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A COMPARISON OF CONSTRAINT QUALIFICATIONS IN INFINITE-DIMENSIONAL CONVEX PROGRAMMING*

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Abstract. In this paper the relationships between various constraint qualifications for infinite-dimensional convex programs are investigated. Using Robinson's refinement of the duality result of Rockafellar, it is demonstrated that the constraint qualification proposed by Rockafellar provides a systematic mechanism for comparing many constraint qualifications as well as establishing new results in different topological environments.

Key words. constraint qualifications, convex programming, duality theory

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1. Introduction.

This paper deals with constraint qualifications for infinite-dimensional convex programs. A constraint qualification is an essential condition needed to establish strong duality results for a pair of optimization problems. The usual constraint qualification is a *Slater-like* condition that requires nonempty interiority of a certain convex set. Unfortunately, this condition often fails for an important class of optimization problems arising in applications, see, e.g., [3].

Recently, many authors have proposed new constraint qualifications for optimization problems in infinite-dimensional vector spaces, see [5]-[7], [2], [15]. Motivated by the recent constraint qualification proposed by Borwein and Wolkowicz [7], in this paper we investigate the relationships between various constraint qualifications. By studying cores and interiors of convex sets, we show that many of the constraint qualifications are equivalent or can be derived from the constraint qualification proposed by Rockafellar [14]. Furthermore, we show that the Rockafellar constraint qualification provides a natural mechanism for establishing new constraint qualifications in various topological environments.

The paper is organized as follows. In § 2 we recall the fundamental constraint qualification proposed by Rockafellar, denoted by (R), and state Robinson's refinement of a result of Rockafellar. In § 3 we demonstrate that condition (R) is instrumental in constructing various constraint qualifications and that many seemingly unrelated constraint qualifications are in fact related to (R). We also derive new results in the general setting of Baire spaces and provide examples.

2. A fundamental constraint qualification.

Let X and Y be real locally convex topological vector spaces and $A: X\to Y$ be a continuous linear operator. Let $f: X\to(-\infty,+\infty]$ and $g: Y\to(-\infty,+\infty]$ be proper, lower semicontinuous convex functions. Consider the primal problem:

$$\text{(P)} \qquad \inf_{x\in X}\{f(x)+g(Ax)\}.$$

The *Fenchel-Rockafellar duality theory*, see Rockafellar [14], associates with (P) the dual problem:

$$\text{(D)} \qquad \sup_{y\in Y^*}\{-g^*(y)-f^*(-A^*y)\}$$

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where $A^*: Y^* \to X^*$ is the adjoint of A and X^*, Y^* are the dual spaces of X and Y, respectively. We recall that for a given function $\phi: X \to (-\infty, +\infty]$, the domain is

$$\operatorname{dom} \phi := \{x \in X: \phi(x) < \infty\}$$

and the *conjugate function* is

$$\phi^*(x^*) := \sup_{x \in \operatorname{dom} \phi} \{\langle x^*, x\rangle - \phi(x)\}, \qquad x^* \in X^*.$$

The main issue regarding the pair of problems (P), (D) is the lack of duality gap, i.e., the proof of the strong duality relation

$$\inf_{x \in X} \{f(x) + g(Ax)\} = \max_{y^* \in Y^*} \{-g^*(y^*) - f^*(-A^*y^*)\} \tag{2.1}$$

which we write, for convenience, as

$$\inf(\mathrm{P})^1 = \max(\mathrm{D}).$$

This can be obtained provided a certain constraint qualification (CQ for short) is satisfied. One of the most popular (CQ) is the so-called *Slater* condition: (see, e.g., [1])

$$0 \in \operatorname{int}(\operatorname{dom} g - A \operatorname{dom} f)). \tag{S}$$

THEOREM 2.1. *Suppose that* (S) *holds. Then* $\inf(\mathrm{P}) = \max(\mathrm{D})$.

Unfortunately, in many important applications the Slater condition fails.

A more general constraint qualification was suggested by Rockafellar [14]. Before stating the condition, we recall the definition of the *core* of a set. For a set $C \subset X$, the *core* of C is defined by

$$\operatorname{core} C := \{c \in C: \forall x \in X\ \exists \varepsilon > 0: \forall \lambda \in [-\varepsilon, \varepsilon],\ c + \lambda x \in C\}.$$

In the context of the pair of problems (P), (D), Rockafellar's (CQ) is

$$0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f). \tag{R}$$

Robinson's refinement [13, Cor. 1] of a result of Rockafellar [14, Thm. 18] leads to the following theorem.

