On the Transpose of a Pseudomonotone Matrix and the LCP

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ABSTRACT

In the study of linear complementary problem, it is known that pseudomonotone matrices belong to the class $\mathbf{P}_0 \cap \mathbf{Q}_0$. In this note we show that under certain conditions, such as invertibility or normality, the transpose of a pseudomonotone matrix belongs to the class $\mathbf{Q}_0$.

1. INTRODUCTION

Given an $n \times n$ real matrix M and a column vector $q \in \mathbb{R}^n$, the *linear complementarity problem*, denoted by LCP(M, q), is to find a vector x such that

(a) $x \geqslant 0$, $Mx + q \geqslant 0$, and
(b) $x^T(Mx + q) = 0$.

Condition (a) refers to the feasibility of LCP(M, q). A matrix M is said to be a $\mathbf{Q}_0$-matrix (or belong to the class $\mathbf{Q}_0$) if for all q, the feasibility of LCP(M, q) implies its solvability. In this note, we address the following question:

For which matrices M in the class $\mathbf{Q}_0$ does M^T belong to $\mathbf{Q}_0$?

Simple examples (see Section 3) show that the transpose of a $\mathbf{Q}_0$-matrix need not be a $\mathbf{Q}_0$-matrix. However, if M is a copositive plus matrix (or a P-matrix

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or a Z-matrix), then $M \in \mathbf{Q}_0$ and $M^T \in \mathbf{Q}_0$ (cf. [6, 8, 2]). In [4] (also in [5]), we introduced pseudomonotone matrices and showed that such matrices are copositive and row sufficient, and (hence) belong to $\mathbf{P}_0 \cap \mathbf{Q}_0$, where $\mathbf{P}_0$ is the class of all matrices with nonnegative principal minors. Within the class of pseudomonotone matrices, we provide a partial answer to the above question. We show for example that under some very simple conditions such as invertibility or normality, the transpose of a pseudomonotone matrix belongs to $\mathbf{Q}_0$. A result of Cottle, Pang, and Venkateswaran [3] shows that for such matrices M, LCP(M^T, q) has nonempty convex solution set whenever LCP(M^T, q) is feasible. Furthermore, Lemke's algorithm can be applied to solve LCP(M^T, q).

2. PRELIMINARIES

We say that a matrix M is pseudomonotone (on $\mathbb{R}^n_+$) if

$$x, y \geqslant 0, \quad (y-x)^T Mx \geqslant 0 \quad \Rightarrow \quad (y-x)^T My \geqslant 0. \tag{2.1}$$

It is easily seen that positive semidefinite matrices are pseudomonotone. It is shown in [4] that pseudomonotone matrices have the *copositive star* property:

$$x^T Mx \geqslant 0 \qquad (\forall x \geqslant 0), \tag{2.2}$$

$$x \geqslant 0, \quad Mx \geqslant 0, \quad x^T Mx = 0 \quad \Rightarrow \quad M^T x \leqslant 0. \tag{2.3}$$

Also, copositive star matrices, i.e., matrices satisfying the above conditions, are in $\mathbf{Q}_0$ [4, Corollary 2]. We refer to [4] for further properties of pseudomonotone matrices. We say that $M \in \mathbf{Q}$ if for all q, LCP(M, q) has a solution. If for a matrix M, the zero vector is the only solution of LCP($M, 0$), then M is said to be an $\mathbf{R}_0$-matrix. Finally, a matrix M is said to be *row sufficient* if

$$x_i(M^T x)_i \leqslant 0 \quad (\forall i = 1,2,\ldots,n) \quad \Rightarrow \quad x_i(M^T x)_i = 0 \quad (\forall i = 1,2,\ldots,n), \tag{2.4}$$

and *column sufficient* if M^T is row sufficient. It is known (see [3, Theorem 6]) that if M is row sufficient, then LCP(M^T, q) has convex solution set for all q. In what follows, for any vector z with components z_i $(i = 1, 2, \dots, n)$, we write z^+ to denote the vector with components $\max\{z_i, 0\}$ $(i = 1, 2, \dots, n)$.

