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Commutation principles for optimization problems on spectral sets in Euclidean Jordan algebras

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Abstract
The commutation principle of Ramírez et al. (SIAM J Optim 23:687–694, 2013) proved in the setting of Euclidean Jordan algebras says that when the sum of a real valued function h and a spectral function Φ is minimized/maximized over a spectral set E, any local optimizer a at which h is Fréchet differentiable operator commutes with the derivative $h'(a)$. In this note, we describe some analogs of the above result by assuming the existence of a subgradient in place of the derivative (of h) and obtaining strong operator commutativity relations. We show, for example: if a solves the problem $\max_E (h + \Phi)$, then a strongly operator commutes with every element in the subdifferential of h at a; If E and h are convex and a solves the problem $\min_E h$, then a strongly operator commutes with the negative of some element in the subdifferential of h at a. These results improve known operator commutativity relations for linear h and for solutions of variational inequality problems. We establish these results via a geometric commutation principle that is valid not only in Euclidean Jordan algebras, but also in a broader setting.

Keywords Euclidean Jordan algebra · Spectral sets/functions · Commutation principle · Variational inequality problem · Normal cone · Subdifferential

1 Introduction

Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product [4] and $\lambda : \mathcal{V} \to \mathcal{R}^n$ denote the eigenvalue map (which takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). For any $a \in \mathcal{V}$, we define its λ-orbit by $[a] := \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\}$. A set E in $\mathcal{V}$ is said to be a *spectral set* if it is of the form $E = \lambda^{-1}(Q)$ for some (permutation invariant) set Q

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in $\mathcal{R}^n$ or, equivalently, a union of λ-orbits. A function $\Phi : \mathcal{V} \to \mathcal{R}$ is said to be a *spectral function* if it is of the form $\Phi = \phi \circ \lambda$ for some (permutation invariant) function $\phi : \mathcal{R}^n \to \mathcal{R}$.

Ramírez et al. [14] prove the following commutation principle in $\mathcal{V}$:

Theorem 1 *Suppose a is a local optimizer of the problem* $\min/\max_E (h + \Phi)$, *where E is a spectral set,* Φ *is a spectral function, and* $h : \mathcal{V} \to \mathcal{R}$. *If h is Fréchet differentiable at a, then a and* $h'(a)$ *operator commute.*

Gowda and Jeong [6] extended the above result by assuming that E and Φ are invariant under the automorphisms of $\mathcal{V}$ and stated an analogous result in the setting of normal decomposition systems. Subsequently, certain modifications (such as replacing the sum by other combinations) and applications were given by Niezgoda [13].

The main objective of this note is to describe some analogs of the above commutation principle by assuming the existence of a subgradient in place of the derivative (of h). In each analog, this change results in a stronger commutativity relation. We derive these analogs via a geometric commutation principle. To elaborate, we first recall some definitions.

- We say that elements a and b *operator commute* in $\mathcal{V}$ if there exists a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ such that the spectral decompositions of a and b are given by

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \quad \text{and} \quad b = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n,$$

  where $a_1, a_2, \ldots, a_n$ are the eigenvalues of a and $b_1, b_2, \ldots, b_n$ are the eigenvalues of b. If, additionally, above decompositions hold with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, we say that a and b *strongly operator commute* (also said to 'simultaneously diagonalizable' [12] or said to have 'similar joint decomposition' [1]).
- Given a (nonempty) set S in $\mathcal{V}$ and $a \in S$, we define the *normal cone* of S at a by

$$N_S(a) := \{d \in \mathcal{V} : \langle d, x - a\rangle \leq 0 \text{ for all } x \in S\}.$$

- Let $h : \mathcal{V} \to \mathcal{R} \cup \{\infty\}$, $S \subseteq \mathcal{V}$, and $a \in S \cap \operatorname{dom} h$. We define the *subdifferential of h at a relative to S* by

$$\partial_S h(a) := \{d \in \mathcal{V} : h(x) - h(a) \geq \langle d, x - a\rangle \text{ for all } x \in S\};$$

  any element of $\partial_S h(a)$ will be called a *S-subgradient of h at a*. Finally, when $S = \mathcal{V}$, we define the *subdifferential of h at a* by

$$\partial h(a) := \{d \in \mathcal{V} : h(x) - h(a) \geq \langle d, x - a\rangle \quad \text{for all} \quad x \in \mathcal{V}\}.$$

  We note that subdifferentials may be empty and $\partial h(a) \subseteq \partial_S h(a)$. We also note [15] that when h (defined on all of $\mathcal{V}$) is convex, the subdifferential is nonempty, compact, and convex; if h is also Fréchet differentiable at a, then $\partial h(a) = \{h'(a)\}$.