THEOREM 2.2. *Let X, Y be Banach spaces and suppose that* (R) *holds. Then* $\inf(\mathrm{P}) = \max(\mathrm{D})$.

We will show below that the *core* constraint qualification of Rockafellar is the *key* for constructing new constraint qualifications and will, as well, explain most of the classical and more recent constraint qualifications existing in the literature. In particular, we will show that many seemingly unrelated constraint qualifications are in fact related to (R) and show how new duality results can be derived from Theorem 2.2.

3. Comparison of constraint qualifications.

In this section we present some constraint qualifications that can be derived from the Rockafellar condition (R). In the first part of this section we discuss the case when A is a continuous linear operator with *finite-dimensional* range, i.e., $A: X \to Y$ with $Y = \mathbb{R}^n$. In the second part we give corresponding results for an operator with infinite-dimensional range.

3.1. *A* is a linear operator with finite-dimensional range $Y = \mathbb{R}^n$.

We first recall the following result, see, e.g., Holmes [9].

[1] Throughout this paper we assume that inf (P) is finite.

PROPOSITION 3.1. *Let X be a real topological vector space and let $C \subset X$ be convex. Then,* $\operatorname{int} C \subset \operatorname{core} C$. *Further,* $\operatorname{int} C = \operatorname{core} C$ *under each of the following conditions*:

(a) $\operatorname{int} C \neq \emptyset$.

(b) *X is finite-dimensional.*

From Proposition 3.1 it follows immediately that if $\operatorname{int}(\operatorname{dom} g - A \operatorname{dom} f) \neq \emptyset$, then

$$0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f) \Leftrightarrow 0 \in \operatorname{int}(\operatorname{dom} g - A \operatorname{dom} f).$$

Recall that for a convex subset $C \subset \mathbb{R}^n$ we have:

$$0 \in \operatorname{int} C \Leftrightarrow \operatorname{cone} C = \mathbb{R}^n. \tag{3.1}$$

where $\operatorname{cone} C = \{\lambda x : \lambda \geqq 0, x \in C\}$. Hence,

$$\begin{aligned} 0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f) &\Leftrightarrow 0 \in \operatorname{int}(\operatorname{dom} g - A \operatorname{dom} f) \\ &\Leftrightarrow \operatorname{cone}(\operatorname{dom} g - A \operatorname{dom} f) = \mathbb{R}^n. \end{aligned} \tag{3.2}$$

It follows that (R) and (S) are equivalent when $Y = \mathbb{R}^n$.

We recall that for a set C in $\mathbb{R}^n$, $y \in \operatorname{ri} C$ if and only if 0 is an interior point of $C - y$ relative to the affine hull of $(C - y)$. It turns out that

$$0 \in \operatorname{ri} C \quad \text{if and only if cone } C \text{ is a (closed) subspace of } \mathbb{R}^n. \tag{3.3}$$

Thus, in view of (3.2), the constraint qualification

$$0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f) \tag{RR}$$

is weaker than (R). However, the following duality result for a Banach space can be deduced from Theorem 2.2. (The proof is omitted since it is similar to the one given for Theorem 3.5.) For a standard proof see, e.g., [6] or [12].

THEOREM 3.1. *Suppose that X is a locally convex space and $Y = \mathbb{R}^n$. If* (RR) *holds, then* $\inf(\mathrm{P}) = \max(\mathrm{D})$.

The remainder of this subsection is devoted to the comparison of constraint qualifications for *linearly constrained convex programs.* In a recent work, Borwein and Wolkowicz [7] introduced a constraint qualification for the linearly constrained convex program:

$$\inf\{f(x) : Ax = b, x \in S\} \tag{L}$$

where S is a convex cone in X, i.e., $S + S \subset S$ and $\lambda S \subset S$ for all $\lambda \geqq 0$ and $b \in \mathbb{R}^n$. The feasible set of (L) is

$$F = \{x \in X : Ax = b, x \in S\}$$

and it is assumed that $F \neq \emptyset$. Note that problem (L) is a special case of problem (P) obtained by replacing f by $f + \delta(\cdot \mid S)$ and g by $\delta(\cdot \mid \{b\})$, where $\delta(\cdot \mid E)$ denotes the indicator function of a given set E. In this setting, the corresponding dual reduces to the concave finite-dimensional problem

$$\sup\{b^T y - (f + \delta(\cdot \mid S))^*(A^* y) : y \in \mathbb{R}^n\}. \tag{DL}$$

In what follows, $\overline{\operatorname{cone}}\, E$ stands for the closure of the cone generated by the set E. The following result is proved in [7].

