic polynomials,
- Normal decomposition systems (Eaton triples).

FTvN systems were originally introduced to transform certain optimization problems over $\mathcal{V}$ to similar problems over $\mathcal{W}$. Semi-FTvN systems turn out to be more flexible than FTvN systems, especially when dealing with subspaces and compositions.

A large class of semi-FTvN systems comes from hyperbolic polynomials.

Hyperbolic polynomials

On a finite dimensional real vector space $\mathcal{V}$, let p be a homogeneous polynomial of degree n (a natural number) and $0 \neq e \in \mathcal{V}$. We say that p is **hyperbolic in the direction of** e if, for for each fixed $x \in \mathcal{V}$, the univariate polynomial $t \mapsto p(x - te)$ has only real roots. Given such a polynomial p and $x \in \mathcal{V}$, we look at the roots of $p(x - te) = 0$ and rearrange them in the decreasing order to get the vector $\lambda(x)$ in $\mathcal{R}^n$. We say that p is **complete** if $\lambda(x) = 0 \Rightarrow x = 0$. Assuming this, Bauschke et al., (2001) have shown that

$$\langle x, y\rangle := \frac{1}{4}\Big[||\lambda(x+y)||^2 - ||\lambda(x-y)||^2\Big]$$

defines an inner product on $\mathcal{V}$ with

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \leq ||x||\,||y|| \quad \forall\, x, y \in \mathcal{V}.$$

Thus,
Every complete hyperbolic polynomial induces a semi-FTvN system.

Examples:

(1) On $\mathcal{R}^n$, let $p(x) = x_1 x_2 \cdots x_n$. Then, $\lambda(x) = x^{\downarrow}$.

(2) On $\mathcal{S}^n$, let $p(X) := \det(X)$. Then, $\lambda(X)$ is the eigenvector of X with $\langle X, Y \rangle = trace(XY)$.

(3) On a Euclidean Jordan algebra $\mathcal{V}$ with Jordan product $x \circ y$, let $p(x) = \det(x)$. Then, $\lambda(x)$ is the vector of eigenvalues of x with $\langle x, y \rangle = trace(x \circ y)$.

Some new concepts

From now on, we fix a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$. Recall that $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces with $\lambda : \mathcal{V} \to \mathcal{W}$ satisfying the condition

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \leq ||x||\, ||y|| \quad \forall\, x, y \in \mathcal{V}.$$

For simplicity, we assume that $\mathcal{V}$ is **finite-dimensional**. Let

$\mathrm{L}(\mathcal{V})$ denote the set of all linear transformations on $\mathcal{V}$,

$\mathrm{GL}(\mathcal{V})$ denote the set of all invertible linear transformations in $\mathcal{V}$.

Majorization in a semi-FTvN system

Recall: In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, for any $a \in \mathcal{V}$, the λ-orbit of a is $[a] := \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\}$.
As λ is Lipschitz-continuous, $[a]$ is closed and bounded (recall $\mathcal{V}$ is finite-dimensional). By Minkowski's Theorem, its convex hull $\mathrm{conv}\,[a]$ is also compact.

Definition

Given $x, y \in \mathcal{V}$, we say that x is **majorized** by y if $x \in \mathrm{conv}\,[y]$.
Notation: $x \prec y$.

Example: In the setting of $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where $\lambda(x) = x^{\downarrow}$, this coincides with the classical definition of majorization (thanks to Hardy-Littlewood-Polya).

Example: In the setting of a Euclidean Jordan algebra, $x \prec y$ if and only if $\lambda(x) \prec \lambda(y)$ in $\mathcal{R}^n$. In particular, this yields a result of T. Ando: In the setting of $\mathcal{H}^n$, $\lambda(A) \prec \lambda(B)$ if and only if A is in the convex hull of matrices of the form UBU^*, where U is a unitary matrix.

Example: In the setting of a semi-FTvN system coming from a complete hyperbolic polynomial, it is known that $x \prec y \Rightarrow \lambda(x) \prec \lambda(y)$. The reverse implication holds when the polynomial is also 'isometric'. The general situation is not known.