Our primary examples of Euclidean Jordan algebras are $\mathcal{R}^n$, $\mathcal{S}^n$, and $\mathcal{H}^n$. In the algebra $\mathcal{R}^n$ (with componentwise multiplication as Jordan product and usual inner product), spectral sets/functions (also called symmetric sets/functions) are precisely those that are invariant under the action of permutation matrices. In this algebra, any two elements operator commute; strong operator commutativity requires simultaneous (permutation) rearrangement with decreasing components. For example, in $\mathcal{R}^2$, the elements $(1, 0)$ and $(0, 1)$ operator commute, but not strongly. In the algebras $\mathcal{S}^n$ (of all $n \times n$ real symmetric matrices) and $\mathcal{H}^n$ (of all $n \times n$ complex Hermitian matrices), the Jordan product and the inner product are given, respectively, by

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle := \mathrm{tr}(XY).$$

In $\mathcal{S}^n$ (in $\mathcal{H}^n$) spectral sets are those that are invariant under linear transformations of the form $X \mapsto UXU^*$, where U is an orthogonal (respectively, unitary) matrix. Also, two matrices X and Y in $\mathcal{S}^n$ (in $\mathcal{H}^n$) operator commute if and only if $XY = YX$, or equivalently, there exists an orthogonal (respectively, unitary) matrix U such that $X = UD_1U^*$ and $Y = UD_2U^*$, where D_1 and D_2 are diagonal matrices consisting, respectively, of eigenvalues of X and Y. If the diagonal vectors of D_1 and D_2 have decreasing components, then X and Y strongly operator commute.

We now state our *geometric commutation principle*:

Theorem 2 *Suppose E is a spectral set in $\mathcal{V}$ and $a \in E$. Then, a strongly operator commutes with every element in the normal cone $N_E(a)$. In particular, a strongly operator commutes with every element in the normal cone $N_{[a]}(a)$, where $[a]$ is the λ-orbit of a.*

When E is also convex, one may see this as a consequence of a result on subgradients of convex spectral functions such as Corollary 31 in [1] or Theorem 5.5 in [2]; (as noted by the Referee) the general case reduces to this via the observation that the closed convex hull of a spectral set is a spectral set. Our objectives in this paper are: to derive Theorem 2 as a simple consequence of (what we call) Fan-Theobald-von Neumann inequality (1) together with its equality case, see Theorem 4 below, and to indicate how it can be formulated in a general setting.

Based on Theorem 2, we derive our commutation principles for optimization problems:

Theorem 3 *Suppose E is a spectral set in $\mathcal{V}$, $\Phi : \mathcal{V} \to \mathcal{R}$ is a spectral function, and $h : \mathcal{V} \to \mathcal{R}$.*

(i) *If a is an optimizer of the problem $\max\limits_{E} (h+\Phi)$, then a strongly operator commutes with every element in $\partial_{[a]} h(a)$, in particular, with those in $\partial_E h(a)$ and $\partial h(a)$.*
(ii) *If E and h are convex and a is an optimizer of the problem $\min\limits_{E} h$, then a strongly operator commutes with the negative of some element in $\partial h(a)$.*

We note that Theorems 1 and 3 are analogous but with different assumptions and conclusions; they cannot, generally, be compared, i.e., neither one implies the other.

Now, specializing h in the above result to a linear function leads to an interesting consequence for variational inequality problems. To elaborate, consider a function $G : \mathcal{V} \to \mathcal{R}$ and a set $E \subseteq \mathcal{V}$. Then, the *variational inequality problem* [3], $\mathrm{VI}(G, E)$, is to find an element $a \in E$ such that

$$\langle G(a), x - a\rangle \geq 0 \quad \text{for all} \quad x \in E.$$

When E is a closed convex cone, this becomes a *cone complementarity problem*. We now state some simple consequences of Theorem 3.