THEOREM 3.2. *Let X be a normed linear space. If*

$$\overline{\operatorname{cone}}(F - S) = X, \tag{BW}$$

then $\inf(\mathrm{L}) = \max(\mathrm{DL})$.

For problem (L), the constraint qualification (R) reads (by (3.2)):

$$\operatorname{cone}(b - A(S)) = \mathbb{R}^n.$$

We show below that (BW) is stronger than (R), i.e., (BW)⇒(R). We assume that $A: X \to \mathbb{R}^n$ is *onto*. This assumption is not really restrictive since we can always assume that Y is the range of A, and, after a unitary change, write $Y = \mathbb{R}^n$. Also, we introduce another interesting constraint qualification (to be called (BW)′) related to (BW) and equivalent to (RR). In what follows, the kernel of A is denoted by Ker A.

THEOREM 3.3. *Consider the following constraint qualifications*:

(BW) $\overline{\operatorname{cone}}(F - S) = X,$

(R) $0 \in \operatorname{core}(b - A(S)),$

(RR) $0 \in \operatorname{ri}(b - A(S)),$

(BW)′ $\operatorname{cone}(F - S) + \operatorname{Ker}(A)$ *is a closed subspace of* X.

Then (BW)⇒(R)⇒(RR)⇔(BW)′.

Proof. (BW)⇒(R): Suppose that (BW) holds. Then,

$$\begin{aligned}\mathbb{R}^n = A(X) &= A(\overline{\operatorname{cone}}(F - S))\\ &\subset \overline{\operatorname{cone}}\, A(F - S)\\ &= \overline{\operatorname{cone}}(b - A(S)) \subset \mathbb{R}^n.\end{aligned}$$

So, the closure of the convex set $\operatorname{cone}(b - A(S))$ is $\mathbb{R}^n$. A simple separation argument shows that $\operatorname{cone}(b - A(S)) = \mathbb{R}^n$. Hence $0 \in \operatorname{core}(b - A(S))$ by (3.2).

(R)⇒(RR). The proof follows from (3.2) and (3.3).

(RR)⇒(BW)′. From (3.3) we see that $\operatorname{cone}(b - A(S))$ is a (closed) subspace of $\mathbb{R}^n$. Hence

$$\begin{aligned}\operatorname{cone}(F - S) + \operatorname{Ker}(A) &= A^{-1}[A \operatorname{cone}(F - S)]\\ &= A^{-1}[\operatorname{cone}(b - A(S))]\end{aligned}$$

is a closed subspace of X.

(BW)′⇒(RR). If $\operatorname{cone}(F - S) + \operatorname{Ker}(A)$ is a subspace of X, then

$$\operatorname{cone}(b - A(S)) = A[\operatorname{cone}(F - S) + \operatorname{Ker}(A)] \text{ is a subspace of } \mathbb{R}^n. \qquad \square$$

In view of the above result, it is clear that Theorem 3.2 is a special case of Theorem 3.1. The following example shows that (R) need not imply (BW) even when X is finite-dimensional.

Example 3.1. Let $X = \mathbb{R}^2$, $S = [-1, 1] \times \{0\}$, $A: \mathbb{R}^2 \to \mathbb{R}$ such that $A(x, y) = x$ and $b = 0$ so that $F = (0, 0)$. Clearly (R) holds since $0 \in \operatorname{core}(b - A(S)) = \operatorname{int}([-1, 1])$ in $\mathbb{R}$, while $\overline{\operatorname{cone}}(F - S) = \mathbb{R} \neq X$.

In a recent work, Borwein and Lewis [6], introduced the notion of *quasi-relative interior.* As we shall see below, this notion is useful in the verification of (RR).

DEFINITION 3.1 [6]. Let X be a topological vector space. For a convex $C \subset X$, the quasi-relative interior of C (qri C) is the set of those $x \in C$ for which $\overline{\operatorname{cone}}(C - x)$ is a subspace.

This notion is studied extensively in [6]. For any set E, in finite dimension, $\overline{\operatorname{cone}}\, E$ is a closed subspace if and only if cone E is a subspace; hence the notion of quasi-relative interior coincides with the relative interior. However, what makes the qri useful is the following important property.

PROPOSITION 3.2 [6]. *Suppose* $A: X \to \mathbb{R}^n$ *is a continuous linear map. Then,* $A(\text{qri } C) \subset \text{ri }(AC)$, *and if* $\text{qri }(C) \neq \emptyset$, *then* $A(\text{qri } C) = \text{ri }(AC)$.