The automorphism group of a semi-FTvN system

Definition

$\mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda) := \{A \in \mathrm{GL}(\mathcal{V}) : \lambda(Ax) = \lambda(x),\ \forall\ x \in \mathcal{V}\}.$

As $||\lambda(x)|| = ||x||$, every A in this set is norm preserving (i.e., orthogonal). In particular, this set is closed in $\mathrm{GL}(\mathcal{V})$. Thus,

$$\mathcal{G} := \mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda)$$

is a **matrix Lie group** with the corresponding **Lie algebra**

$$\mathrm{Lie}(\mathcal{G}) := \{D \in \mathrm{L}(\mathcal{V}) : exp(tD) \in \mathcal{G} \text{ for all } t \in \mathcal{R}\}.$$

Example: In the setting of a Euclidean Jordan algebra, $\mathcal{G}$ is the group of all Jordan algebra automorphisms and $\mathrm{Lie}(\mathcal{G})$ is the set of all derivations.

Commutativity in a semi-FTvN system

Definition

Let $x, y \in \mathcal{V}$. We say that x and y **commute** if $\langle Dx, y \rangle = 0$ for all $D \in \mathrm{Lie}(\mathcal{G})$.

Example: In the setting of the semi-FTvN system $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where $\lambda(x) = x^{\downarrow}$, $\mathcal{G}$ is the permutation group and its Lie algebra is $\{0\}$. Thus, in this system, any two elements commute.

Example: In the setting of $(\mathcal{S}^n, \mathcal{R}^n, \lambda)$, above commutativity is the same as the usual commutativity of two matrices.

Example: In the setting of a Euclidean Jordan algebra, above commutativity is the same as operator commutativity.

Strong commutativity in a semi-FTvN system

Definition

Let $x, y \in \mathcal{V}$. We say that x and y **strongly commute** if

$$\langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle.$$

Theorem

Strong commutativity $\Rightarrow$ commutativity.

Example: In $(\mathcal{R}^2, \mathcal{R}^2, \lambda)$ with $\lambda(x) = x^{\downarrow}$, vectors $x = (-1, 2)$ and $y = (-2, 3)$ strongly commute. However, $u = (1, 0)$ and $v = (0, 1)$ commute, but not strongly.

Example: In a Euclidean Jordan algebra, strong commutativity is the same as strong operator commutativity.

A commutation principle for $\mathcal{S}^n$

- $\mathcal{S}^n$ = All $n \times n$ real symmetric matrices, $\langle X, Y \rangle = tr(XY)$.
- For $X \in \mathcal{S}^n$, $\lambda(X)$ = (decreasing) vector of eigenvalues of X.
- $E \subseteq \mathcal{S}^n$ is a **spectral set** if $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$.
- $F : \mathcal{S}^n \to \mathcal{R}$ is a **spectral function** if $F = f \circ \lambda$ for some $f : \mathcal{R}^n \to \mathcal{R}$.

Theorem

(Iusem and Seeger - 2007)
In $\mathcal{S}^n$, let E be a spectral set, F be a spectral function, and $C \in \mathcal{S}^n$. If A is a local optimizer of

$$\underset{X \in E}{max/min} \quad \langle C, X \rangle + F(X),$$

then A and C commute, that is, $AC = CA$.

Theorem of Ramírez, Seeger, and Sossa (2013)

- $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ = Euclidean Jordan algebra of rank n.
- For $x \in \mathcal{V}$, $\lambda(x)$ = (decreasing) vector of eigenvalues of x.
- $E \subseteq \mathcal{V}$ is a **spectral set** if $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$.
- $F : \mathcal{V} \to \mathcal{R}$ is a **spectral function** if $F = f \circ \lambda$ for some $f : \mathcal{R}^n \to \mathcal{R}$.

Theorem

In $\mathcal{V}$, let E be a spectral set, F be a spectral function, and $\Phi : \mathcal{V} \to \mathcal{R}$ be Fréchet differentiable. If a is a local optimizer of

$$\underset{x \in E}{max/min} \quad \Phi(x) + F(x),$$

then a and $b := \Phi'(a)$ operator commute, that is, $L_a L_b = L_b L_a$.

