Simultaneous spectral decomposition in Euclidean Jordan algebras and related systems

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Abstract

This article deals with necessary and sufficient conditions for a family of elements in a Euclidean Jordan algebra to have simultaneous (order) spectral decomposition. Motivated by a well-known matrix theory result that any family of pairwise commuting complex Hermitian matrices is simultaneously (unitarily) diagonalizable, we show that in the setting of a general Euclidean Jordan algebra, any family of pairwise operator commuting elements has a simultaneous spectral decomposition, i.e., there exists a common Jordan frame $\{e_1, e_2, \ldots, e_n\}$ relative to which every element in the given family has the eigenvalue decomposition of the form $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$. The simultaneous order spectral decomposition further demands the ordering of eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We characterize this by pairwise strong operator commutativity condition $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$, or equivalently, $\lambda(x + y) = \lambda(x) + \lambda(y)$, where $\lambda(x)$ denotes the vector of eigenvalues of x written in the decreasing order. Going beyond Euclidean Jordan algebras, we formulate commutativity conditions in the setting of the so-called Fan-Theobald-von Neumann system that includes normal decomposition systems (Eaton triples) and certain systems induced by hyperbolic polynomials.

Key Words: Euclidean Jordan algebra, operator commutativity, strong operator commutativity, FTvN system

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1 Introduction

A well-known result in matrix theory asserts that any family $\mathcal{F}$ of pairwise commuting Hermitian matrices is simultaneously (unitarily) diagonalizable, that is, there exists a unitary matrix U such that for every $A \in \mathcal{F}$, UAU^* is a diagonal matrix ([8], Theorem 2.5.5). Related to this, there is a large body of literature dealing with generalizations and operator versions. Observing that $\mathcal{H}^n$, the algebra of all $n \times n$ complex Hermitian matrices, is a Euclidean Jordan algebra, we raise the question whether the above result has an analog in general Euclidean Jordan algebras. To elaborate, consider a Euclidean Jordan algebra $\mathcal{V}$ of rank n equipped with the trace inner product. Given two elements $a, b \in \mathcal{V}$, we say that a and b *operator commute* if $L_a L_b = L_b L_a$, where L_a is the operator on $\mathcal{V}$ defined by $L_a(x) := a \circ x$, etc. It is known ([3], Lemma X.2.2) that a and b operator commute if and only if a and b have their spectral decompositions with respect to a common Jordan frame. This means that there is a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ such that

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \quad \text{and} \quad b = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n, \tag{1}$$

where $a_1, a_2, \ldots, a_n$ are the eigenvalues of a, etc. Motivated by the above matrix theory result, we ask: If $\mathcal{F}$ is family of elements in $\mathcal{V}$ such that any two elements of $\mathcal{F}$ operator commute, does it follow that all the elements in $\mathcal{F}$ have their spectral decompositions with respect to a common Jordan frame? While the (expected) answer is 'yes' and perhaps known, for lack of ready reference, we state and prove the result, see Theorem 3.1 below. Our proof is based on an induction argument on the rank of the Euclidean Jordan algebra and Peirce decomposition of the algebra relative to a Jordan frame (which itself comes from the operator version of above matrix theory result, see [3], Theorem IV.2.1).

Going back to the matrix theory result, we may ask when a family $\mathcal{F}$ in $\mathcal{H}^n$ is simultaneously *order* diagonalizable, i.e., when there is a unitary matrix U such that for all $A \in \mathcal{F}$,

$$UAU^* = \mathrm{Diag}(a_1, a_2, \ldots, a_n) \quad \text{with} \quad a_1 \geq a_2 \geq \cdots \geq a_n, \tag{2}$$

where $\mathrm{Diag}(a_1, a_2, \ldots, a_n)$ denotes the diagonal matrix with diagonal vector $(a_1, a_2, \ldots, a_n)$. Clearly, a necessary condition – formulated in terms of the trace inner product in $\mathcal{H}^n$ and the usual inner product in $\mathcal{R}^n$ – is

$$\langle A, B \rangle = \langle \lambda(A), \lambda(B) \rangle \text{ for all } A, B \in \mathcal{F},$$

where $\lambda(A)$ is the vector of eigenvalues of A written in the decreasing order, etc. We show that this 'pairwise strong commutativity' condition is also sufficient. In fact, we formulate these statements in general Euclidean Jordan algebras and prove their equivalence. In transitioning to Euclidean Jordan algebras, we employ the term *strong operator commute* for two elements a and b to mean

that

$$\langle a, b\rangle = \langle \lambda(a), \lambda(b)\rangle, \tag{3}$$

where $\lambda(a)$ is the vector of eigenvalues of a written in the decreasing order, etc. It is known, see [11, 1, 7], that (3) is equivalent to *simultaneous order spectral decomposition* of a and b: There exists a common Jordan frame $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ such that (1) holds with

$$a_1 \geq a_2 \geq \cdots \geq a_n \quad \text{and} \quad b_1 \geq b_2 \geq \cdots \geq b_n.$$