Using the above proposition, we see that when $\text{qri } S \neq \emptyset$, the constraint qualification (RR), namely $b \in \text{ri } A(S)$, reads

$$\text{(BL)} \qquad \exists x \in \text{qri }(S) \text{ such that } Ax = b.$$

Thus in view of Theorem 3.1 we have the following theorem.

THEOREM 3.4. *Suppose that X is locally convex and* $\text{qri } S \neq \emptyset$. *If* (BL) *holds then* $\inf (\text{L}) = \max (\text{DL})$.

We wish to emphasize the importance of this result. The importance lies in the fact that in many applications (for example, when S is the nonnegative cone L_p^+, $1 \leqq p < \infty$), $\text{qri } S \neq \emptyset$ while $\text{ri } S = \emptyset$.

3.2. *A* is a linear operator with infinite-dimensional range. We have shown in the previous subsection that the core condition provides a systematic mechanism for constructing old and new constraint qualifications. The natural question is now to see whether similar results can be derived for the general case. Unless otherwise specified, in the sequel we assume that X and Y are Banach spaces. We recall that for a convex subset C of an infinite-dimensional vector space X:

$$0 \in \text{core } C \Leftrightarrow \text{cone } C = X \quad \text{and} \quad 0 \in \text{int } C \Rightarrow \text{cone } C = X.$$

When $A: X \to Y$ with $Y = \mathbb{R}^n$, we were able to relax (R) by (RR) using the notion of relative interior instead of that of interior and core. Following the same methodology, by introducing the concept of *intrinsic core* we may establish an appropriate (CQ) when Y is a Banach space.

DEFINITION 3.2 [9]. The core of C relative to aff C, the affine hull of C, is called the intrinsic core of C and is written icr C.

When $C \subset X$ is convex and X is finite-dimensional we have

$$\text{icr } C = \text{ri } C.$$

Recall that aff C is $x + \text{span }(C - x)$ for any fixed $x \in C$, where $\text{span }(C - x)$ is the smallest subspace of X that contains $(C - C)$.

PROPOSITION 3.3. *Let C be a convex subset of X. Then,*

$$x \in \text{icr } C \Leftrightarrow \text{cone }(C - x) = \text{aff }(C - x) = \text{aff }(C - C).$$

Proof. By Definition 3.2, we have $x \in \text{icr } C \Leftrightarrow \text{cone }(C - x) = \text{aff }(C - x)$. But, when $x \in C$, $\text{aff }(C - x) = \text{aff }((C - x) - (C - x)) = \text{aff }(C - C)$. □

In the finite-dimensional setting, $x \in \text{ri } C$ if and only if $\text{cone }(C - x) = \text{aff }(C - C)$ and further, $\text{aff }(C - C)$ is closed. Thus, a natural (CQ) in a general setting should now be:

$$x \in \text{icr } C \quad \text{and} \quad \text{aff }(C - C) \quad \text{is a closed subspace.}$$

For the problem (P) this general constraint qualification reads:

$$\text{(GCQ)} \qquad 0 \in \text{icr }(\text{dom } g - A \text{ dom } f) \quad \text{and}$$

$$\text{aff }(\text{dom } g - A \text{ dom } f) \quad \text{is a closed subspace.}$$

From Theorem 2.2, we know that a strong duality result is guaranteed if

$$0 \in \text{core }(\text{dom } g - A \text{ dom } f).$$

Since $(\operatorname{dom} g - A \operatorname{dom} f)$ is a convex subset of Y, condition (R) is equivalent to:

$$\operatorname{cone}(\operatorname{dom} g - A \operatorname{dom} f) = Y.$$

In the context of problem (P), if $0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f)$, then

$$0 \in \operatorname{icr}(\operatorname{dom} g - A \operatorname{dom} f) \quad \text{and} \quad \operatorname{aff}(\operatorname{dom} g - A \operatorname{dom} f) = Y \ \text{(a closed subspace)}.$$

Thus, the constraint qualification (R) is stronger than the constraint qualification (GCQ). However, as we see below, the strong duality result under (GCQ) can be deduced from Theorem 2.2. In this sense, (GCQ) and (R) are equivalent.