Theorem of Gowda and Jeong (2017)

- $\mathcal{V}$ = Euclidean Jordan algebra of rank n.
- $\mathrm{Aut}(\mathcal{V})$ = **Automorphism group** of $\mathcal{V}$.
- $E \subseteq \mathcal{V}$ is a **weakly spectral set** if $A(E) \subseteq E$ for all $A \in \mathrm{Aut}(\mathcal{V})$.
- $F : \mathcal{V} \to \mathcal{R}$ is a **weakly spectral function** if $F(Ax) = F(x)$ for all $A \in \mathrm{Aut}(\mathcal{V})$.

Theorem

In $\mathcal{V}$, let E be a weakly spectral set, F be a weakly spectral function, and $\Phi : \mathcal{V} \to \mathcal{R}$ be Fréchet differentiable. If a is a local optimizer of

$$\underset{x \in E}{max/min} \quad \Phi(x) + F(x),$$

then a and $b := \Phi'(a)$ operator commute, that is, $L_a L_b = L_b L_a$.

A commutation principle for semi-FTvN systems

Theorem

Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system and $\mathcal{G} = \mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$. Suppose

- $E \subseteq \mathcal{V}$ *is $\mathcal{G}$-invariant, i.e., $A(E) \subseteq E$ for all $A \in \mathcal{G}$,*
- $F : \mathcal{V} \to \mathcal{R}$ *is $\mathcal{G}$-invariant, i.e., $F(Ax) = F(x)$ for all $A \in \mathcal{G}$ and $x \in \mathcal{V}$, and*
- $\Phi : \mathcal{V} \to \mathcal{R}$ *is Fréchet differentiable.*

If a is a local optimizer of

$$\max_{x \in E}/\min \quad \Phi(x) + F(x),$$

then a commutes with $b := \Phi'(a)$ in the semi-FTvN system.

Commutativity in a variational inequality problem

In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, let $E \subseteq \mathcal{V}$ and $h : \mathcal{V} \to \mathcal{R}$. Then, the variational inequality problem $VI(h, E)$ is to find $a \in E$ such that

$$\langle h(a), x - a \rangle \geq 0 \text{ for all } x \in E.$$

Corollary

Let E be $\mathcal{G}$-invariant. If a solves the variational inequality problem $\mathrm{VI}(h, E)$, then a and $h(a)$ commute.

Commutativity in a linear complementarity problem

Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system, K be a closed convex cone in $\mathcal{V}$ that is $\mathcal{G}$-invaraint. (For example, this could be the hyperbolicity cone corresponding to a complete hyperbolic polynomial or a symmetric cone in a Euclidean Jordan algebra.) Let $h : K \to \mathcal{V}$. Consider the *complementarity problem* $\mathrm{CP}(h, K)$, which is to find

$$a \in K \text{ such that } h(a) \in K^* \text{ and } \langle h(a), a \rangle = 0,$$

where $K^* := \{y \in \mathcal{V} : \langle y, x \rangle \geq 0, \forall\, x \in K\}$ is the dual of K. If a is such a solution, by the previous corollary specialized to $E = K$, we see that a commutes with $h(a)$. In particular, in the setting of a Euclidean Jordan algebra with its symmetric cone K, we (recover) the operator commutativity of a and $h(a)$.

Strong commutativity in FTvN systems

Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is an FTvN system, where $\mathcal{V}$ is finite dimensional. Then, for any $c \in \mathcal{V}$,

$$\max_{[a]} \langle c, x \rangle = \langle \lambda(c), \lambda(a) \rangle.$$

Suppose the maximum above is attained at $u \in [a]$. Then, c strongly commutes with u. Also, in this setting, when E is a $\mathcal{G}$-invariant set and $h : E \to \mathcal{V}$, if a solves $\mathrm{VI}(h, E)$, then a and $-h(a)$ strongly commute.

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