(The terms 'simultaneously order diagonalizable' [11] and 'similar joint decomposition' [1] have also been used in the literature.) Extending this, in Theorem 4.1, we show that any family $\mathcal{F}$ of pairwise strong operator commuting elements in $\mathcal{V}$ will have a simultaneous order spectral decomposition, i.e., there exists a common Jordan frame $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ such that for all $a \in \mathcal{F}$,

$$a = a_1e_1 + a_2e_2 + \cdots + a_ne_n \text{ with } a_1 \geq a_2 \geq \cdots \geq a_n.$$

Our induction proof is crucially based on the Hardy-Littlewood-Pólya rearrangement theorem (see below). As an application of this result, we consider the problem of maximizing a linear function $f(x) := \langle c, x\rangle$ over a (spectral) set of the form $E = \lambda^{-1}(Q)$, where $Q \subseteq \mathcal{R}^n$. It is known that any optimizer of this problem strongly operator commutes with c. We show in Corollary 4.3 that if c has n distinct eigenvalues (n being the rank of $\mathcal{V}$), then any two optimizers of this problem strongly operator commute.

There is another easily verifiable necessary condition for simultaneous order spectral decomposition of a family $\mathcal{F}$ in $\mathcal{V}$:

$$\lambda(a + b) = \lambda(a) + \lambda(b) \text{ for all } a, b \in \mathcal{F}. \tag{4}$$

It turns out that this pairwise λ-additivity condition is actually equivalent to pairwise strong operator commutativity of $\mathcal{F}$, hence also sufficient. We will establish such an equivalence in the broader setting of *Fan-Theobald-von Neumann systems* (FTvN systems, for short). Such a system was introduced in [4] as a unifying framework for studying certain optimization problems over Euclidean Jordan algebras, systems induced by hyperbolic polynomials, and normal decomposition systems (Eaton triples). A FTvN system is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a norm-preserving map satisfying the property

$$\max\Big\{\langle x, z\rangle : z \in [y]\Big\} = \langle \lambda(x), \lambda(y)\rangle \quad (\forall\, x, y \in \mathcal{V}), \tag{5}$$

with $[y] := \{z \in \mathcal{V} : \lambda(z) = \lambda(y)\}$. This property is a combination of an inequality and a condition for equality. The inequality $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$ (that comes from (5) and originated in the works of Fan [2], Theobald [13], and von Neumann [14]) is referred to as the *Fan-Theobald-von Neumann*

inequality and the equality

$$\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$$

defines *commutativity* of x and y in this system. In the case of a Euclidean Jordan algebra $\mathcal{V}$ of rank n carrying the trace inner product, the triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$ is a FTvN system, where $\lambda : \mathcal{V} \to \mathcal{R}^n$ is the eigenvalue map that takes any element of $\mathcal{V}$ to its vector of eigenvalues written in the decreasing order. Commutativity in this FTvN system is the same as strong operator commutativity in the algebra $\mathcal{V}$. In Theorem 5.2, we show that $\mathcal{F}$ is a family of pairwise commuting elements in the FTvN system if and only if

$$\lambda(x_1 + x_2 + \cdots + x_k) = \lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_k), \tag{6}$$

for every finite set $\{x_1, x_2, \ldots, x_k\} \subseteq \mathcal{F}$. We also show that simultaneous order spectral decomposition of a finite family in a Euclidean Jordan algebra can be described by a single condition of the form (6).

There is yet another reformulation of the order decomposition result (2) in $\mathcal{H}^n$ via automorphisms. Define, for any fixed unitary matrix U, the map $\Phi : \mathcal{H}^n \to \mathcal{H}^n$ by $\Phi(A) := U^*AU$. Then, Φ is an automorphism of the algebra $\mathcal{H}^n$ (that is, it is an invertible linear map preserving the Jordan product on $\mathcal{H}^n$); it is known that any automorphism of $\mathcal{H}^n$ arises this way. For any $A \in \mathcal{H}^n$, we let $\gamma(A)$ denote the diagonal matrix with $\lambda(A)$ on the diagonal. Then, the existence of a unitary U with order decomposition (2) for all $A \in \mathcal{F}$ amounts to the existence of an automorphism Φ of $\mathcal{H}^n$ such that

$$A = \Phi(\gamma(A)) \text{ for all } A \in \mathcal{F}.$$

Lewis [10] proves such a result in the setting of finite dimensional normal decomposition systems (that includes $\mathcal{H}^n$ as well as the space of all $m \times n$ complex matrices with the singular value map in place of the eigenvalue map). This result is applicable to simple Euclidean Jordan algebras, as they are normal decomposition systems [11, 12]. In order to accommodate general Euclidean Jordan algebras, we modify the proof of Lewis (given for Theorem 2.2 in [10]), thereby giving another proof of Theorem 4.1.

2 Preliminaries

Let $\mathcal{R}^n$ denote the Euclidean n-space of column (or row) vectors with the usual inner product. Given $p \in \mathcal{R}^n$, we write $p^{\downarrow}$ for the decreasing rearrangement of entries in p. We recall the following *Hardy-Littlewood-Pólya rearrangement inequality/theorem* ([9], Theorem 368).

Theorem 2.1 *For any two vectors p and q in $\mathcal{R}^n$,*

$$\langle p, q\rangle \leq \langle p^{\downarrow}, q^{\downarrow}\rangle.$$

Equality holds if and only if there is a permutation that simultaneously takes p to $p^{\downarrow}$ and q to $q^{\downarrow}$.