Corollary 1 *Suppose E is a spectral set in $\mathcal{V}$, Φ is a spectral function, and $h : \mathcal{V} \to R$ is convex and Fréchet differentiable. Let $c \in \mathcal{V}$ and $G : \mathcal{V} \to \mathcal{R}$. Then, the following statements hold:*

(i) *If a is an optimizer of $\max\limits_E (h + \Phi)$, then a strongly operator commutes with $h'(a)$.*
(ii) *If $h(x) := \langle c, x\rangle$ on $\mathcal{V}$ and a is an optimizer of $\max\limits_E (h + \Phi)$, then a strongly operator commutes with c. Moreover, the maximum value is $\langle \lambda(c), \lambda(a)\rangle + \Phi(a)$.*
(iii) *If $h(x) := \langle c, x\rangle$ on $\mathcal{V}$ and a is an optimizer of $\min\limits_E (h+\Phi)$, then a strongly operator commutes with $-c$. Moreover, the minimum value is $\langle \widetilde{\lambda}(c), \lambda(a)\rangle + \Phi(a)$, where $\widetilde{\lambda}(c) := -\lambda(-c)$.*
(iv) *If a solves $VI(G, E)$, then a strongly operator commutes with $-G(a)$.*

In our proofs, we employ standard ideas/results from convex analysis [15] and the following key result from Euclidean Jordan algebras [1,8,12]:

Theorem 4 *For all $x, y \in \mathcal{V}$,*

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle. \tag{1}$$

Equality holds in (1) if and only if x and y strongly operator commute.

While our results are stated in the setting of Euclidean Jordan algebras for simplicity and ease of proofs, it is possible to describe these in a general setting/system. This general system is formulated by turning (1) into an axiom and defining the concept of commutativity via the equality in (1). The precise formulation is as follows. A *Fan-Theobald-von Neumann system* (FTvN system, for short) [5], is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a norm preserving map satisfying the property

$$\max\Big\{\langle c, x\rangle : x \in [u]\Big\} = \langle \lambda(c), \lambda(u)\rangle \quad (\forall\, c, u \in \mathcal{V}), \tag{2}$$

with $[u] := \{x \in \mathcal{V} : \lambda(x) = \lambda(u)\}$. This property is a combination of an inequality and a condition for equality. The inequality $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$ (that comes from (2)) is referred to as the *Fan-Theobald-von Neumann inequality* and the equality

$$\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$$

defines *commutativity* of x and y in this system. Spectral sets in this system are defined as sets of the form $E = \lambda^{-1}(Q)$ for some $Q \subseteq W$; spectral functions are of the form $\Phi = \phi \circ \lambda$ for some $\phi : W \to \mathcal{R}$. Examples of such systems include [5]:

- The triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$, where $\mathcal{V}$ is a Euclidean Jordan algebra of rank n carrying the trace inner product with $\lambda : \mathcal{V} \to \mathcal{R}^n$ denoting the eigenvalue map. Commutativity in this FTvN system reduces to strong operator commutativity in the algebra $\mathcal{V}$.
- The triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$, where $\mathcal{V}$ is a finite dimensional real vector space and p is a real homogeneous polynomial of degree n that is hyperbolic with respect to a vector $e \in \mathcal{V}$, complete and isometric, with $\lambda(x)$ denoting the vector of roots of the univariate polynomial $t \to p(te - x)$ written in the decreasing order [2].
- The triple $(\mathcal{V}, \mathcal{W}, \gamma)$, where $(\mathcal{V}, \mathcal{G}, \gamma)$ is a normal decomposition system (in particular, an Eaton triple) and $\mathcal{W} := \text{span}(\gamma(\mathcal{V}))$ [11].

Based on the property (2), one can show—see Remark 1 below—that an analog of Theorem 2 holds in any FTvN system. *Consequently, all of our stated results, with appropriate modifications, can be extended to FTvN systems.*

2 Preliminaries

Throughout, we let $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ denote a Euclidean Jordan algebra of rank n with unit element e [4,7]. Additionally, we assume that the inner product is the trace inner product, that is, $\langle x, y \rangle = \text{tr}(x \circ y)$, where 'tr' denotes the trace of an element (which is the sum of its eigenvalues). In this setting, every Jordan frame in $\mathcal{V}$ is orthonormal and the eigenvalue map $\lambda : \mathcal{V} \to \mathcal{R}^n$ is an isometry.

