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APPLICATIONS OF DEGREE THEORY TO LINEAR COMPLEMENTARITY PROBLEMS

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In this paper, we consider two applications of degree theory to linear complementarity problems. In the first application, we study the stability of an LCP at a solution point. Specifically we prove the stability of an LCP corresponding to a $\mathbf{P}_0$-matrix at an isolated solution. Using a recent degree formula due to Stewart 1991, we strengthen a stability result of Gowda and Pang 1992. In the second application, we use the same degree formula of Stewart to describe the number of solutions of LCP(M, q) when M is a negative almost **N**-matrix. This analysis leads to a Lipschitzian characterization of the solution map Φ: $q \mapsto \text{SOL}(M, q)$ corresponding to a nondegenerate negative matrix.

1. Introduction. Given a matrix $M \in R^{n \times n}$ and a vector $q \in R^n$, the Linear Complementarity Problem LCP(M, q) is to find a vector x in R^n such that

$$(1) \qquad x \geqslant 0, \qquad Mx + q \geqslant 0, \quad \text{and} \quad x^T(Mx + q) = 0.$$

The advantage of studying such a problem is well documented in the literature. See, e.g., Murty 1987 and Cottle, Pang and Stone 1992.

The degree theory (see, e.g., Lloyd 1978 or Ortega and Rheinboldt 1970) has been effectively used to study linear complementarity problems specifically dealing with the existence of solutions and cardinality of the solution set (see Kojima and Saigal 1979, 1981, Howe and Stone 1983, Howe 1983, Garcia, Gold and Turnbull 1983) and also with the stability issues (see Ha 1987).

In this paper, we present two applications of degree theory. The first application deals with the stability of an LCP at a solution point. (For definitions, see §3.) Our main result here is that when M is a $\mathbf{P}_0$-matrix, a solution x^* is stable for LCP(M, q) if and only if x^* is isolated. As a consequence, we show that when M is a $\mathbf{P}_0$-matrix and q is any vector, the number of solutions of LCP(M, q) is either zero or one or infinity, a result that was first observed by Cottle and Guu 1991. Moreover, by using a degree formula due to Stewart 1991, we strengthen a stability result of Gowda and Pang 1992.

In our second application, we once again use the degree formula of Stewart 1991 to completely describe the number of solutions of LCP(M, q) when M is either an **N**-matrix of first category or a negative almost **N**-matrix. This analysis improves the earlier results of Kojima and Saigal 1981, Mohan and Sridhar 1989, and Mohan, Parthasarathy and Sridhar 1991.

The motivation for our second application comes from the problem of finding matrices M for which the (multivalued) solution map Φ: $q \mapsto \text{SOL}(M, q)$ is Lipschitzian. (See §6 for the definition.) It was shown in Gowda 1992 that **P**-matrices and negative **N**-matrices have this Lipschitzian property and that every **G**-matrix

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satisfying the Lipschitzian property is a **P**-matrix. Using our precise knowledge of the number of solutions of LCP(M, q) corresponding to a negative almost **N**-matrix, we show that when M is a nondegenerate negative matrix, the solution map Φ is Lipschitzian if and only if M is an **N**-matrix.

A few words about the notation: Throughout this paper, $\|q\|$ denotes the Euclidean norm of the vector q and $\|M\|$ denotes the operator norm of the matrix M. The unit ball in R^n is denoted by B. We write $|E|$ for the number of elements in the set E. We denote the set of all solutions of LCP(M, q) by SOL(M, q). Corresponding to any solution x^* of LCP(M, q), we let

$$I = \{i: x_i^* > 0\}, \qquad J = \{i: (Mx^* + q)_i > 0\}, \tag{2}$$
$$K = \{i: x_i^* = (Mx^* + q)_i = 0\}.$$

The set I will be called the support set of x^* and the submatrix M_{II} of M will be called the *supporting submatrix* of x^*. (If I is empty, we let M_{II} be the identity matrix.) For any set S in R^n, $\bar{S}$ and ∂S denote, respectively, the closure and the boundary of S. Also, $\text{dist}(0, S) := \inf\{\|s\|: s \in S\}$.

2. Degree theory. The purpose of this section is to review the basic concepts of degree theory, introduce LCP-related degree concepts and to state the recent degree formula of Stewart.

The standard references for degree theory are Lloyd 1978 and Ortega and Rheinboldt 1970.

Following Ortega and Rheinboldt (1970, Chapter 6), corresponding to a bounded open set Ω in R^n, a continuous function $F: \bar{\Omega} \to R^n$, and an n-vector $p \notin F(\partial\Omega)$ we denote the *degree of F at p relative to* Ω by $\deg(F, \Omega, p)$. (In this situation, we shall say that $\deg(F, \Omega, p)$ is defined.) Since p is always zero in our discussion, we write $\deg(F, \Omega)$ as a shorthand for $\deg(F, \Omega, 0)$. We list below a few properties that are relevant to our discussion.

PROPERTIES OF DEGREE.