For basic definitions and results on Euclidean Jordan algebras, we refer to [3, 6]. Let $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ denote a Euclidean Jordan algebra with unit element e; here for any two elements x, y, the Jordan product and inner product are denoted, respectively, by $x \circ y$ and $\langle x, y\rangle$. It is well-known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra. We recall that in the algebra $\mathcal{H}^n$, the Jordan product and the inner product are given by

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y\rangle := tr(XY).$$

A nonzero element c in $\mathcal{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is not the sum of two other idempotents. A *Jordan frame* $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ consists of primitive idempotents that are mutually orthogonal and with sum equal to the unit element e. With n denoting the rank of $\mathcal{V}$ (which is the number of elements in any Jordan frame), we have the *spectral decomposition theorem* ([3], Theorem III.1.2): *Every x in $\mathcal{V}$ can be written as*

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n,$$

where the real numbers $x_1, x_2, \ldots, x_n$ are the eigenvalues of x and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. Corresponding to the above decomposition, we define the trace of x as $\mathrm{tr}(x) := x_1 + x_2 + \cdots + x_n$. It is known that $(x, y) \mapsto \mathrm{tr}(x \circ y)$ defines another inner product on $\mathcal{V}$ that is compatible with the Jordan product. *Throughout this paper we assume that the inner product on $\mathcal{V}$ is this trace inner product, that is, $\langle x, y\rangle = \mathrm{tr}(x \circ y)$.* In this inner product, any Jordan frame is orthonormal. For $x \in \mathcal{V}$, we let $\lambda(x)$ denote the vector of eigenvalues of x written in the decreasing order. We let $\lambda : \mathcal{V} \to \mathcal{R}^n$ denote the *eigenvalue map* on $\mathcal{V}$. Note that on the algebra $\mathcal{R}^n$ (with componentwise product and usual inner product), $\lambda(p) = p^{\downarrow}$.

Given an idempotent c and $\gamma \in \{1, \frac{1}{2}, 0\}$, we let

$$\mathcal{V}(c, \gamma) := \{x \in \mathcal{V} : x \circ c = \gamma\, x\}$$

and note that $\mathcal{V}(c, 1)$ is a subalgebra of $\mathcal{V}$ ([3], Prop. IV.1.1). If $\{c_1, c_2, \ldots, c_k\}$ is a *complete system of orthogonal idempotents* in $\mathcal{V}$ (meaning that c_is are mutually orthogonal idempotents adding up

to e), then we have the *Peirce orthogonal decomposition* (see the proof of Theorem IV.2.1 in [3]):

$$\mathcal{V} = \sum_{i \le j} \mathcal{V}_{ij},$$

where $\mathcal{V}_{ii} := \mathcal{V}(c_i, 1)$, and for $i \neq j$, $\mathcal{V}_{ij} := \mathcal{V}(c_i, \frac{1}{2}) \cap \mathcal{V}(c_j, \frac{1}{2})$. Moreover, if $\{c_1, c_2, \ldots, c_k\}$ is a Jordan frame (in which case, $k = n$ and each c_i is primitive), then $\mathcal{V}_{ii} = \mathcal{R}\, e_i$ for all i ([3], Theorem IV.2.1).

The following unique decomposition result is useful.

Theorem 2.2 ([3], Theorem III.1.1): *For any $x \in \mathcal{V}$, there exist a unique set of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, all distinct, and a unique complete system of orthogonal idempotents $c_1, c_2, \ldots, c_k$ such that $x = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k$.*

Given a Jordan frame $\{e_1, e_2, \ldots, e_n\}$, sometimes the ordering of its entries becomes important. To handle this, we define the related ordered Jordan frame

$$\mathcal{E} := \big(e_1, e_2, \ldots, e_n\big).$$

Then, for any $p = (p_1, p_2, \ldots, p_n) \in \mathcal{R}^n$, we define

$$p * \mathcal{E} := p_1 e_1 + p_2 e_2 + \cdots + p_n e_n.$$

By the spectral decomposition theorem, every $x \in \mathcal{V}$ can be written in the above form for some p and $\mathcal{E}$.

As mentioned in the Introduction, two elements $a, b \in \mathcal{V}$ operator commute (that is, $L_a L_b = L_b L_a$) if and only if there exist an ordered Jordan frame $\mathcal{E}$ and vectors $p, q \in \mathcal{R}^n$ such that

$$a = p * \mathcal{E} \quad \text{and} \quad b = q * \mathcal{E}.$$

The Hardy-Littlewood-Pólya result mentioned above has been extended to real/complex matrices by Fan and Theobald [2, 13]; an analogous result due to von Neumann [14] deals with singular values in place of eigenvalues. The result given below is formulated in the setting of Euclidean Jordan algebras. The simple algebra case is found in [11, 7] and the general case in [1]; the general case can also be obtained by combining the simple algebra case with the Hardy-Littlewood-Pólya result mentioned above.