It is well known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra.

It is known [10] that when $\mathcal{V}$ is simple, spectral sets are precisely those that are invariant under automorphisms of $\mathcal{V}$ (which are invertible linear transformations from $\mathcal{V}$ to $\mathcal{V}$ that preserve Jordan products).

For an element $a \in \mathcal{V}$, we abbreviate the spectral decomposition $a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$ as $a = q * \mathcal{E}$, where $q = (a_1, a_2, \ldots, a_n) \in \mathcal{R}^n$ and $\mathcal{E} := (e_1, e_2, \ldots, e_n)$ is an ordered Jordan frame. Note that (by rearranging the entries of q and $\mathcal{E}$, if necessary), we can always write $a = \lambda(a) * \mathcal{E}$ for some $\mathcal{E}$.

3 Proofs

Proof of Theorem 2 Let E be a (nonempty) spectral set in $\mathcal{V}$ and $a \in E$. Then, $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$. As $a \in E$,

$$[a] = \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\} \subseteq E.$$

Since $N_E(a) \subseteq N_{[a]}(a)$, it is enough to show that a strongly operator commutes with every element in $N_{[a]}(a)$. (This will also prove the second part of the theorem.) Let $d \in N_{[a]}(a)$ so that $\langle d, x - a\rangle \leq 0$ for all $x \in [a]$. Rewriting this, we see

$$\langle x, d\rangle \leq \langle a, d\rangle \quad \text{for all} \quad x \in [a]. \tag{3}$$

Now, writing the spectral decomposition of d as $d = \lambda(d) * \mathcal{E}$ for some ordered Jordan frame $\mathcal{E}$, we define $x := \lambda(a) * \mathcal{E}$. Then, $\lambda(x) = \lambda(a)$, $x \in [a]$, and (because every Jordan frame is orthonormal) $\langle x, d\rangle = \langle \lambda(x), \lambda(d)\rangle = \langle \lambda(a), \lambda(d)\rangle$. Hence, from (3),

$$\langle \lambda(a), \lambda(d)\rangle \leq \langle a, d\rangle.$$

From Theorem 4, we see the equality $\langle a, d\rangle = \langle \lambda(a), \lambda(d)\rangle$ and the strong operator commutativity of a and d. □

Remark 1 In the above proof, the part beyond (3) essentially says that $\max\{\langle d, x\rangle : x \in [a]\} = \langle \lambda(d), \lambda(a)\rangle$, which is our defining property of a FTvN system. This shows that an analog of Theorem 2 holds in any FTvN system.

Remark 2 A simple consequence of Theorem 2 leads to the following modification of Theorem 1: *suppose a is a local optimizer of the problem $\max_E h$, where E is a convex spectral set and $h : \mathcal{V} \to \mathcal{R}$ is Fréchet differentiable at a. Then, a and $h'(a)$ strongly operator commute.* This is seen by observing:

$$\langle h'(a), x - a\rangle = \lim_{t \downarrow 0} \frac{h(a + t(x - a)) - h(a)}{t} \leq 0 \quad (x \in E)$$

so that $h'(a) \in N_E(a)$. Clearly, by replacing 'max' by 'min' one gets the strong operator commutativity of a and $-h'(a)$.

Proof of Theorem 3 (i) Suppose a solves the problem $\max_E (h + \Phi)$, where $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$ and $\Phi = \phi \circ \lambda$ for some function $\phi : \mathcal{R}^n \to \mathcal{R}$. Then, $a \in E$ and

$$h(a) + \Phi(a) \geq h(x) + \Phi(x) \quad \text{for all} \quad x \in E.$$

Now, $[a] = \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\} \subseteq E$ and so

$$h(a) + \Phi(a) \geq h(x) + \Phi(x) \quad \text{for all} \quad x \in [a]. \tag{4}$$

Since $\Phi(a) = \phi(\lambda(a)) = \phi(\lambda(x)) = \Phi(x)$ for all $x \in [a]$, the above expression simplifies to

$$h(a) \geq h(x) \quad \text{for all} \quad x \in [a]. \tag{5}$$

Now, take any $d \in \partial_{[a]} h(a)$. Then,

$$h(x) - h(a) \geq \langle d, x - a \rangle \quad \text{for all} \quad x \in [a].$$

So, (5) leads to $\langle d, x - a \rangle \leq 0$ for all $x \in [a]$, i.e., to $d \in N_{[a]}(a)$. By Theorem 2, a strongly operator commutes with d.