(1) If $\deg(F, \Omega) \neq 0$, then the equation $F(x) = 0$ has a solution in Ω.

(2) Suppose that $\deg(F, \Omega)$ is defined. If G is a continuous function on $\bar{\Omega}$ such that

$$\sup_{\Omega} \|G(x) - F(x)\| < \text{dist}(0, F(\partial\Omega)) \tag{3}$$

then $\deg(G, \Omega)$ is defined and is equal to $\deg(F, \Omega)$.

(3) (Homotopy invariance property). Suppose that $H: [0, 1] \times \bar{\Omega} \to R^n$ is continuous and $0 \notin H(t, \partial\Omega)$ for all $t \in [0, 1]$. Then $\deg(H(0, \cdot), \Omega) = \deg(H(1, \cdot), \Omega)$.

(4) (Domain decomposition property). Suppose that $\deg(F, \Omega)$ is defined and Ω is a disjoint union of finite number of open sets Ω_i. Then $\deg(F, \Omega) = \sum_i \deg(F, \Omega_i)$.

(5) (Excision property). Suppose that $\deg(F, \Omega)$ is defined and Δ is a compact subset of Ω such that there are no solutions of $F(x) = 0$ in Δ. Then $\deg(F, \Omega) = \deg(F, \Omega \setminus \Delta)$.

(6) Let x^* be an isolated solution of the equation $F(x) = 0$. Then $\deg(F, \Omega)$ is the same for any bounded open set Ω containing x^* with the additional property that $\bar{\Omega}$ contains no other solution of $F(x) = 0$. In this situation, we call $\deg(F, \Omega)$ the index of F at x^* and denote it by index(F, x^*). If F is differentiable at x^* with a nonsingular Jacobian matrix $F'(x^*)$, then

$$\text{index}(F, x^*) = \text{sgn}\det F'(x^*). \tag{4}$$

STEWART'S FORMULA. Let $M \in R^{n \times n}$ and $q \in R^n$. In order to make use of the degree theory, we formulate LCP(M, q) as an equation $F(x) = 0$ where

$$F(x) := x \wedge (Mx + q) := \big(\min\{x_i, (Mx + q)_i\}\big) \tag{5}$$

(i.e., $F(x)$ is the componentwise minimum of x and $Mx + q$).

If x^* is an isolated solution of LCP(M, q), then index(F, x^*) is defined. To indicate the dependence of F on M and q, we write index(M, q, x^*) for index(F, x^*).

Suppose that the isolated solution x^* is also *nondegenerate* (which means that $x^* + Mx^* + q > 0$). Then (cf. Mangasarian 1980, Corollary 3.2) the supporting matrix M_{II} (with I as in (2)) is nonsingular and F is differentiable at x^* with $\det F'(x^*) = \det M_{II}$. In this situation,

$$\text{index}(M, q, x^*) = \operatorname{sgn} \det M_{II}. \tag{6}$$

(We remark that the above formula holds whether I is empty or not.)

The new formula due to Stewart 1991 makes it possible to compute the index even when the solution x^* is not nondegenerate. Let us say that an isolated solution x^* of LCP(M, q) is *semi-nondegenerate* if the corresponding supporting submatrix is nonsingular. We assume that x^* is such a solution and I, J, and K are given by (2). Let M_{II}^S denote the Schur complement of M_{II} in the matrix

$$\begin{bmatrix} M_{II} & M_{IK} \\ M_{KI} & M_{KK} \end{bmatrix},$$

i.e., let $M_{II}^S = M_{KK} - M_{KI} M_{II}^{-1} M_{IK}$. (If K is empty, we let M_{II}^S be the identity matrix. If I is empty, M_{II}^S is taken as M_{KK}.) Then Stewart's formula is

$$\text{index}(M, q, x^*) = (\operatorname{sgn} \det M_{II})\,\text{index}\big(M_{II}^S, 0, 0\big). \tag{7}$$

We note that in the above formula, index$(M_{II}^S, 0, 0)$ is defined because when x^* is an isolated solution, LCP$(M_{II}^S, 0)$ has only one solution, namely, zero (cf. Mangasarian 1980, Theorem 3.8).

DEGREE OF AN $\mathbf{R}_0$-MATRIX. A matrix M with the property that the zero vector is the only solution of LCP$(M, 0)$ is called an $\mathbf{R}_0$-matrix. It is easy to show (see for example, Proposition 3.9.23 in Cottle, Pang and Stone 1992) that corresponding to such a matrix, SOL(M, q) is uniformly bounded as q varies over a bounded set in R^n. In particular, SOL(M, q) is bounded for every q. Taking any bounded open set Ω containing SOL(M, q), we see that deg(F, Ω) (with F given by (5)) is defined and, in view of the Excision property of degree, is independent of Ω. We claim that this degree is independent of q also. To see this, consider any two vectors q_1 and q_2 and define the homotopy $H(t, x) = x \wedge \{Mx + tq_1 + (1 - t)q_2\}$. Let D be any bounded open set containing all the sets SOL$(M, tq_1 + (1 - t)q_2)$ as t varies over the interval $[0, 1]$. The claim follows from the Homotopy invariance property applied to the pair (H, D). Hence corresponding to any $\mathbf{R}_0$-matrix M we can associate an integer, called the degree of M, by