THEOREM 3.5. *Let X, Y be Banach spaces and let $A: X \to Y$ be a continuous map. Let $f: X \to (-\infty, +\infty]$ and let $g: Y \to (-\infty, +\infty]$ be proper, lower semicontinuous convex functions. Suppose that* (GCQ) *holds, i.e.,*

$$0 \in \operatorname{icr}(\operatorname{dom} g - A \operatorname{dom} f) \quad \textit{and} \quad \operatorname{aff}(\operatorname{dom} g - A \operatorname{dom} f) \quad \textit{is a closed subspace.}$$

Then, $\inf(\mathrm{P}) = \max(\mathrm{D})$.

Proof. Let $x_0 \in \operatorname{dom} f$ such that $Ax_0 \in \operatorname{dom} g$. Define

$$F(x) := f(x + x_0) \quad \text{and} \quad G(y) := g(y + Ax_0).$$

Then, we have $\operatorname{dom} F = \operatorname{dom} f - x_0$, $\operatorname{dom} G = \operatorname{dom} g - Ax_0$,

$$F^*(x^*) = f^*(x^*) - \langle x^*, x_0\rangle, \qquad G^*(y^*) = g^*(y^*) - \langle y^*, Ax_0\rangle,$$

$$\inf_{x\in X} \{f(x) + g(Ax)\} = \inf_{x \in X} \{F(x) + G(Ax)\},$$

and

$$\sup_{y^* \in Y^*} \{-g^*(y^*) - f(-{}^*y^*)\} = \sup_{y^*\in Y^*} \{-G^*(y^*) - F^*(-A^*y^*)\}.$$

Further, if in the last equation, sup is attained in the right-hand side, then sup is attained in the left-hand side. Also,

$$\operatorname{dom} G - A(\operatorname{dom} F) = \operatorname{dom} g - A(\operatorname{dom} f).$$

We see that $F(0) = f(x_0)$ and $G(0) = g(Ax_0)$ are real numbers. Thus, without loss of generality, we can assume that

$$0 \in \operatorname{dom} f \quad \text{and} \quad 0 \in \operatorname{dom} g. \tag{3.4}$$

Since (GCQ) holds, by Proposition 3.3,

$$M := \operatorname{cone}(\operatorname{dom} g - A \operatorname{dom} f) = \operatorname{aff}(\operatorname{dom} g - A \operatorname{dom} f).$$

It is given that M is a closed subspace of Y. Then $\hat{X} = A^{-1}(M)$ is a Banach space, $\operatorname{dom} f \subset \hat{X}$, $\operatorname{dom} g \subset M$ from (3.4). We replace X by $\hat{X}$, Y by M in problem (P) and regard A as a mapping from $\hat{X} \to M$. For the corresponding pair of transformed problems (P′), (D′)

$$\text{(P}'\text{)} \qquad \inf_{x \in \hat{X}} \{f(x) + g(Ax)\}$$

$$\text{(D}'\text{)} \qquad \sup_{y^* \in M^*} \{-g^*(y^*) - f^*(-A^*y^*)\}$$

the condition (R) holds, namely,

$$0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f).$$

By Theorem 2.2, we see that inf (P′) = max (D′). To complete the proof it remains to show that inf (P′) = inf (P) and max (D′) = max (D). Clearly, since dom $f \subset \hat{X}$, it follows that inf (P′) = inf (P).

Let $v^* \in M^*$ be such that

$$\max (\mathrm{D}') = -g^*(v^*) - f^*(-A^*v^*).$$

For any $y^* \in Y^*$, let Γy^* denote the restriction of y^* to M. It is easily seen that

$$g^*(y^*) = g^*(\Gamma y^*) \quad (\text{since dom } g \subset M)$$

and

$$f^*(A^*(\Gamma y^*)) = f^*(A^*y^*) \quad (\text{since } A\hat{X} \subset M).$$

Therefore,

$$\begin{aligned}\sup_{y^* \in Y^*} \{-g^*(y^*) - f^*(-A^*y^*)\} &= \sup_{y^* \in Y^*} \{-g^*(\Gamma y^*) - f^*(-A^*(\Gamma y^*))\} \\ &= \sup_{w^* \in \Gamma(Y^*)} \{-g^*(w^*) - f^*(-A^*w^*)\}.\end{aligned}$$

By the Hahn–Banach extension theorem, $\Gamma(Y^*) = M^*$ and hence

$$\sup (D) = \sup_{w^* \in M^*} \{-g^*(w^*) - f^*(-A^*w^*)\} = \sup (\mathrm{D}') = -g^*(v^*) - f^*(-A^*v^*).$$

But it is clear that sup (D) is attained by any continuous linear extension of v^* to Y. Hence the above equality gives max (D) = max (D′). □