3. RESULTS

We start with an example to show that the transpose of a pseudomonotone matrix need not be in $\mathbf{Q}_0$ and hence need not be pseudomonotone.

EXAMPLE 1. Let

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Given $x, y, u, v \geqslant 0$ and

$$\begin{bmatrix} u - x \\ v - y \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (v - y)(x + y) \geqslant 0, \tag{3.1}$$

it follows that

$$\begin{bmatrix} u - x \\ v - y \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = (v - y)(u + v) \geqslant 0. \tag{3.2}$$

Hence M is pseudomonotone. However, it is easily seen that the problem LCP(M^T, q) if feasible but not solvable. Thus $M^T \notin \mathbf{Q}_0$. Since pseudomonotone matrices belong to $\mathbf{Q}_0$ [4, Corollary 3], it follows that M^T is not pseudomonotone.

THEOREM 1. *Suppose that M is pseudomonotone. Then, under each of the following conditions, M^T satisfies the copositive star property and hence belongs to $\mathbf{Q}_0$.*

(a) *The diagonal of M consists only of zeros.*
(b) *The system $0 \neq d \geqslant 0$, $M^T d = 0$ has no solution.*
(c) *M is invertible.*
(d) *$M \in \mathbf{R}_0$.*
(e) *M is normal, i.e., $MM^T = M^T M$.*

Proof. Since a pseudomonotone matrix is copositive [4, Proposition 1], we need to show that the condition (2.3) holds for M^T. Let

$$0 \neq d \geqslant 0, \qquad M^T d \geqslant 0, \quad \text{and} \quad d^T M^T d = 0. \tag{3.3}$$

We show that under each of the above conditions (a)–(e), $Md \leqslant 0$.

(a): In this case, we have for every coordinate vector e_i (which has one at the ith spot and zeros elsewhere),

$$(d - e_i)^T M e_i = e_i^T M^T d \geqslant 0.$$

By pseudomonotonicity, $(d - e_i)^T Md \geqslant 0$, i.e., $e_i^T Md \leqslant 0$. Thus $Md \leqslant 0$.

(b): When this holds, $(M^T d)_i > 0$ for some i. Ignoring the trivial case $n = 1$, let $x = e_i + \lambda e_j$, where $j \neq i$ and $\lambda \geqslant 0$ are arbitrary. Then, for all small $\epsilon > 0$, we have

$$(d - \epsilon x)^T M(\epsilon x) = \epsilon\left[(M^T d)_i + \lambda (M^T d)_j - \epsilon x^T M x\right] \geqslant 0.$$

By pseudomonotonicity, $(d - \epsilon x)^T Md \geqslant 0$, i.e, $x^T Md \leqslant 0$. This gives

$$(Md)_i + \lambda (Md)_j \leqslant 0.$$

Since λ is arbitrary, $(Md)_i \leqslant 0$ and $(Md)_j \leqslant 0$. Hence $Md \leqslant 0$.

(c): When this holds, (b) holds, and once again $Md \leqslant 0$.

(d): When $M \in \mathbf{R}_0$, we merely show that (b) holds. Let $d \geqslant 0$, $M^T d = 0$. Then $d^T Md = 0$. By the copositivity of M, we have for any $x \geqslant 0$

$$x^T (M + M^T) d = \lim_{\lambda \to 0^+} \lambda^{-1} (\lambda x + d)^T M (\lambda x + d) \geqslant 0.$$

This gives $(M + M^T) d \geqslant 0$, so $Md \geqslant 0$. Hence d is a solution of the problem LCP$(M, 0)$. Since $M \in \mathbf{R}_0$, $d = 0$.

(e): If $M^T d = 0$, then by normality, $Md = 0$. (Recall that $d^T M M^T d = d^T M^T M d$.) If $M^T d \neq 0$, then $(M^T d)_i > 0$ for some i. In this situation, we proceed as in (b) and get $Md \leqslant 0$. ■

Since a pseudomonotone matrix is copositive star, one is led to ask whether the transpose of a pseudomonotone matrix is pseudomonotone under any of conditions (a)–(e). We answer this question for 2×2 matrices; the answer is not known for higher order matrices.