$$\deg M := \deg(F, \Omega)$$

where q is any vector, F is defined by (5) and Ω is any bounded open set containing SOL(M, q). Now let M be an $\mathbf{R}_0$-matrix and q be a vector such that SOL(M, q) is finite. Then by the Domain decomposition and the Excision properties,

$$\text{(8)} \qquad \deg M = \sum_{x \in \mathrm{SOL}(M,q)} \operatorname{index}(M, q, x).$$

When q is nondegenerate with respect to M (i.e., each solution of LCP(M, q) is nondegenerate), it is known that LCP(M, q) has a finite number of solutions and the supporting submatrix of each solution is nonsingular. In this case the above formula reads, because of (6),

$$\text{(9)} \qquad \deg M = \sum \operatorname{sgn} \det M_{II}$$

where the sum varies over all solutions of LCP(M, q). Stewart's extension (Stewart 1991) consists in replacing a nondegenerate vector by a semi-nondegenerate vector (it is a vector q for which LCP(M, q) has finite number of solutions and the supporting submatrix of each solution is nonsingular). Thus if q is semi-nondegenerate with respect to M, then from (7) and (8),

$$\text{(10)} \qquad \deg M = \sum (\operatorname{sgn} \det M_{II}) \deg M_{II}^{S}.$$

where the summation varies over all solutions of LCP(M, q).

The above formula is particularly useful when M is a nondegenerate matrix, i.e., when all the principal minors of M are nonzero. In this situation, every vector q, for which LCP(M, q) is solvable, is semi-nondegenerate (Murty 1972, Theorem 3.2).

REMARK. While describing the LCP-related degree concepts above, we used the 'min' function (5). In the LCP theory, other mappings have also been used. For example, LCP(M, q) is equivalent to the equation $M(x^{+}) + q - x^{-} = 0$ where $x^{+} = \max\{x, 0\}$, etc. One can study degree theory via (variants of) the mapping $x \mapsto M(x^{+}) + q - x^{-}$. See for example, Kojima and Saigal 1979, 1981, Howe 1983, Howe and Stone 1983, Cottle, Pang and Stone 1992, and Garcia, Gould and Turnbull 1983, Ha 1987. It turns out that in all these formulations, formula (6) is used as the basis of degree analysis. In view of this, the formulas (6)—(10) are valid for mappings other than the 'min' mapping. This observation allows us to use the degree theoretic results given in the references cited above.

3. Stability at a solution point. Let x^* be a solution of LCP(M, q). We say that LCP(M, q) is *stable* at x^* (Ha 1985, Jansen and Tijs 1987) (or that x^* is a stable solution of LCP(M, q)) if x^* is an isolated solution of LCP(M, q) and corresponding to every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(x^* + \epsilon B) \cap \mathrm{SOL}(M', q') \neq \varnothing$$

for all $(M', q') \in R^{n \times n} \times R^n$ satisfying the condition $\|M' - M\| + \|q' - q\| < \delta$. This simply means that when (M', q') is close to (M, q), LCP(M', q') will have solutions near x^*.

In order to explain the connection between stability and degree theory, suppose that x^* is the only solution of LCP(M, q) in the closure of some bounded open set Ω. Then index(M, q, x^*) (which is equal to $\deg(F, \Omega)$ for F given in (5)) is defined. Furthermore, if corresponding to the matrix M' and the vector q' the function

$$G(x) = x \wedge (M'x + q')$$

satisfies the condition (3), then $\deg(G, \Omega) = \deg(F, \Omega)$. From the simple inequality $|a \wedge b - a \wedge c| \leqslant |b - c|$ which holds for all real numbers a, b, and c, it is easily seen that for $x \in \Omega$,

$$\begin{aligned} \|F(x) - G(x)\| &= \|x \wedge (Mx + q) - x \wedge (M'x + q')\| \\ &\leqslant \|(Mx + q) - (M'x + q')\| \\ &\leqslant (L + 1)(\|M - M'\| + \|q - q'\|) \end{aligned}$$

where L is a bound on $\|x\|$ as x varies over Ω. It is clear that if the pair (M', q') is close to (M, q), then (3) holds. Thus, if $\deg(F, \Omega)$ is nonzero then $\deg(G, \Omega)$ is nonzero for all (M', q') close to (M, q). It follows that $G(x) = 0$ has a solution in Ω. *This amounts to saying that if x^* is isolated and* index(M, q, x^*) *is nonzero, then x^* is a stable solution of* LCP(M, q). (Precisely the same result is obtained by Ha 1987 using a "Minty-like" map.)