Theorem 2.3 ([11, 1, 7]) *Let $\mathcal{V}$ be a Euclidean Jordan algebra carrying the trace inner product. Then, for any $a, b \in \mathcal{V}$, $\langle a, b\rangle \le \langle \lambda(a), \lambda(b)\rangle$; the equality holds if and only if there exists an ordered Jordan frame $\mathcal{E}$ such that $a = \lambda(a) * \mathcal{E}$ and $b = \lambda(b) * \mathcal{E}$.*

The inequality $\langle a, b\rangle \le \langle \lambda(a), \lambda(b)\rangle$ will be referred to as the Fan-Theobald-von Neumann inequality;

the equality defines the strong operator commutativity of a and b. The motivation for defining FTvN systems (mentioned in the Introduction) comes from these concepts.

3 Operator commutativity in Euclidean Jordan algebras

Theorem 3.1 *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product and $\mathcal{F}$ be a nonempty subset of $\mathcal{V}$. Then the following two statements are equivalent:*

(i) Any two elements of $\mathcal{F}$ operator commute.

(ii) There exists a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ such that every $a \in \mathcal{F}$ has spectral decomposition of the form $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$.

Proof. The implication $(ii) \Rightarrow (i)$ follows from Lemma X.2.2 in [3].
We prove $(i) \Rightarrow (ii)$ by induction on n, the rank of $\mathcal{V}$. As the implication is obvious for $n = 1$, we fix $n > 1$ and assume that the implication holds for all Euclidean Jordan algebras of rank less than n. Consider $\mathcal{F} \subset \mathcal{V}$ in which any two elements operator commute. If every element of $\mathcal{F}$ is a multiple of the unit element e, then (ii) is immediate as e can be written as the sum of elements in any Jordan frame. So, we assume that there is some element $x \in \mathcal{F}$ which is not a multiple of e, hence has at least two distinct eigenvalues. Applying Theorem 2.2, we write

$$x = \alpha_1c_1 + \alpha_2c_2 + \cdots + \alpha_kc_k,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are all distinct, $1 < k \leq n$, and $\{c_1, c_2, \ldots, c_k\}$ is a complete system of orthonormal idempotents. Now consider mutually orthogonal subalgebras $\mathcal{V}(c_1, 1), \ldots, \mathcal{V}(c_k, 1)$, each of which has rank less than n. We claim the following:

(1) Each $y \in \mathcal{F}$ has the direct sum decomposition $y = y_{11} + y_{22} + \cdots + y_{kk}$, where $y_{ii} \in \mathcal{V}(c_i, 1)$ for $i = 1, 2, \ldots, k$.

(2) For each i, the family $\mathcal{F}_i := \{y_{ii} : y \in \mathcal{F}\}$ consists of pairwise operator commuting elements in the Euclidean Jordan algebra $\mathcal{V}(c_i, 1)$.

(3) There is a common Jordan frame in $\mathcal{V}(c_i, 1)$ relative to which the family $\mathcal{F}_i$ has simultaneous spectral decomposition.

We now proceed to justify these statements. Writing $x_{ii} := \alpha_ic_i$, we have

$$x = x_{11} + x_{22} + \cdots + x_{kk}.$$

Now, take any $y \in \mathcal{F}$. We employ an argument used in the proof of Lemma X.2.2 in [3]. Consider the Peirce orthogonal decomposition of y with respect to the the system $\{c_1, c_2, \ldots, c_k\}$:

$$y = \sum_{i \leq j} y_{ij}, \quad y_{ij} \in \mathcal{V}_{ij}.$$

This results in, see ([3], page 191),

$$0 = (L_y L_x - L_x L_y)(x) = \sum_{i<j} \Big(\frac{\alpha_i - \alpha_j}{2}\Big)^2 y_{ij}.$$

We see that $y_{ij} = 0$ when $i \neq j$. Thus, $y = \sum_{i=1}^k y_{ii}$, where $y_{ii} \in \mathcal{V}_{ii} = \mathcal{V}(c_i, 1)$ proving our first claim. Similarly, for $z \in \mathcal{F}$ we can write $z = z_{11} + z_{22} + \cdots + z_{kk}$, where $z_{ii} \in \mathcal{V}(c_i, 1)$ for all i. Since, $\mathcal{V}(c_i, 1) \circ \mathcal{V}(c_j, 1) = \{0\}$ for all $i \neq j$, we see that for all $u \in \mathcal{V}(c_1, 1)$,

$$(L_{y_{11}} L_{z_{11}} - L_{z_{11}} L_{y_{11}})(u) = (L_y L_z - L_z L_y)(u) = 0.$$

This means that the elements in the family $\mathcal{F}_1 := \{L_{y_{11}} : y \in \mathcal{F}\}$ pairwise operator commute in the algebra $\mathcal{V}(c_1, 1)$. A similar statement can be made for other families $\mathcal{F}_i$, $i = 2, 3, \ldots, k$, proving our second claim. By our induction hypothesis, there is a common Jordan frame $\mathcal{E}_i$ in $\mathcal{V}(c_i, 1)$ with respect to which each element $y_{ii} \in \mathcal{F}_i$ has a spectral decomposition, proving our third claim. Now, by noting that $\mathcal{E} := \bigcup_{i=1}^k \mathcal{E}_i$ is a Jordan frame in $\mathcal{V}$, we see that each element $y = \sum_i^k y_{ii} \in \mathcal{F}$ has a spectral decomposition with respect to $\mathcal{E}$. This completes the proof. □

We highlight one special case, which is, actually, a consequence of Lemma X.2.2 in [3].

