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Algebraic Univalence Theorems for Nonsmooth Functions

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A well known univalence result due to D. Gale and H. Nikaido (1965, *Math. Ann.* **159,** 81–93) asserts that if the Jacobian matrix of a differentiable function from a closed rectangle K in R^n into R^n is a **P**-matrix at each point of K, then f is one-to-one on K. In this paper, by introducing the concepts of H-differentiability and H-differential of a function (as a set of matrices), we generalize the Gale–Nikaido result to nonsmooth functions. Our results further extend those of other authors valid for compact rectangles. We show that our results are applicable when the H-differential is any one of the following: the Jacobian matrix of a differentiable function, the generalized Jacobian of a locally Lipschitzian function, the Bouligand subdifferential of a semismooth function, and the C-differential of L. Qi (1993, *Math. Oper. Res.* **18,** 227–244).

1. INTRODUCTION

A (global) univalence theorem describes conditions for a continuous function from one (metric, Euclidean, or Banach) space into another to be one-to-one. An excellent account of global univalence theorems up to 1983 is given in Parthasarathy's book [18]. For subsequent work see Pourciau [20–22], Radulescu and Radulescu [26, 27], Scholtes [31], Warga [35–37],

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Zangwill [38], and the references therein. For an extensive literature related to polynomial maps, see [16]. References to applications to univalence theorems to economics, statistics, optimization, electrical networks, stability theory of differential equations, etc., can be found in the cited papers. Various univalence theorems that exist in the literature can roughly be classified into three groups: analytical univalence theorems, topological univalence theorems, and algebraic univalence theorems. The classical result of Hadamard [9] is an example of an analytical univalence theorem. It says that if $f : R^n \to R^n$ is continuously differentiable, has nonsingular Jacobian $Jf(x)$ at each $x \in R^n$, and

$$\sup_{x \in R^n} \|Jf(x)^{-1}\| < \infty$$

or more generally,

$$\int_0^\infty \inf_{\|x\| \le t} \frac{1}{\|Jf(x)^{-1}\|}\, dt = \infty$$

then f is a (onto) homeomorphism of R^n; see [2] for a Banach space version. Pourciau [20, 21] extended Hadamard's theorem to a locally Lipschitzian function on R^n by replacing the Jacobian by the generalized Jacobian of Clarke [4]. Warga [37] further extended this result (to Banach spaces) by replacing the generalized Jacobian of a locally Lipschitzian function by a "local derivate container." A topological univalence theorem, attributed to Hadamard (and the metric space version due to Banach and Mazur [1]) says that a function from R^n into itself is a (onto) homeomorphism if and only if it is proper and a local homeomorphism. The above mentioned results of Hadamard can be deduced as a consequence of a topological result due to Plastock [19].

Our main aim in this paper is to generalize a well known algebraic univalence result due to Gale and Nikaido which asserts that if the Jacobian matrix of a differentiable function from a closed rectangle K in R^n into R^n is a **P**-matrix at each point of K, then f is one-to-one on K. In doing so we will also generalize the following result due to Garcia and Zangwill [6], Mas-Colell [15], and Robinson [28] valid for compact rectangles: if the Jacobian matrix of a *continuously/strongly* Fréchet differentiable function from a compact rectangle K in R^n into R^n has a positive determinant in the interior of K and is a **P**-matrix at each point of the boundary of K, then f is one-to-one on K. By introducing the concepts of H-differentiability and H-differential we generalize the above mentioned results, see Section 5, by replacing a differentiable function by an H-differentiable function and by replacing the Jacobian matrix by an H-differential which is a set of matrices. In particular, our results are valid when the H-differential is any one of the following: the Jacobian matrix of a differentiable function, the generalized Jacobian of a locally Lipschitzian function, the Bouligand subdifferential of a semismooth function, and the C-differential of Qi [24]. Our analysis in this paper relies on degree theory [13] and is along the arguments of Mas-Colell [15] and Kojima and Saigal [12] who considered generalizations of the Gale–Nikaido result to compact polyhedral sets and to piecewise smooth functions based on polyhedral subdivisions.

The organization of the paper is as follows. In Section 3, we introduce the H-differential of a function and relate it to various other known differentials. In Section 4, we give elementary properties of H-differentials and state the chain rule for H-differentials. Section 5 contains our main results.