Our first result concerns a $\mathbf{P}_0$-matrix. To be specific, we show below that when M is a $\mathbf{P}_0$-matrix and x^* is an isolated solution of LCP(M, q), index$(M, q, x^*) = \mathbf{1}$ so that x^* is a stable solution. Before stating the result we recall that a *matrix M is a $\mathbf{P}_0$-matrix if every principal minor of M is nonnegative and it is a* $\mathbf{P}$*-matrix if every principal minor is positive.*

THEOREM 1. *Let M be a $\mathbf{P}_0$-matrix and x^* be a solution of LCP(M, q). Then x^* is stable if and only if it is isolated.*

PROOF. It is enough to prove the 'if' part. Suppose that x^* is an isolated solution of LCP(M, q) and let Ω be any bounded open neighbourhood of x^* such that x^* is the only solution of LCP(M, q) in $\overline{\Omega}$. Define $(\hat{M}, \hat{q})$ by

$$\hat{M}x := Mx + \epsilon x, \qquad \hat{q} := q - \epsilon x^*$$

where ϵ is a small positive number. Let

$$F(x) := x \wedge (Mx + q), \qquad \hat{F}(x) := x \wedge \left(\hat{M}x + \hat{q}\right).$$

Then for all small $\varepsilon > 0$, condition (3) holds with $G = \hat{F}$ so that $\deg(F, \Omega) = \deg(\hat{F}, \Omega)$. Now the matrix $\hat{M}$ is a $\mathbf{P}$-matrix. Furthermore, x^* is the unique solution of LCP$(\hat{M}, \hat{q})$ which is in Ω and which is nondegenerate. It is easily seen from (6) that $\deg(\hat{F}, \Omega) = \mathbf{1}$. It follows that index$(M, q, x^*) = \deg(F, \Omega) = \mathbf{1}$ and the proof is complete. □

The next result was first observed by Cottle and Guu (1991). Here we give a degree-theoretic proof.

THEOREM 2. *Let M be a $\mathbf{P}_0$-matrix and q be any vector. Then the number of solutions of LCP(M, q) is either zero or one or infinity.*

PROOF. It is clear that LCP(M, q) has a solution that is not isolated, then LCP(M, q) has infinitely many solutions. We prove the result by showing that LCP(M, q) has at most one isolated solution. Suppose the contrary and let x^* and u^* be two distinct isolated solutions of LCP(M, q). By Theorem 1, both solutions are stable. This means that if a matrix $\hat{M}$ is sufficiently close to M, then LCP$(\hat{M}, q)$ must have solutions close to both x^* and u^*. In particular, LCP$(\hat{M}, q)$ must have at least two solutions. But this is obviously false for the $\mathbf{P}$-matrix $\hat{M}$ defined in the proof of the previous theorem. This completes the proof. □

Our next theorem, although trivial to prove, contains as special cases some known stability results.

THEOREM 3. *Suppose that* x^* *is an isolated solution of* $LCP(M, q)$ *such that* M_{II} *is nonsingular and* $\deg M_{II}^S \neq 0$. *Then* x^* *is stable.*

PROOF. By (7), we see that index(M, q, x^*) is nonzero. By the remarks made prior to Theorem 1, x^* is stable. □

REMARKS. In order to apply the above theorem, one has to know that the degree of M_{II}^S is nonzero. We list below three instances when the degree of an $\mathbf{R}_0$-matrix M is nonzero. (See, Howe and Stone 1983, Kojima and Saigal 1981.)

(1) M is a **P**-matrix in which case $\deg M = 1$.

(2) M is a **G**-matrix which means that for some positive vector d, LCP(M, d) has a unique solution, namely, zero. In this case, $\deg M = 1$. Note that the class **G** includes semimonotone matrices which are defined by the condition that for all positive vectors d, LCP(M, d) has a unique solution, namely, zero. We specifically note that copositive matrices and positive semidefinite matrices are semimonotone.

(3) M is an **N**-matrix of first category which means that every principal minor of M is negative and $M \not< 0$. Here $\deg M = -1$.

COROLLARY 1 (HA 1985). *Suppose that* x^* *is an isolated and nondegenerate solution of* $LCP(M, q)$. *Then* x^* *is stable.*

PROOF. By (Mangasarian 1980, Corollary 3.2), M_{II} is invertible and M_{II}^S is the identity matrix. Since the degree of the identity matrix is one, the stability of x^* follows. □

Our next result deals with a fully semimonotone matrix and strengthens a result of Gowda and Pang 1992, Corollary 2. By definition, a matrix is *fully semimonotone* if every principal pivot transform Cottle, Pang and Stone 1992 of M is semimonotone.

COROLLARY 2. *Suppose that M is fully semimonotone and* x^* *is an isolated solution* $LCP(M, q)$ *such that the supporting submatrix is* M_{II} *nonsingular. Then* index$(M, q, x^*) = \pm 1$.