THEOREM 2. *Let $M \in \mathbb{R}^{2\times 2}$ and pseudomonotone. Then, under each of conditionss* (a)–(e) *in Theorem* 1, *M^T is pseudomonotone.*

LEMMA 1. *Let the matrix*

$$N = \begin{bmatrix} 1 & c \\ b & 0 \end{bmatrix}$$

be such that

(i) $b + c \geqslant 0$,
(ii) $b \leqslant 0$ *when* $c \geqslant 0$,
(iii) $c \leqslant 0$ *when* $b \geqslant 0$.

Then N and N^T are pseudomonotone.

Proof. We show that N is pseudomonotone; the proof of the pseudomonotonicity of N^T is similar. Let $x \geqslant 0$, $y \geqslant 0$ with $(y-x)^T Nx \geqslant 0$, so

$$(y-x)_1(x_1 + cx_2) + (y-x)_2 bx_1 \geqslant 0. \tag{3.4}$$

Since N is copositive [from (i)], we can assume that $(y-x) \notin \mathbb{R}^2_+ \cup (-\mathbb{R}^2_+)$. We consider two cases.

Case 1: $(y-x)_1 > 0$, $(y-x)_2 < 0$. We have $y_1 > x_1 \geqslant 0$, $x_2 > y_2 \geqslant 0$.

Subcase 1.1: $b \geqslant 0$. By (iii), $c \leqslant 0$. Then (3.4) gives $(y-x)_1 x_1 + (y-x)_2 bx_1 \geqslant 0$. If $x_1 > 0$, then $(y-x)_1 + (y-x)_2 b \geqslant 0$ and so

$$\begin{aligned}(y-x)^T Ny &= (y-x)_1(y_1 + cy_2) + (y-x)_2 by_1 \\ &= y_1\{(y-x)_1 + (y-x)_2 b\} + (y-x)_1 cy_2 \\ &\geqslant x_1\{(y-x)_1 + (y-x)_2 b\} + (y-x)_1 cx_2 \\ &= (y-x)^T Nx \geqslant 0.\end{aligned}$$

If $x_1 = 0$, then (3.4) gives [in view of $c \leqslant 0$, $x_2 > 0$, $(y-x)_1 > 0$] $c = 0$. From (ii), $b = 0$. But then, $(y-x)^T Ny = (y-x)_1 y_1 \geqslant 0$.

Subcase 2: $b < 0$. By (i), $c > 0$. From $(y-x)_2 < 0$, $b < 0$, $c > 0$, we get

$$(y-x)^T Ny = (y-x)_1(y_1 + cy_2) + (y-x)_2 by_1 \geqslant 0.$$

Case 2: $(y-x)_1 < 0$, $(y-x)_2 > 0$. If $b < 0$, then by (i), $c > 0$. This leads to $(y-x)_1(x_1 + cx_2) + (y-x)_2 bx_1 = 0$. But this gives $y_1 = x_1 = 0$, contradicting $(y-x)_1 < 0$. Thus $b \geqslant 0$. In this case, $c \leqslant 0$. Since $x_1 > 0$, $y_1 < x_1$, $y_2 > x_2$, $c(y-x)_1 \geqslant 0$, we have

$$\begin{aligned}(y-x)^T Ny &= (y-x)_1 y_1 + c(y-x)_1 y_2 + (y-x)_2 by_1 \\ &\geqslant (y-x)_1 y_1 + c(y-x)_1 x_2 \frac{y_1}{x_1} + (y-x)_2 by_1 \\ &= \frac{y_1}{x_1}\{(y-x)_1 x_1 + c(y-x)_1 x_2 + (y-x)_2 bx_1\} \\ &= \frac{y_1}{x_1}(y-x)^T Nx \geqslant 0.\end{aligned}$$