Corollary 3.2 *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product. Suppose $x \in \mathcal{V}$ has n distinct eigenvalues and has a spectral decomposition with respect to an ordered Jordan frame $\mathcal{E}$. If $y \in \mathcal{V}$ operator commutes with x, then y has a spectral decomposition with respect to $\mathcal{E}$. Consequently, if another $z \in \mathcal{V}$ operator commutes with x, then y and z operator commute.*

Note: By Theorem 2.2, $\mathcal{E}$ in the above corollary is unique (up to permutations).

Proof. Let $\mathcal{E} = \big(e_1, e_2, \ldots, e_n\big)$ and let y operator commute with x. We follow the proof of the above theorem (or the proof of Lemma X.2.2 in [3]) by letting $k = n$ and $c_i = e_i$ for all i. Then $\mathcal{V}(c_i, 1) = \mathcal{R}\, e_i$ for all i and so $y = y_{11} + y_{22} + \cdots + y_{nn}$, where $y_{ii} \in \mathcal{R}\, e_i$ for all $i = 1, 2, \ldots, n$. Thus, y has a spectral decomposition relative to the ordered Jordan frame $\mathcal{E}$. The same holds for any z that commutes with x. Thus, y and z operator commute. □

4 Strong operator commutativity in Euclidean Jordan algebras

We recall that two elements $a, b \in \mathcal{V}$ *strongly operator commute* if $\langle a, b\rangle = \langle \lambda(a), \lambda(b)\rangle$. As mentioned in the Introduction, this is stronger than operator commutativity of a and b. The converse fails even in $\mathcal{R}^2$: The coordinate vectors $(1, 0)$ and $(0, 1)$ operator commute, but not strongly.

Theorem 4.1 *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product and $\mathcal{F}$ be a nonempty subset of $\mathcal{V}$. Then the following statements are equivalent:*

(*i*) *Any two elements of* $\mathcal{F}$ *strongly operator commute.*

(*ii*) *There exists an ordered Jordan frame* $\mathcal{E}$ *such that every* $a \in \mathcal{F}$ *has the spectral decomposition* $a = \lambda(a) * \mathcal{E}$.

Proof. By Theorem 2.3, $(ii) \Rightarrow (i)$.

We proceed to prove $(i) \Rightarrow (ii)$ by induction on n (the rank of $\mathcal{V}$). We fix $n > 1$ and assume that the implication is true for all Euclidean Jordan algebras of rank less than n. If every element of $\mathcal{F}$ is a multiple of e, we have the implication (ii) for any ordered Jordan frame. Hence, we fix $\mathcal{F} \subseteq \mathcal{V}$ satisfying (i) (in particular, any two elements operator commute) and assume that there is an an $x \in \mathcal{F}$ with at least two distinct eigenvalues. As in the proof of the previous theorem, we use Theorem 2.2 to write $x = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k$, where $1 < k \leq n$, $\{c_1, c_2, \ldots, c_k\}$ is a complete system of orthogonal idempotents and, without loss of generality,

$$\alpha_1 > \alpha_2 > \cdots > \alpha_k.$$

We follow the notation and proof of the previous theorem. We consider subalgebras $\mathcal{V}_{ii} := \mathcal{V}(c_i, 1)$, define $x_{ii} := \alpha_i c_i$, and decompose arbitrary $y, z \in \mathcal{F}$ as

$$y = y_{11} + y_{22} + \cdots + y_{kk} \quad \text{and} \quad z = z_{11} + z_{22} + \cdots + z_{kk},$$

where $y_{ii}, z_{ii} \in \mathcal{V}_{ii}$ for $i = 1, 2, \ldots, k$. As observed previously, y_{ii} and z_{ii} operator commute in $\mathcal{V}_{ii}$ for every i. In view of the previous theorem, for each i, there exists a common ordered Jordan frame $\mathcal{E}_i$ relative to which the the family $\mathcal{F}_i = \{y_{ii} : y \in \mathcal{F}\}$ has a simultaneous spectral decomposition in $\mathcal{V}_{ii}$. Now, relative to $\mathcal{E}_i$ and $\mathcal{V}_{ii}$, we write

$$x_{ii} = p_{ii} * \mathcal{E}_i, \; y_{ii} = q_{ii} * \mathcal{E}_i, \text{ and } z_{ii} = r_{ii} * \mathcal{E}_i,$$

where the (row) vectors p_{ii}, q_{ii}, and r_{ii} consist of eigenvalues of x_{ii}, y_{ii}, and z_{ii} respectively. We also write $\lambda_r(y_{ii}) := q_{ii}^{\downarrow}$ for the the decreasing rearrangement of the vector of eigenvalues of y_{ii} *relative* to $\mathcal{V}_{ii}$. By abuse of notation, we write $p = \big(p_{11}, p_{22}, \ldots, p_{kk}\big)$ and define q and r similarly; we note that the eigenvalues of x are made of up entries in p, $\lambda(x) = p^{\downarrow}$, etc.

Since the algebras $\mathcal{V}_{ii}$ are mutually orthogonal and each Jordan frame $\mathcal{E}_i$ is an orthonormal set, we have

$$\langle p, q \rangle = \sum_{i=1}^{k} \langle p_{ii}, q_{ii} \rangle = \sum_{i=1}^{k} \langle x_{ii}, y_{ii} \rangle = \langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle = \langle p^{\downarrow}, q^{\downarrow} \rangle,$$

where the fourth equality comes from the strong operator commutativity of x and y.