PROOF. By the index formula (7), we have

$$\text{index}(M, q, x^*) = (\operatorname{sgn} \det M_{II}) \deg M_{II}^S.$$

We show that the degree of M_{II}^S is 1 from which the result follows. Clearly, M_{II}^S is a submatrix of the principal transform of M obtained by pivoting on M_{II}. Since any principal pivot transform of M is semimonotone and any submatrix of a semimonotone matrix is semimonotone, we see that M_{II}^S is semimonotone (in addition to being an $\mathbf{R}_0$-matrix). As mentioned earlier, the degree of an $\mathbf{R}_0$ semimonotone matrix is 1. Thus we see that index$(M, q, x^*) = \pm 1$. □

At this particular stage, we do not know if the nonsingularity assumption on the supporting submatrix can be removed.

It is interesting to note that Theorem 3 settles, affirmatively, a conjecture of Broyden 1991. Let us say that a nondegenerate matrix M is *odd* if there is a nondegenerate vector q such that LCP(M, q) has odd number of solutions. Broyden conjectures that if the matrix M_{II}^S corresponding to a solution x^* of LCP(M, q) (with M nondegenerate) is odd, then the solution is stable. This is immediate from the theorem because when M_{II}^S is odd, the degree of M_{II}^S is nonzero. (If the sum in (9) has an odd number of terms, then the left-hand side is nonzero.)

4. Cardinality of SOL(M, q) corresponding to an N-matrix. From stability results we move on to results describing the cardinality of the solution set of a linear complementarity problem. We restrict our attention to nondegenerate matrices so that we can use Stewart's degree formula (10). For a given matrix M and any q, we write

$$\Phi(q) := \mathrm{SOL}(M, q).$$

It is well known that when M is a **P**-matrix, $|\Phi(q)| = 1$ for every vector q. The situation changes when M is an **N**-matrix, i.e., when all the principal minors of M are negative. If M is an **N**-matrix of second category (which means that $M < 0$), then it is known (Kojima and Saigal 1979) that

$$|\Phi(q)| = \begin{cases} 0 & \text{if } q \ngeq 0, \\ 1 & \text{if } q \geqslant 0,\, q \ngtr 0, \\ 2 & \text{if } q > 0. \end{cases} \tag{11}$$

Concerning a matrix M which is an **N**-matrix of first category (means that $M \nless 0$) Kojima and Saigal 1981 (see also Mohan and Sridhar 1989) show, using degree theoretic arguments, that

$$|\Phi(q)| = \begin{cases} 1 & \text{if } q \ngeq 0 \text{ or } q = 0, \\ 1 \text{ or } 2 & \text{if } q \geqslant 0,\, q \ngtr 0, \\ 3 & \text{if } q > 0. \end{cases} \tag{12}$$

As a refinement of the above, we describe in the following result, the precise number of solutions when $0 \leqslant q \neq 0$. We freely use the known results Kojima and Saigal 1981 that the degree of a **P**-matrix is one and that *the degree of an* **N**-*matrix is zero if the matrix is of second category and* -1 *otherwise*. We also observe that when x^* is a nonzero solution of LCP(M, q) where M is an **N**-matrix, the Schur complement M_{II}^S is a **P**-matrix and hence index$(M, q, x^*) = (\mathrm{sgn}\det M_{II})\deg M_{II}^S = -1$.

THEOREM 4: *Let M be an* **N**-*matrix such that $M \nless 0$. Then for any q,*

$$|\Phi(q)| = \begin{cases} 1 & \textit{if } q \ngeq 0 \textit{ or } q = 0, \\ 1 & \textit{if } q \geqslant 0,\, q \neq 0,\, L := \{i\colon q_i = 0\} \neq \varnothing,\, M_{LL} \nless 0, \\ 2 & \textit{if } q \geqslant 0,\, q \neq 0,\, L := \{i\colon q_i = 0\} \neq \varnothing,\, M_{LL} < 0, \\ 3 & \textit{if } q > 0. \end{cases} \tag{13}$$

PROOF. In view of (12), we need only to deal with a nonzero nonnegative vector q for which $L = \{i\colon q_i = 0\} \neq \varnothing$. Let q be such a vector. Since M is nondegenerate, there are a finite number of solutions to LCP(M, q) and all the solutions are isolated. Let $\alpha = \mathrm{index}(M, q, 0)$. By the index formula (7), with $x^* = 0$, $I = \varnothing$, and $K = L$, we have $\alpha = \mathrm{index}(M_{LL}, 0, 0) = \deg M_{LL}$. Since M_{LL} is an **N**-matrix, α is zero if $M_{LL} < 0$ and -1 otherwise.