2. PRELIMINARIES

A closed (open) rectangle in R^n is a Cartesian product of n closed (respectively, open) intervals in R; *we always assume that rectangles under consideration have nonempty interiors*. We denote the interior and the boundary of a set A by int A and bdy A. For any $r > 0$, $B(x^*; r)$ denotes the open (Euclidean) ball in R^n with center x^* and radius r.

We use the notation $r(x) = o(\|x - a\|)$ as $x \to a$ to mean that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|r(x)\| \leq \varepsilon \|x - a\|$ whenever $\|x - a\| \leq \delta$. (For sequences, $r(x^k) = o(\|x^k - a\|)$ as $x^k \to a$ means that for all $\varepsilon > 0$ there exists a natural number N such that $\|r(x^k)\| \leq \varepsilon \|x^k - a\|$ whenever $n \geq N$.)

We say that a matrix $M \in R^{n \times n}$ is a $\mathbf{P}_0$-matrix ($\mathbf{P}$-matrix) if every principle minor of M is nonnegative (resp., positive). (A typical principle minor of M is given by the determinant of the principle submatrix $M_{\alpha\alpha}$ where $\alpha \subseteq \{1, 2, \dots, n\}$.) Since for a $\mathbf{P}_0$-matrix M, $M + \varepsilon I$ is a $\mathbf{P}$-matrix and hence nonsingular for every $\varepsilon > 0$, we see that *when M is a nonsingular $\mathbf{P}_0$-matrix* (*in* [18], *they are called weak* $\mathbf{P}$-*matrices*), *for every $t \in [0, 1]$, the matrix $tM + (1 - t)I$ is nonsingular*. For issues related to degree theory, we refer to [13]. For a continuous function $f: \overline{\Omega} \to R^n$ where Ω is a bounded open set in R^n and a $p \in R^n$, we use the notation deg (f, Ω, p) to denote the degree of f at p relative to Ω. If x^* is an isolated solution of the equation $f(x) = p$, then for all small $\varepsilon > 0$, deg $(f, B(x^*, \varepsilon), f(x^*))$ is the same; we denote this common value by index(f, x^*).

We say that a function f from an open set $\Omega \subseteq R^n$ to R^n is *weakly univalent* if there exists a sequence of continuous one-to-one functions

$f_k : \Omega \to R^n$ such that $f_k \to f$ uniformly on each bounded subset of Ω. We have the following result from [8]:

PROPOSITION 1. *Suppose Ω is open in R^n, $f : \Omega \to R^n$ is weakly univalent, and $q \in f(\Omega)$. If x^* is an isolated solution of the equation $f(x) = q$ in Ω, then it is the only solution.*

3. H-DIFFERENTIAL

DEFINITION 1. Let $a \in R^n$, Ω be a neighborhood of a, and $f : \Omega \to R^m$. We say that a (nonempty) subset $T(a)$ of $R^{m \times n}$ is an *H*-differential of f at a if for every sequence $\{x^k\}$ converging to a, there exists a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(a)$ such that

$$f(x^{k_j}) - f(a) - A(x^{k_j} - a) = o(\|x^{k_j} - a\|). \tag{1}$$

We say that f is *H*-differentiable at a if f has an *H*-differential at a.

Remarks. (1) As we see in the examples below, the *H*-differential of a function at a point need not be unique.

(2) Clearly, the *H*-differentiability implies continuity.

(3) The *H*-differentiability condition can be equivalently described as follows: For every sequence $\{a + t_k d^k\}$ with $t_k \downarrow 0$ and $\|d^k\| = 1$, there exist $d^{k_j} \to d \in R^n$ and $A \in T(a)$ such that

$$\frac{f\left(a + t_{k_j} d^{k_j}\right) - f(a)}{t_{k_j}} \to Ad.$$

(4) Recall that a function f defined from an open set Ω of R^n into R^m is *directionally differentiable* at a point $a \in \Omega$ if for each $d \in R^n$, the limit

$$f'(a; d) := \lim_{t \downarrow 0} \frac{f(a + td) - f(a)}{t}$$

exists; f is *Hadamard directionally differentiable* at a, if for any $d^k \to d$ and $t_k \downarrow 0$,

$$\lim_{k \to \infty} \frac{f\left(a + t_k d^k\right) - f(a)}{t_k} = f'(a; d).$$

The similarity between this limit and the one in Remark 3 prompted us to use the terms "*H*-differentiability" and "*H*-differential" in Definition 1.

(Although the prefix H really stands for Hadamard, we refrain from using the term "Hadamard differentiability" since this term has already been used in classical analysis. We note that in the finite dimensional setting, Hadamard differentiability is the same as the Fréchet differentiability; see [4, Sect. 2.2].)