A proof of the above result for the special case, $X = Y$ and A = Identity, appears in Attouch and Brezis [2]. The proof there is based on the Banach–Dieudonne–Krein–Smulian theorem [8, Thm. V.5.7]. Based on this special case, using the notion of strong quasi-relative interior (see Definition 3.3 below), Borwein et al. [5] prove Theorem 3.5. Using a completely different approach, Zalinescu [15, Cor. 4] shows that the above theorem of Attouch and Brezis is valid when X and Y are Fréchet spaces. It is important to note that, by modifying the argument of [5, Thm. 3.1], *Theorem* 3.5 *will remain valid when X and Y are Fréchet spaces.* At this juncture, we wish to mention an earlier work with applications to perturbational duality by Borwein [4]. We thank one of the referee's for bringing this reference to our attention.

In [5], the notion of strong quasi-relative interior is introduced as a natural extension of the quasi-relative interior.

DEFINITION 3.3. For a convex subset $C \subset X$, the strong quasi-relative interior of C is the set of those $x \in C$ for which cone $(C - x)$ is a closed subspace.

When X is finite-dimensional we have

$$\operatorname{sqri} C = \operatorname{ri} C = \operatorname{qri} C = \operatorname{icr} C.$$

In the context of problem (P), the following (CQ) is proposed in [5]:

$$0 \in \operatorname{sqri} (\operatorname{dom} g - A \operatorname{dom} f).$$

As the following proposition shows, the above constraint qualification given in terms of the strong quasi-relative interior is equivalent to the constraint qualification (GCQ).

PROPOSITION 3.4.

$$\left\{\begin{array}{c} x \in \operatorname{icr}(C) \\ \operatorname{aff}(C - x) \textit{ is a closed subspace} \end{array}\right\} \Leftrightarrow x \in \operatorname{sqri} C.$$

Proof. If $x \in \text{icr } C$ and $\text{aff}(C-x)$ is a closed subspace, then by Proposition 3.3 $\text{cone}(C-x) = \text{aff}(C-x)$ is a closed subspace, and hence $x \in \text{sqri } C$ from Definition 3.3. On the other hand, if $x \in \text{sqri } C$, then $\text{cone}(C-x)$ is a closed subspace. But, in this situation, $\text{cone}(C-x) \subset \text{aff}(C-x)$ and $\text{aff}(C-x) \subset \text{cone}(C-x)$. Hence $\text{aff}(C-x) = \text{cone}(C-x)$ and thus $x \in \text{icr } C$ by Proposition 3.3. □

The name strong quasi-relative interior (sqri) has been introduced as a natural generalization of the quasi-relative interior. However, our results below demonstrate that in fact the strong quasi-relative interior is "closer" to the relative interior. Recall that for a set C in a topological space X, $y \in \text{ri } C$ if and only if 0 is an interior point of $C-y$ relative to the closure of the affine hull of $C-y$, see [11].

We prove the following results in the general setting of Baire spaces. Recall that X is a Baire space if it is locally convex and the intersection of every countable collection of dense open subsets of X is dense in X. Every closed subspace of such a space is Baire and such a space is barrelled, i.e., each absorbing, convex, circled, and closed subset of X is a neighborhood of the origin. Examples of Baire spaces are Fréchet spaces and Banach spaces (see [10]).

THEOREM 3.6. *Let X be a Baire space and C be a closed convex set in X. Then*

$$\text{sqri } C \subset \text{ri } C.$$

Proof. If $\text{sqri } C = \emptyset$, then there is nothing to prove. Let $\hat{x} \in \text{sqri } C$ so that $Y := \text{cone}(C - \hat{x})$ is a closed subspace of X. Let $K := C - \hat{x}$. Note that $0 \in K$ and K is absorbing in Y. Let $B = \bigcap_{|\lambda| \geqq 1} \lambda K$ be the balanced core of K (see [10, p. 80]). We note that

(i) $0 \in B \subset K$,

(ii) B is balanced, closed convex in Y,

(iii) B is absorbing in Y.

Statement (i) follows immediately from the definition of B and (ii) follows since each λK is closed and convex. To see (iii), let $y \in Y$. Since K is absorbing we can find $\mu > 0$ such that $\pm \mu y \in K$. Then from the convexity of K it follows that for every $|\lambda| \geqq 1$, $\mu y / \lambda \in K$, and so $\mu y \in B$.