It is clear that there could be at most one solution $x^\#$ with $I \neq \varnothing$, $I \cup K = \{1, 2, \ldots, n\}$. Define β as zero if there is no $x^\#$ with this property or as index$(M, q, x^\#)$ if there is such an $x^\#$. In view of the observation preceeding the theorem, β is either zero (when there is no $x^\#$) or -1. Finally we let γ be $\Sigma\,\mathrm{index}(M, q, x)$ where the sum is taken over the set E of all solutions x with $I \neq \varnothing$, $I \cup K \neq \{1, 2, \ldots, n\}$. (If there is no such solution, we let γ be zero.) Once again, γ is either a negative integer

($= -|E|$) or zero. The degree formula (10) now reads,

$$\alpha + \beta + \gamma = -1.$$

In view of the values taken by α, β and γ, we see that the triple (α, β, γ) is either $(-1, 0, 0)$ or $(0, -1, 0)$ or $(0, 0, -1)$. In the first case, there is only one solution, namely zero. In the last two cases, apart from the zero solution there is another solution. These considerations complete the proof. □

From considering matrices, all of whose minors have the same sign, namely **P** and **N**-matrices, we move on to almost **P** and almost **N**-matrices. In addition to having interesting LCP properties, these matrices appear in the study of univalent functions (see Parthasarathy and Ravindran 1990, Mohan, Parthasarathy and Sridhar 1991, Olech, Parthasarathy and Ravindran 1991). We say that a matrix M is

(a) an *almost* **P**-*matrix* if M has negative determinant and every proper principal minor of M is positive,

(b) an *almost* **N**-*matrix* if M has positive determinant and every proper principal minor of M is negative.

The analysis of an almost **P**-matrix M is somewhat simplified in view of the observation that the inverse of such a matrix is an **N**-matrix. Since for any invertible $\mathbf{R}_0$-matrix M, $\deg M^{-1} = (\operatorname{sgn}\det M)\deg M$ (Cottle, Pang and Stone 1992, Theorem 6.6.23), we see that *the degree of an almost* **P**-*matrix is zero if* $M^{-1} < 0$ *and one if* $M^{-1} \nless 0$.

5. The number of solutions corresponding to a negative almost N-matrix. Let M be a negative almost **N**-matrix. We note to begin with that LCP(M, q) has no solution when $q \ngeqslant 0$ and hence $\deg M = 0$. We let pos$(-M)$ denote the set $-M(R^n_+)$. It is clear that for $q \in \operatorname{pos}(-M)$, there is only one $x \geqslant 0$ with $q = -Mx$ so that M^S_{II} is uniquely defined where I denotes the support of x.

THEOREM 5. *Let $M < 0$ be an almost* **N**-*matrix. Then*

$$(14)\quad |\Phi(q)| = \begin{cases} 1 & \text{if } q \in \mathscr{A} := \{r : r \geqslant 0,\ r \ngtr 0\}, \\ 4 & \text{if } q \in \mathscr{B} := \{r : r > 0,\ M^{-1}r < 0\}, \\ 3 & \text{if } q \in \mathscr{C} := \left\{r : r > 0,\ M^{-1}r \nless 0,\ r \in \operatorname{pos}(-M),\ \left(M^S_{II}\right)^{-1} < 0\right\}, \\ 2 & \text{if } q \in \mathscr{D} := \left\{r : r > 0,\ M^{-1}r \nless 0,\ r \in \operatorname{pos}(-M),\ \left(M^S_{II}\right)^{-1} \nless 0\right\}, \\ 2 & \text{if } q \in \mathscr{E} := \{r : r > 0,\ r \notin \operatorname{pos}(-M)\}. \end{cases}$$

PROOF. Since $M < 0$, $|\Phi(q)| = 1$ for $q \in \mathscr{A}$. Now let $q \in \mathscr{B}$. Clearly, the zero vector and $-M^{-1}q$ are solutions of LCP(M, q). Since $\det M > 0$, the formula (6) gives,

$$\operatorname{index}(M, q, 0) = 1 = \operatorname{index}(M, q, -M^{-1}q).$$

Let x be any other solution of LCP(M, q) with index sets I and K defined as in (2), so that $I \neq \varnothing$, and $I \cup K \neq \{1, 2, \ldots, n\}$. Since the Schur complement of a proper submatrix of an **N**-matrix is a **P**-matrix, it follows that the degree of M^S_{II} is 1. Since $\operatorname{sgn}\det M_{II} = -1$, we see that index$(M, q, x) = -1$. As noted before, the degree of M is zero. The degree formula (10) now shows that apart from the zero vector and $-M^{-1}q$, there must be two more solutions to LCP(M, q). This shows that $|\Phi(q)| = 4$ when $q \in \mathscr{B}$.

Now let $q \in \mathscr{C} \cup \mathscr{D}$. Then there is a unique $x \geqslant 0$ such that $q = -Mx$, $0 \neq x \ngtr 0$. If I and K are defined as in (2), then $I \neq \varnothing \neq K$, $I \cup K = \{1, 2, \ldots, n\}$. It is easily seen that M_{II}^S is an almost **P**-matrix and sgn det $M_{II} = -1$. From the index formula (7), we see that index(M, q, x) is equal to zero if $q \in \mathscr{C}$ and -1 if $q \in \mathscr{D}$. Let y be a nonzero solution of LCP(M, q) different from x. Let us once again use the letters I, J, and K to denote the index sets corresponding to y. Now $I \neq \varnothing$, $I \cup K \neq \{1, 2, \ldots, n\}$. Since M is an almost **N**-matrix, sgn det $M_{II} = -1$ and M_{II}^S is a **P**-matrix. Thus index$(M, q, y) = -1$. If $q \in \mathscr{C}$, then the degree formula (10) (with deg $M = 0$) shows that in addition to the solutions zero and x, there must be one more solution. Thus there are three solutions when $q \in \mathscr{C}$. Similar argument shows that there are exactly two solutions when $q \in \mathscr{D}$.