By the Hardy-Littlewood-Pólya rearrangement theorem, there is a permutation which simultaneously takes p to $p^{\downarrow}$ and q to $q^{\downarrow}$. Since $p_{ii} = \alpha_i$ times a vector of ones and $\alpha_1 > \alpha_2 \cdots > \alpha_k$, $p = p^{\downarrow}$, and each entry in p_{11} exceeds every entry in p_{22} etc., we see that this permutation can only permute entries in any p_{ii}, but cannot take an entry in a p_{ii} to an entry in p_{jj} for $i \neq j$. Thus, this permutation (when applied to q) rearranges entries in any q_{ii} and cannot take an entry in q_{ii} to an

entry in q_{jj} for $i \neq j$. This means that when $i > j$, any entry in q_{ii} is greater than or equal to any entry in q_{jj}. Thus,

$$\lambda(y) = q^{\downarrow} = (q_{11}^{\downarrow}, q_{22}^{\downarrow}, \ldots, q_{kk}^{\downarrow}).$$

Similarly,

$$\lambda(z) = r^{\downarrow} = (r_{11}^{\downarrow}, r_{22}^{\downarrow}, \ldots, r_{kk}^{\downarrow}).$$

We now claim that for each i, y_{ii} strongly operator commutes with z_{ii} in $\mathcal{V}_{ii}$. We observe

$$\langle y, z\rangle = \sum_{i=1}^{k} \langle y_{ii}, z_{ii}\rangle = \sum_{i=1}^{k} \langle q_{ii}, r_{ii}\rangle \leq \sum_{i=1}^{k} \langle q_{ii}^{\downarrow}, r_{ii}^{\downarrow}\rangle = \langle \lambda(z), \lambda(z)\rangle,$$

where the inequality is due to the Hardy-Littlewood-Pólya inequality. Since y and z strongly operator commute, we have equality throughout and so, for each i, $\langle y_{ii}, z_{ii}\rangle = \langle q_{ii}^{\downarrow}, r_{ii}^{\downarrow}\rangle = \langle \lambda_r(y_{ii}), \lambda_r(z_{ii})\rangle$. This means that for each index i, any two elements in the family $\mathcal{F}_i = \{y_{ii} : y \in \mathcal{F}\}$ strongly operator commute. As the rank of each $\mathcal{V}_{ii}$ is less than n, by our induction hypothesis, there is an ordered Jordan frame in $\mathcal{V}_{ii}$ – we continue to denote it by $\mathcal{E}_i$ – with respect to which every element in $\mathcal{F}_i$ has an ordered spectral decomposition. We can now consider the union of these ordered Jordan frames $\mathcal{E}_i$ (where we list, consecutively, elements of $\mathcal{E}_1$, $\mathcal{E}_2$, etc.) to get an ordered Jordan frame $\mathcal{E}$ in $\mathcal{V}$. Since for any $y \in \mathcal{F}$, $\lambda(y) = (\lambda_r(y_{11}), \lambda_r(y_{22}), \ldots, \lambda_r(y_{kk}))$, we see that with respect to $\mathcal{E}$, every element in $\mathcal{F}$ has an ordered spectral decomposition. This completes the proof. □

Analogous to the previous corollary, we have the following.

Corollary 4.2 *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product. Suppose $x \in \mathcal{V}$ has n distinct eigenvalues and $x = \lambda(x) * \mathcal{E}$ for some ordered Jordan frame $\mathcal{E}$. If $y \in \mathcal{V}$ strongly operator commutes with x, then $y = \lambda(y) * \mathcal{E}$. Consequently, if another $z \in \mathcal{V}$ strongly operator commutes with x, then y and z strongly operator commute.*

Proof. Let $\mathcal{E} = \big(e_1, e_2, \ldots, e_n\big)$ and let y strongly operator commute with x. We follow the proof of the above theorem by letting $k = n$ and $c_i = e_i$ for all i. Then $\mathcal{V}(c_i, 1) = \mathcal{R}\, e_i$ for all i and so $y = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$, where $\beta_1 \geq \beta_2 \cdots \geq \beta_n$. Thus, $y = \lambda(y) * \mathcal{E}$. We can make a similar statement for any z that strongly commutes with x. Thus, y and z themselves strongly operator commute. □

Note: An alternate proof of the above corollary can be given by using Theorem 2.3 and the uniqueness of $\mathcal{E}$ (Cf. Theorem 2.2).

We now describe a simple application of the above corollary. Consider a *spectral set* E in $\mathcal{V}$, which, by definition is a set of the form $\lambda^{-1}(Q)$, where $Q \subseteq \mathcal{R}^n$. For any given $c \in \mathcal{V}$, we consider the

problem

$$\max_{x \in E} \langle c, x \rangle. \tag{7}$$

It has been shown in [5], Corollary 1, that if a is an optimizer of the above problem, then c and a strongly operator commute. Combining this with the previous corollary, we have the following result which provides an instance where any two optimizers strongly operator commute.