Since X is Baire, Y is also Baire and hence barrelled. B, being an absorbing, balanced, closed, and convex set in Y, is a neighborhood of 0 in Y, and hence

$$0 \in \text{int}_Y B \subset \text{int}_Y K = \text{int}_Y (C - \hat{x})$$

where int_Y denotes the interior relative to Y. Therefore $\hat{x} \in \text{ri } C$ and the proof is complete. □

We remark that the above result may not hold for barrelled spaces since a closed subspace of a barrelled space need not be barrelled.

COROLLARY 3.1. *Let E be a convex set in X where X is a Baire space. Then*

$$\text{sqri } E \subset \text{ri } \bar{E}.$$

Proof.

$$\begin{aligned} \hat{x} \in \text{sqri } E &\Rightarrow \text{cone}(E - \hat{x}) =: Y \text{ is a closed subspace} \\ &\Rightarrow \text{cone}(\overline{E - \hat{x}}) = Y \text{ (closure with respect to } X) \\ &\Rightarrow \text{cone}(\bar{E} - \hat{x}) = Y \\ &\Rightarrow \hat{x} \in \text{sqri } \bar{E} \subset \text{ri } \bar{E}. \end{aligned}$$

Since $\bar{E}$ is closed convex in X, the last inclusion follows from the previous theorem. □

We have seen, as a consequence of Proposition 3.4, that the constraint qualification expressed in terms of the strong quasi-relative interior is equivalent to (GCQ). The above corollary suggests looking at the following weaker condition to guarantee the strong duality result:

$$0 \in \operatorname{ri} \overline{(\operatorname{dom} g - A \operatorname{dom} f)}. \tag{3.5}$$

The following examples demonstrate that

(i) equality may not hold in Corollary 3.1, and

(ii) the strong duality result may not hold under (3.5).

Example 3.2. Let X be an infinite-dimensional Banach space. Let $\phi: X \to \mathbb{R}$ be a noncontinuous linear functional so that $S := \operatorname{Ker} \phi$ is a dense subspace of X. For any $e \in X \setminus S$, we see that

$$X = S + \mathbb{R}e \quad \text{and} \quad S \cap \mathbb{R}e = \{0\}.$$

Let $E := S + [0, 1]e$ where $[0, 1] = \{\lambda: 0 \leqq \lambda \leqq 1\}$. Clearly, $\bar{E} \supset \bar{S} = X$ and hence $\bar{E} = X$, so that $\operatorname{ri} \bar{E} = \operatorname{ri} X = \operatorname{int} X = X$ contains 0.

To get a contradiction, suppose that $0 \in \operatorname{sqri} E$. Then $0 \in \operatorname{icr} E$, i.e., $0 \in \operatorname{core} E$ relative to aff E. Now $-e \in \operatorname{aff} E$ and hence there exists $\lambda > 0$ such that

$$-\lambda e \in E = S + [0, 1]e.$$

Thus, $-\lambda e = s + \mu e$ for some $s \in S$ and $\mu \in [0, 1]$. This implies that $e \in S$, a contradiction. Thus $0 \in \operatorname{ri} \bar{E}$ while $0 \notin \operatorname{sqri} E$. We note that aff $E = X$ is closed while $0 \notin \operatorname{icr} E$.

Example 3.3. As in [11, p. 77] we consider the following setting:

$$X = l_2 = \left\{ x = (x_1, \cdots, x_n, \cdots): x_n \in \mathbb{R}, \sum_1^\infty x_n^2 < \infty \right\},$$

$$C = \{x \in l_2: x_{2n-1} + x_{2n} = 0, \forall n = 1, 2, \cdots,\},$$

$$S = \{x \in l_2: x_{2n} + x_{2n+1} = 0, \forall n = 1, 2, \cdots,\}.$$

Clearly, C and S are closed subspaces of X and $C \cap S = \{0\}$. Define f and g on X by $f(x) = \delta(x \mid C)$ and $g(x) = x_1$ if $x \in S$ and ∞ otherwise. It is easily seen that f and g are convex and lower semicontinuous on X with dom $f = C$ and dom $g = S$. We now compute the conjugates of f and g. Since C is a subspace it is easy to see that

$$f^*(x^*) = \delta(x^* \mid C^\perp)$$

where $C^\perp$ is the orthogonal complement of C. Also we have,

$$\begin{aligned} g^*(x^*) &= \sup_{x \in S} \{\langle x, x^* \rangle - x_1\} \\ &= \sup_{x \in S} \langle x^* - e_1, x \rangle \quad (\text{where } e_1 = (1, 0, \cdots)) \\ &= \begin{cases} 0 & \text{if } x^* - e_1 \in S^\perp \\ \infty & \text{if } x^* - e_1 \notin S^\perp \end{cases} \\ &= \delta(x^* \mid e_1 + S^\perp). \end{aligned}$$