Finally, let $q \in \mathscr{E}$. Since deg $M = 0$, apart from the zero vector, there must be at least one more solution to LCP(M, q). Let x be any nonzero solution with I and K defined as in (2). Since $q \notin \text{pos}(-M)$, $I \cup K \neq \{1, 2, \ldots, n\}$. We note that $I \neq \varnothing$, det $M_{II} = -1$, and M_{II}^S is a **P**-matrix. Thus index$(M, q, x) = -1$ for any nonzero solution of LCP(M, q). Once again from the degree formula we conclude that apart from the zero vector there is exactly one other solution to LCP(M, q). This argument shows that $|\Phi(q)| = 2$ when $q \in \mathscr{E}$. □

Remarks. The above theorem thus completely describes the number of solutions for any q when M is any $n \times n$ negative almost **N**-matrix. Parts of the above result were known. Mohan, Parthasarathy and Sridhar 1991 describe, for $n \geqslant 4$, the number of solutions of LCP(M, q) when $q \in \mathscr{A} \cup \mathscr{B} \cup \mathscr{E}$. Interestingly enough, they also give a characterization (for $n \geqslant 4$) of negative almost **N**-matrices in terms of the number of solutions of LCP(M, q). In view of Theorem 5 one may ask whether their characterization goes through even for $n = 2$ and $n = 3$. We do not know the answer.

6. A Lipschitzian characterization of negative N-matrices.

For a given matrix M we say that the solution mapping Φ: $q \mapsto \text{SOL}(M, q)$ is *Lipschitzian* if there is a positive number C such that the inclusion

$$\Phi(q) \subseteq \Phi(q') + C\|q' - q\|B \tag{15}$$

holds for all q and q' with $\Phi(q) \neq \varnothing \neq \Phi(q')$. The above inclusion simply means that for each solution x of LCP(M, q) there is a solution x' of LCP(M, q'), such that $\|x' - x\| \leqslant C\|q' - q\|$.

As noted in Gowda 1992, mappings corresponding to **P**-matrices and negative **N**-matrices have this property. If the matrix M is a **G**-matrix, then the matrix is a **P**-matrix if and only if the corresponding solution mapping is Lipschitzian (Gowda 1992). Our aim in this section is to characterize negative **N**-matrices via the Lipschitzian property. We begin with a Lemma.

Lemma 1. *Let M be an $n \times n$ negative almost* **N**-*matrix with $n \geqslant 2$. Then the solution mapping Φ corresponding to M can never be Lipschitzian.*

Proof. We first prove the existence of a vector q in the set $\mathscr{C}$ defined in Theorem 5. We let p be the vector in R^n consisting of ones in the first $(n-1)$ spots and zero in the last spot and let $q = (-M)p$. It is clear that $0 < q \in \text{pos}(-M)$ and $M^{-1}q \nless 0$. Clearly, p is a solution of LCP(M, q) with $I = \{1, 2, \ldots, (n-1)\}$ and $K = \{n\}$. Hence M_{II}^S is a 1×1 matrix. Since det M is positive and det M_{II} is negative, we see by the Schur determinant formula, det $M = (\det M_{II}^S)(\det M_{II})$, that M_{II}^S is a 1×1 negative matrix. Hence $(M_{II}^S)^{-1} < 0$. Thus we have shown that q belongs to the set $\mathscr{C}$.

By Theorem 5, $\Phi(q)$ has three elements. Apart from the zero vector, let u and v be the other two solutions. By scaling the vector q (if necessary), we can assume that

$$1 \leqslant \min\{\|u\|, \|v\|, \|u - v\|\}. \tag{16}$$

Now suppose, to get a contradiction, that Φ is Lipschitzian so that for some constant $C > 0$, the inclusion (15) holds for all q' with $\Phi(q') \neq \varnothing$. Now pick a q' in the set $\mathscr{E}$ (defined in Theorem 5) such that

$$\|q - q'\| \leqslant \frac{1}{3C}.$$

(We can simply let $q' = q + M(\epsilon e_n)$ where ϵ is small and positive and e_n denotes the vector with zeros in the first $(n - 1)$ spots and one in the last spot.) By Theorem 5, $\Phi(q')$ has two elements one of which is zero. Let x be the other element. The inclusion (15) now reads

$$\{0, u, v\} \subseteq \{0, x\} + C\|q' - q\|B \subseteq \{0, x\} + \frac{1}{3}B.$$

Since $1 \leqslant \min\{\|u\|, \|v\|\}$ we must have $\|u - x\| \leqslant \frac{1}{3}$ and $\|v - x\| \leqslant \frac{1}{3}$. But this leads to $\|u - v\| \leqslant \frac{2}{3}$ which contradicts (16). This shows that the mapping Φ cannot be Lipschitzian. □

We now come to our final result.