Corollary 4.3 *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product. If c has n distinct eigenvalues, then any two optimizers of the problem (7) strongly operator commute.*

This result is clearly false when c does not have n distinct eigenvalues: For example, in $\mathcal{V} = \mathcal{R}^2$ (whose rank is 2), let $c = (1, 1)$ and consider the spectral set $E = \{(1, 0), (0, 1)\}$. Clearly, c strongly operator commutes with both optimizers $(1, 0)$ and $(0, 1)$, but $(1, 0)$ does not strongly operator commute with $(0, 1)$.

5 Commutativity in FTvN-systems

We now take up results similar to the above for FTvN systems. We recall that commutativity of two elements x, y in a FTvN system is defined by: $\langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle$. In [4], Proposition 2.6, it is shown that this is equivalent to $\lambda(x + y) = \lambda(x) + \lambda(y)$ and also to $||\lambda(x + y)|| = ||\lambda(x) + \lambda(y)||$. In what follows, we extend this to any finite set of elements. We require the following sublinear inequality:

$$\Big\langle \lambda(x + y), \lambda(c) \Big\rangle \leq \Big\langle \lambda(x) + \lambda(y), \lambda(c) \Big\rangle \text{ for all } x, y, c \in \mathcal{V}.$$

This is seen by observing, for any $z \in [c]$,

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \leq \langle \lambda(x), \lambda(z) \rangle + \langle \lambda(y), \lambda(z) \rangle = \langle \lambda(x) + \lambda(y), \lambda(c) \rangle$$

and taking the maximum over z in $[c]$. The following general version is easily proved via induction:

$$\Big\langle \lambda(x_1 + x_2 + \cdots + x_k), \lambda(c) \Big\rangle \leq \Big\langle \lambda(x_2) + \cdots + \lambda(x_k), \lambda(c) \Big\rangle \text{ for all } x_1, x_2, \ldots, x_k, c \in \mathcal{V}. \tag{8}$$

Lemma 5.1 *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a FTvN system and consider elements $x_1, x_2, \ldots, x_k \in \mathcal{V}$. Then, the following are equivalent:*

(*a*) $\lambda(x_1 + x_2 + \cdots + x_k) = \lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_k)$.

(*b*) $||\lambda(x_1 + x_2 + \cdots + x_k)|| = ||\lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_k)||$.

(*c*) *Any two elements in $\{x_1, x_2, \ldots, x_k\}$ commute, that is, $\langle x_i, x_j \rangle = \langle \lambda(x_i), \lambda(x_j) \rangle$ for all $i, j \in \{1, 2, \ldots, k\}$.*

Proof. (*a*) $\Rightarrow$ (*b*): Obvious.

$(b) \iff (c)$: As λ is norm-preserving, (b) holds if and only if

$$||x_1 + x_2 + \cdots + x_k||^2 = ||\lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_N)||^2,$$

or, equivalently,

$$\sum_{i=1}^{k} ||x_i||^2 + 2 \sum_{i,j=1;i\neq j}^{k} \langle x_i, x_j\rangle = \sum_{i=1}^{k} ||\lambda(x_i)||^2 + 2 \sum_{i,j=1;i\neq j}^{k} \langle \lambda(x_i), \lambda(x_j)\rangle.$$

Using the norm-preserving property of λ, we cancel appropriate terms and rewrite this as

$$\sum_{i,j=1;i\neq j}^{k} \langle x_i, x_j\rangle = \sum_{i,j=1;i\neq j}^{k} \langle \lambda(x_i), \lambda(x_j)\rangle.$$

Since $\langle x_i, x_j\rangle \leq \langle \lambda(x_i), \lambda(x_j)\rangle$ in our FTvN system, the above equality can hold if and only if $\langle x_i, x_j\rangle = \langle \lambda(x_i), \lambda(x_j)\rangle$ for all $i \neq j$.
$(b) \Rightarrow (a)$: Assume (b) and let $u := \lambda(x_1 + x_2 + \cdots + x_k)$ and $v := \lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_k)$. Given $||u|| = ||v||$, we show that $\langle u, v\rangle = ||u|| \, ||v||$. Then, by the equality case in Cauchy-Schwarz inequality, we get $u = v$. Now, we apply (8) by putting $c = x_1 + x_2 + \cdots + x_k$:

$$||u||^2 = \Big\langle \lambda(x_1+x_2+\cdots+x_k), \lambda(x_1+x_2+\cdots+x_k) \Big\rangle \leq \Big\langle \lambda(x_1)+\cdots+\lambda(x_k), \lambda(c) \Big\rangle = \langle v, u\rangle \leq ||v|| \, ||u||.$$

Since $||v|| \, ||u|| = ||u||^2$, we have $\langle u, v\rangle = ||u|| \, ||v||$; consequently, $u = v$. This gives (a). □

An immediate consequence is the following.

Theorem 5.2 *Consider a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ and let $\mathcal{F}$ be a nonempty subset of $\mathcal{V}$. Then, the following are equivalent:*

(i) Any two elements of $\mathcal{F}$ commute, that is, for all $x, y \in \mathcal{F}$, $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$.

(ii) For any finite set $\{x_1, x_2, \ldots, x_k\}$ in $\mathcal{F}$, $\lambda(x_1 + x_2 + \cdots + x_k) = \lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_k)$.