(ii) Suppose E is convex (in addition to being spectral) and a solves the problem $\min\limits_E h$, where h is also convex. Let $\mathcal{X}$ denote the indicator function of E (i.e., it takes the value zero on E and infinity outside of E). Then, a is a optimizer of the (global) convex problem $\min\limits_{\mathcal{V}} (h + \mathcal{X})$ and so

$$0 \in \partial\, (h + \mathcal{X})(a) = \partial\, h(a) + \partial\, \mathcal{X}(a),$$

where the equality comes from the subdifferential sum formula [15, Theorem 23.8]. Hence, there is a $c \in \partial\, h(a)$ such that $-c \in \partial\, \mathcal{X}(a)$. This c will have the property that

$$\langle -c, x - a \rangle \leq 0 \quad \text{for all} \quad x \in E,$$

that is, $-c \in N_E(a)$. By Theorem 2, a strongly operator commutes with $-c$. This completes the proof. □

Remark 3 In the proof of Item (i) above, we went from (4) to (5) by canceling the common term $\Phi(a)$. This type of cancellation can be carried out in certain other situations—for example, when we consider the product $h(x)\,\Phi(x)$ with $\Phi(x) > 0$ for all $x \in E$. Thus, the above proof could be modified to get results similar to (i) for other appropriate combinations of h and Φ.

Proof of Corollary 1 (i) Suppose a is an optimizer of $\max\limits_E (h + \Phi)$. As h is assumed to be convex and differentiable, $h'(a)$ is the only element in $\partial\, h(a)$. The stated assertion comes from Theorem 3, Item (i).

(ii) The strong operator commutativity part comes from (i). Also, the maximum value is

$$h(a) + \Phi(a) = \langle c, a \rangle + \Phi(a) = \langle \lambda(c), \lambda(a) \rangle + \Phi(a),$$

where the second equality comes from Theorem 4.

(iii) When $h(x) = \langle c, x \rangle$ for all x, and a solves $\min\limits_E (h + \Phi)$, we consider the problem $\max\limits_E (-h - \Phi)$ and apply (ii) by observing that $-\Phi$ is a spectral function. Also, the minimum value is

$$-\Big(\langle -c, a \rangle - \Phi(a)\Big) = -\langle \lambda(-c), \lambda(a) \rangle + \Phi(a) = \langle \widetilde{\lambda}(c), \lambda(a) \rangle + \Phi(a).$$

(iv) Suppose a solves $\mathrm{VI}(G, E)$ so that $\langle G(a), x-a\rangle \geq 0$ for all $x \in E$. Then, a solves the problem $\min_E h$, where $h(x) := \langle G(a), x\rangle$ for all $x \in \mathcal{V}$. By Item (iii), a and $-G(a)$ strongly operator commute. □

Remark 4 We note that strong operator commutativity of a and b implies the operator commutativity of a and $\pm b$. Hence, Items (ii)–(iv) in Corollary 1 improve known operator commutativity relations [14, Theorem 2 and Proposition 8] for linear h and variational inequalities. We also note that this Corollary is similar to Theorem 1.3 in [6], which is applicable to *simple* Euclidean Jordan algebras.

We now provide some illustrative examples.

Example 1 This example shows that in Theorem 1, differentiability alone is not enough to give strong operator commutativity. In the Euclidean Jordan algebra $\mathcal{R}^2$, spectral sets are just permutation invariant sets. So the set $E = \{(1,0),\ (0,1)\}$ is spectral. For the function $h(x,y) := \frac{1}{2}x^2 - x + x(y^2+y)$, we have $h(1,0) = -\frac{1}{2}$ and $h(0,1) = 0$. Also, $h'(x,y) = (x-1+y^2+y,\ 2xy+x)$. So, $h'(1,0) = (0,1)$ and $h'(0,1) = (1,0)$. We note that the elements $(1,0)$ and $(0,1)$ operator commute in $\mathcal{R}^2$, but not strongly. Thus, if a denotes either a minimizer or a maximizer of h on E, then a and $h'(a)$ do not strongly operator commute.