THEOREM 6. *Let M be any $n \times n$ matrix. In the following statements, any two will imply the third.*

(a) *M is a negative nondegenerate matrix.*

(b) *M is an* **N**-*matrix.*

(c) *Φ corresponding to M is Lipschitzian.*

PROOF. The implication (a) + (b) ⇒ (c) follows from Theorem 14 in Gowda 1992. To see the implication (b) + (c) ⇒ (a), suppose that M is not a negative matrix. Then (by (12)) for all $q \not\geqslant 0$, LCP(M, q) has a unique solution. In particular, LCP(M, q) has a unique solution for a nondegenerate q. By Corollary 5 in Gowda 1992, M is a **P**-matrix which contradicts (b). We now come to the proof of (a) + (c) ⇒ (b). We prove the implication by induction on n. Clearly the result is true for $n = 1$. Assume the result for all matrices of order $k \times k$ ($k = 1, 2, \ldots, n - 1$) where $n \geqslant 2$. We now fix a matrix $M \in R^{n \times n}$ satisfying (a) and (c). We first show that every proper principal submatrix N of M is an **N**-matrix. Let $N = M_{\alpha\alpha}$ (where α is a proper subset of $\{1, 2, \ldots, n\}$) and let $\Lambda(r) = \mathrm{SOL}(N, r)$ denote the corresponding solution mapping. Without loss of generality let α be the first $|\alpha|$ natural numbers. Let β be the complement of α in $\{1, 2, \ldots, n\}$, $r, s \in R^{|\alpha|}$, and e be the vector of ones in $R^{|\beta|}$. For $m = 1, 2, \ldots,$ let

$$r^m = \begin{bmatrix} r \\ me \end{bmatrix}, \qquad s^m = \begin{bmatrix} s \\ me \end{bmatrix}, \qquad e^m = \begin{bmatrix} 0 \\ me \end{bmatrix}.$$

We claim that for all large m,

$$\Phi(r^m) = \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \Lambda(r) \right\}. \tag{17}$$

To see this, we observe, from the Lipschitzian property (15), the inclusion

$$\Phi(r^m) \subseteq \Phi(e^m) + C\|r\|B.$$

Since $M < 0$, $\Phi(e^m) = \{0\}$ for all m, we see from the above inclusion that $\Phi(r^m)$ is uniformly bounded for all m. It follows that for large m, the β-part of any solution of LCP(M, r^m) is zero. This gives one inclusion towards the equality of sets in (17). The reverse inclusion follows immediately from the observation that since N is nondegenerate, it is an $\mathbf{R}_0$-matrix and hence $\Lambda(r)$ is bounded. We thus have (17) for large m. Now fix r and s with $\Lambda(r) \neq \emptyset \neq \Lambda(s)$. The Lipschitzian property of M gives

$$\Phi(r^m) \subseteq \Phi(s^m) + C\|r - s\|B$$

for all m. By choosing a large m we can apply (17) and conclude that

$$\Lambda(r) \subseteq \Lambda(s) + C\|r - s\|B'$$

where B' denotes the open unit ball in $R^{|\alpha|}$. The above inclusion says that Λ is Lipschitzian. Thus, the given proper principal submatrix N of M has properties (a) and (c). By induction, N is an **N**-matrix. At this stage, we have shown that every proper principal minor of M is negative. Since the solution mapping Φ of M is Lipschitzian, by the above Lemma, M is not an almost **N**-matrix. Therefore, the determinant of M is nonpositive. Since M is also nondegenerate, the determinant of M must be negative so that M is an **N**-matrix. This completes the induction argument. We thus have the implication (a) + (c) $\Rightarrow$ (b). □

7. Concluding remarks and open problems. In this article, we have used degree theory to study stability and cardinality of the solution set in linear complementarity problems. The analysis, as done in this paper, is by no means complete. For example, the question of describing the number of solutions of LCP(M, q) when $M \not< 0$ is an almost **N**-matrix, was not dealt with in this paper. We intend, in a separate article, to go beyond the known description for such matrices (Mohan, Parthasarathy and Sridhar 1991).

As we mentioned earlier, the motivation for studying negative almost **N**-matrices came from the problem of characterizing those matrices M for which the corresponding solution mapping Φ is Lipschitzian. This characterization problem is wide open. In fact, one specific unsolved problem is: Suppose that M is a **Q**-matrix such that the corresponding Φ is Lipschitzian. Can we say that M is a **P**-matrix? A related problem pertaining to negative matrices is to decide whether the implication (a) + (c) $\Rightarrow$ (b) holds in Theorem 6 without the nondegeneracy assumption.

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