Another consequence, via Theorem 4.1, is stated for Euclidean Jordan algebras.

Corollary 5.3 *Let $\mathcal{V}$ be a Euclidean Jordan algebra carrying the trace inner product. For any finite set $\{x_1, x_2, \ldots, x_k\}$, the following are equivalent:*

(a) The family $\{x_1, x_2, \ldots, x_k\}$ has a simultaneous order spectral decomposition.

(b) $\lambda(x_1 + x_2 + \cdots + x_k) = \lambda(x_1) + \lambda(x_2) + \cdots + \lambda(x_k)$.

6 Normal decomposition systems and an alternate proof of Theorem 4.1

Previously, we gave a direct induction based proof of Theorem 4.1. In this section, we provide an alternate proof by modifying a proof of Lewis [10] given for a family of commuting elements in a finite dimensional normal decomposition system. Recall that a *normal decomposition system* (NDS, for short) is a tripe $(X, \mathcal{G}, \gamma)$, where X is a inner product space, $\mathcal{G}$ is a closed subgroup of the orthogonal group of X, and $\gamma : X \to X$ is a map satisfying the following properties:

(i) $\gamma(Ax) = \gamma(x)$ for all $x \in X$ and $A \in \mathcal{G}$.

(ii) For every $x \in X$, there is a $A \in \mathcal{G}$ such that $x = A\gamma(x)$.

(iii) $\langle x, y\rangle \leq \langle \gamma(x), \gamma(y)\rangle$ for all $x, y \in X$.

As observed in [4], every NDS is a FTvN system: we let $\mathcal{V} = X$, $\lambda = \gamma$, and $\mathcal{W} = \lambda(\mathcal{V}) - \lambda(\mathcal{V})$. Thus, we can define commutativity in a NDS via the equation $\langle x, y\rangle = \langle \gamma(x), \gamma(y)\rangle$. In the setting of a finite dimensional NDS, it is shown by Lewis ([10], Theorem 2.2) that every family $\mathcal{F}$ of pairwise commuting elements admits a 'simultaneous decomposition': there exists an $A \in \mathcal{G}$ such that $x = A\gamma(x)$ for all $x \in \mathcal{F}$. As every simple Euclidean Jordan algebra is a finite dimensional normal decomposition system [11], this result can be used to get another proof of Theorem 4.1 when the algebra is simple. However, a general Euclidean Jordan algebra need not be a normal decomposition system [12], hence this result cannot be directly used. In what follows, we modify Lewis' proof to accommodate general Euclidean Jordan algebras.

Another proof of Theorem 4.1:
The implication $(ii) \Rightarrow (i)$ comes from Theorem 2.3. Suppose (i) holds. Consider $\mathrm{conv}(\mathcal{F})$, the convex hull of $\mathcal{F}$. Let z be in the *relative interior* of $\mathrm{conv}(\mathcal{F})$. Choose an ordered Jordan frame $\mathcal{E}$ such that

$$z = \lambda(z) * \mathcal{E}.$$

We *claim* that $a = \lambda(a) * \mathcal{E}$ for all $a \in \mathcal{F}$. Suppose, if possible, there exists an $a \in \mathcal{F}$ such that $a \neq \lambda(a) * \mathcal{E}$. Let

$$b := \lambda(a) * \mathcal{E}.$$

Then, $\lambda(b) = \lambda(a)$ and for all $x \in \mathcal{V}$, $\langle b, x\rangle \leq \langle \lambda(b), \lambda(x)\rangle = \langle \lambda(a), \lambda(x)\rangle$. As $||b|| = ||\lambda(a) * \mathcal{E}|| = ||\lambda(a)|| = ||a||$ and $a \neq b$, we see that

$$\langle a, b\rangle < ||a||\,||b|| = ||a||^2.$$

Now, z, which is in the relative interior of $\mathrm{conv}(\mathcal{F})$, lies on the open line segment joining a and some $c \in \mathrm{conv}(\mathcal{F})$. Hence we can write z as a convex combination of a and c, and c as a convex

combination a finite number of elements in $\mathcal{F}$. Thus, we can write

$$z = \alpha_0 a + \alpha_1 a^{(1)} + \cdots + \alpha_k a^{(k)},$$

where $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ are in $\mathcal{F}$ and α_is are *positive* with sum 1. As a and $a^{(i)}$ strongly commute for every i,

$$\begin{array}{rcl} \langle \lambda(a), \lambda(z) \rangle & = & \langle \lambda(a) * \mathcal{E}, \lambda(z) * \mathcal{E} \rangle = \langle b, z \rangle = \alpha_0 \langle b, a \rangle + \sum_1^k \alpha_i \langle b, a^{(i)} \rangle \\ & < & \alpha_0 ||a||^2 + \sum_1^k \alpha_i \big\langle \lambda(a), \lambda(a^{(i)}) \big\rangle \\ & = & \alpha_0 ||a||^2 + \sum_1^k \alpha_i \langle a, a^{(i)} \rangle \\ & = & \langle a, z \rangle \leq \langle \lambda(a), \lambda(z) \rangle. \end{array}$$

This yields a contradiction. Hence, $a = \lambda(a) * \mathcal{E}$ for all $a \in \mathcal{F}$. This completes the proof. □

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