We claim that the following are true:

(i) $0 \in \operatorname{ri} \overline{(\operatorname{dom} g - \operatorname{dom} f)}$,

(ii) $0 \notin \operatorname{sqri} (\operatorname{dom} g - \operatorname{dom} f)$,

(iii) $\inf_{x \in X} \{f(x) + g(x)\} = 0$,

(iv) $\sup_{x^* \in X^*} \{-g^*(x^*) - f^*(-x^*)\} = -\infty$.

It follows from these that the strong duality result fails to hold under the weaker constraint qualification

$$0 \in \operatorname{ri} \overline{(\operatorname{dom} g - \operatorname{dom} f)}.$$

To see (i), we show that $(\text{dom } g - \text{dom } f) = S - C$ is dense in X. To this end, let $x = (x_n)$ be orthogonal to $S - C$. Since $e_{2n-1} - e_{2n} \in C$ and $e_{2n} - e_{2n+1} \in S$ for all $n = 1, 2, \cdots$, we see that $x_{2n-1} - x_{2n} = 0$ and $x_{2n} - x_{2n+1} = 0$ for all n. Since $x \in l_2$ we must have $x = 0$ so that $S - C$ is dense in X.

Statement (ii) follows immediately from the observation that

$$\text{aff}\,(\text{dom } g - \text{dom } f) = S - C \quad \text{is not closed.}$$

Note however that $0 \in \text{icr}\,(\text{dom } g - \text{dom } f)$.

Now $\inf_{x \in X} \{f(x) + g(x)\} = \inf_{x \in \text{dom } g \cap \text{dom } f} = f(0) + g(0) = 0$ gives (iii). We now show that

$$\text{dom } g^* \cap \text{dom } f^* = \emptyset \tag{3.6}$$

so that

$$\sup_{x^* \in X^*} \{-g(x^*) - f^*(-x^*)\} = \sup_{x^* \in \text{dom } g^* \cap \text{dom } f^*} \{-g(x^*) - f^*(-x^*)\} = -\infty$$

giving (iv). To see (3.6), suppose that

$$(x_n) = x \in \text{dom } g^* \cap \text{dom } f^* = (e_1 + S^\perp) \cap C^\perp.$$

Then, as in the proof of (i), we get $x_{2n-1} - x_{2n} = 0$ and $x_{2n} - x_{2n+1} = 0$ for all $n = 1, 2, \cdots$. Hence, $x = 0$. But then $0 \in e_1 + S^\perp$ implies $-e_1 \in S^\perp$, which is false since $e_1 \in S$.

Our last result resembles Proposition 3.2 and partially addresses the question of verifying (GCQ).

PROPOSITION 3.5. *Let X be a locally convex topological vector space and let Y be a Baire space. Let $A: X \to Y$ be a continuous linear operator and C be a convex set in X. Then*

$$\text{sqri } A(C) \subset \overline{A(\text{qri } C)}$$

whenever $\text{qri } C \neq \emptyset$.

Proof. From Corollary 3.1 we have

$$\text{sqri } A(C) \subset \text{ri } \overline{A(C)}.$$

Let $x_1 \in \text{qri } C$ and $y \in \text{ri } \overline{A(C)}$. Since y and Ax_1 belong to $\overline{A(C)}$, we have for some $\varepsilon > 0$, $\varepsilon(y - Ax_1) \in \overline{A(C)} - y$, i.e.,

$$y - \varepsilon(Ax_1 - y) \in \overline{A(C)}.$$

Let V be any convex, balanced neighborhood of 0. Then there exists $u \in V$ such that

$$y - \varepsilon(Ax_1 - y) + u = Ax_2 \quad \text{for some } x_2 \in C$$

and thus

$$y + \frac{u}{1+\varepsilon} = \frac{Ax_2 + \varepsilon Ax_1}{1+\varepsilon}$$

from which it follows that

$$y + \frac{u}{1+\varepsilon} \in A(\text{qri } C)$$

since $(x_2 + \varepsilon x_1)/(1+\varepsilon) \in \text{qri } C$ by [6, Lemma 2.9]. Now

$$u/(1+\varepsilon) \in V \quad \text{and hence} \quad (y + V) \cap A \text{ qri } C \neq \emptyset$$

implying that $y \in \overline{A(\text{qri } C)}$. $\square$

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