Example 2 In the Euclidean Jordan algebra $\mathcal{R}^2$, let $E = \mathcal{R}^2$ and $h(x,y) := |x-1| + |y|$. Then, both E and h are convex, and $\min_E h$ is attained at $(1,0)$. Note that $(0,1)$ and $(0,-1)$ are in $\partial h(1,0)$; also, $(1,0)$ strongly operator commutes with $-(0,1)$, but not with $-(0,-1)$. This example illustrates Theorem 3(ii) and highlights the difference between the maximization and minimization problems.

Example 3 Consider two $n\times n$ complex Hermitian matrices C and A with eigenvalues $c_1 \geq c_2 \geq \cdots \geq c_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. In the algebra $\mathcal{H}^n$, consider the spectral set

$$E := \{UAU^* : U \in \mathcal{C}^{n\times n} \text{ is unitary}\}.$$

As this is also compact, the linear function $\langle C, X\rangle$ attains its maximum on this set at some matrix D in E. By Corollary 1, Item (ii), C and D strongly operator commute and

$$\max_{X\in E} \langle C, X\rangle = \langle C, D\rangle = \langle \lambda(C), \lambda(D)\rangle = \langle \lambda(C), \lambda(A)\rangle = \sum_{i=1}^n c_i a_i.$$

Thus we get the classical result of Fan, namely,

$$\max\left\{\mathrm{tr}(CUAU^*) : U \in \mathcal{C}^{n\times n} \text{ is unitary}\right\} = \sum_{i=1}^n c_i a_i.$$

Example 4 In $\mathcal{V}$, an element c is said to be an *idempotent* if $c^2 = c$. It is known that zero and one are the only possible eigenvalues of such an element. If c has exactly k nonzero eigenvalues (namely, ones), then we say that c has *rank* k. Every idempotent of rank k is of the form $e_1 + e_2 + \cdots + e_k$ for some Jordan frame $\{e_1, e_2, \ldots, e_n\}$. Now, consider the set of all idempotents of rank k, where $1 \leq k \leq n$. This set is a spectral set in $\mathcal{V}$; it is also known to be compact. Now, for any $c \in \mathcal{V}$, we maximize $\langle c, x \rangle$ over this spectral set. By Corollary 1, the maximum is attained at some a which strongly operator commutes with c. So, this maximum $= \langle c, a \rangle = \langle \lambda(c), \lambda(a) \rangle = \lambda_1(c) + \lambda_2(c) + \cdots + \lambda_k(c)$ since $\lambda(a) = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$. Thus, *for any $c \in \mathcal{V}$, the sum of the largest k eigenvalues equals the maximum of $\langle c, x \rangle$ over the set of all idempotents of rank k*. This is a well-known *variational principle*, see [1]. We remark that Theorem 1 falls short of justifying this principle. For a broader result in the setting of certain hyperbolic systems, see [2], Corollary 5.6.

Example 5 In $\mathcal{V}$, let K be a closed convex cone that is also a spectral set. For example, $K = \lambda^{-1}(Q)$, where Q is a permutation invariant closed convex cone in $\mathcal{R}^n$ [9]. For a function $f : \mathcal{V} \to \mathcal{V}$, consider the cone complementarity problem, $\mathrm{CP}(f, K)$, which is to find $x \in \mathcal{V}$ such that

$$x \in K,\ y := f(x) \in K^*, \quad \text{and} \quad \langle x, y \rangle = 0,$$

where K^* denotes the dual of K in $\mathcal{V}$. We specialize Corollary 1, Item (iv) to get: if a solves $\mathrm{CP}(f, K)$, then a strongly operator commutes with $-f(a)$. This means that

$$a \in K, \quad b := f(a) \in K^*, \quad \text{and} \quad 0 = \langle a, b \rangle = \langle \lambda(a), \widetilde{\lambda}(b) \rangle,$$

where $\widetilde{\lambda}(b) = -\lambda(-b)$.

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