Hence in all cases, $(y-x)^T Ny \geqslant 0$. So N is pseudomonotone. ∎

Proof of Theorem 2. We suppose that M is pseudomonotone and satisfies one of conditions (a)–(e). We first note that $P^T MP$ is pseudomonotone for any nonnegative matrix P. In particular, $P^T MP$ is pseudomonotone when P is a permutation matrix and when P is a nonnegative diagonal matrix. (When P is a nonnegative diagonal matrix, we refer to the transformation $M \mapsto P^T MP$ as *scaling*. We note that when P is a positive diagonal matrix, a matrix N is pseudomonotone if and only if $P^T NP$ is pseudomonotone.) Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $M \in \mathbf{P}_0$ [4, Proposition 2],

$$a \geqslant 0 \quad \text{and} \quad d \geqslant 0. \tag{3.5}$$

If $a = d$, then $M^T = P^TMP$ for some permutation matrix P. In this case, M^T is pseudomonotone. If $a \neq d$ we consider three cases:

Case 1: $a \neq 0 \neq d$. By (3.5), $a > 0$ and $d > 0$. Let

$$P = \begin{bmatrix} 1/\sqrt{a} & 0 \\ 0 & 1/\sqrt{d} \end{bmatrix}.$$

Then $N = P^TMP$ is pseudomonotone and has equal matrices (namely, 1) on the main diagonal. By the previous argument, N^T is pseudomonotone and hence $M^T = (P^{-1})^T N^T P^{-1}$ is also pseudomonotone (since P^{-1} is nonnegative).

Case 2: $a \neq 0$, $d = 0$. From (3.5) we have $a > 0$. By suitable scaling, we can assume that $a = 1$. Then

$$M = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}.$$

Since M is copositive, $b + c \geqslant 0$. We know that M is copositive star, and by Theorem 1, that M^T is also copositive star. Since $e_2^T M e_2 = 0$, we conclude that $b \leqslant 0$ when $c \geqslant 0$ and $c \leqslant 0$ when $b \geqslant 0$. Thus the matrix $N := M^T$ satisfies the conditions of the previous lemma. Therefore, M^T is pseudomonotone.

Case 3: $a = 0$, $d \neq 0$. In this case, by suitable scaling, we can make $d = 1$, so that

$$M = \begin{bmatrix} 0 & b \\ c & 1 \end{bmatrix}.$$

For a suitable permutation matrix P, P^TMP looks like the matrix of case 2. Since M is assumed to satisfy one of conditions (a)–(e), P^TMP also satisfies one of conditions (a)–(e). By case 2, P^TM^TP is pseudomonotone. Thus $M^T = PP^TM^TPP^T$ is also pseudomonotone.

So in all cases, M^T is pseudomonotone. This completes the proof. ∎

Remarks.

(1) In view of a result of Pang [9, Theorem 4], condition (d) in Theorem 1 can be replaced by the equivalent condition $M \in \mathbf{Q}$.

(2) The normality condition in Theorem 1 can be replaced by the hyponormality condition (defined by $\|M^Tx\| \geqslant \|Mx\|$ for all x).

(3) One might ask whether $M^T \in \mathbf{Q}$ when M is pseudomonotone and $M \in \mathbf{Q}$. This is false even for positive semidefinite matrices. For example, let

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

It is clear that M is positive semidefinite and belongs to $\mathbf{R}_0$ and hence to $\mathbf{Q}$ [9, Lemma 1], while LCP(M^T, $-e_1$) is not even feasible.

(4) It is shown in [5, Corollary 3] that every pseudomonotone matrix M is row sufficient, and hence M^T is column sufficient. By Theorem 6 in [3], the solution set of LCP(M^T, q) is convex for all q. When M is pseudomonotone and one of conditions (a)–(e) in Theorem 1 holds, then LCP(M^T, q) has nonempty convex solution set for every feasible q. Since in this case $M^T \in \mathbf{P}_0 \cap \mathbf{Q}_0$, LCP($M^T, q$) can be solved by Lemke's algorithm [1].

(5) The arguments used in the proof of Theorem 1 along with Lemma 1 reveal the following result: A 2×2 pseudomonotone matrix has a pseudomonotone transpose if and only if it is not permutation similar to

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$

with $a > 0$, $0 \neq (b, c) \geqslant 0$. (This observation is due to one of the referees.)

We conclude this note by posing two *open problems*:

(1) Suppose that M is pseudomonotone and invertible (or normal). Can we say that M^T is pseudomonotone? row sufficient?

(2) Suppose that M is row sufficient and invertible (or normal). Can we say that M^T is in $\mathbf{Q}_0$?

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