We observe that with $T(a) = R^{m \times n}$, Hadamard directional differentiability implies H-differentiability. However, the function $f: R \to R$, defined by

$$f(x) = \begin{cases} x, & x \in E \\ 0, & x \notin E, \end{cases}$$

where $E = \bigcup_1^\infty [\frac{1}{2n}, \frac{1}{2n-1}]$ is H-differentiable at the origin with $T(0) = \{0, 1\}$ but not even directionally differentiable there.

(5) It is easily seen that if a function $f: \Omega \subseteq R^n \to R^m$ is H-differentiable at a point a, then there exist a constant $L > 0$ and a neighborhood $B(a, \delta)$ of a with

$$\| f(x) - f(a) \| \le L \|x - a\|, \qquad \forall x \in B(a, \delta). \tag{2}$$

Conversely, if condition (2) holds, then $T(a) := R^{m \times n}$ can be taken as an H-differential of f at a. We thus have, in (2), an alternate description of H-differentiability. But, as we see in the sequel, it is the identification of an appropriate H-differential that becomes important and relevant. Clearly any function locally Lipschitzian at a will satisfy (2). For real valued functions, condition (2) is known as the "calmness" of f at a. This concept has been well studied in the literature of nonsmooth analysis (see [30, Chap. 8]).

EXAMPLE 1. Consider a function $f: R^n \to R^m$ which is Fréchet differentiable at $a \in R^n$ with Fréchet derivative ($=$ Jacobian matrix) $Jf(a) \in R^{n \times m}$ so that

$$f(x) - f(a) - Jf(a)(x - a) = o(\|x - a\|).$$

Clearly, $\{Jf(a)\}$ is an H-differential of f and so f is H-differentiable.

EXAMPLE 2. Consider a function $f: \Omega \to R^m$ which is locally Lipschitzian at each point of an open set $\Omega \subseteq R^n$ (so that on some neighborhood of each point of Ω, f is Lipschitzian). Then the well known Rademacher's theorem asserts that f is Fréchet differentiable almost everywhere (in the Lebesgue sense) in Ω. Let Ω_f be the set of all points in Ω where f is Fréchet differentiable. Then, at any $a \in \Omega$, the (Clarke) generalized Jacobian [4]

$$\partial f(a) = \mathrm{co}\left\{\lim Jf(x^k) : x^k \to a, x^k \in \Omega_f\right\}, \tag{3}$$

exists and is nonempty, compact, and convex. (We note that [4, p. 30] when f is continuously differentiable or strictly differentiable at a point a, the generalized Jacobian at a coincides with $\{Jf(a)\}$, and that when f is merely Fréchet differentiable at a, as in the one variable function $f(x) = x^2 \sin \frac{1}{x}$ at the origin, the generalized Jacobian may be much more than $\{Jf(a)\}$. Note also that the function $g(x) := |x|^{3/2} \sin \frac{1}{x}$ is Fréchet differentiable on R but not even locally Lipschitzian at the origin.) The set

$$\partial_B f(a) := \left\{\lim Jf(x^k) : x^k \to a, x^k \in \Omega_f\right\} \tag{4}$$

is called the Bouligand (sub)differential of f at a.

We claim that (*the locally Lipschitzian function*) f *is H-differentiable at* $a \in \Omega$ *with* $T(a) = \partial f(a)$ *as an H-differential*. To see the claim, we adopt an argument given in the proof of Proposition 3.1 in [10]. Consider a sequence $\{a + t_k d^k\}$ with $t_k \downarrow 0$ and $\|d^k\| = 1$ for all k. By the mean value theorem for locally Lipschitzian functions, see Proposition 2.6.5 in [4], we may write

$$f(a + t_k d^k) - f(a) \in \mathrm{co}\, \partial f([a, a + t_k d^k])(a + t_k d^k - a)$$

which, in view of the Carathéodory theorem [29, Theorem 17.1], reduces to

$$f(a + t_k d^k) - f(a) = \left(\sum_{i=1}^{n+1} \lambda_i^k V_i^k\right)(a + t_k d^k - a),$$

where the λ_i^k's are nonnegative numbers adding up to one and $V_i^k \in \partial f(b_i^k)$ with b_i^k lying in the line segment joining a and $a + t_k d^k$. Since the multivalued mapping $x \mapsto \partial f(x)$ is upper semicontinuous and closed on Ω [4, Proposition 2.6.2], we may assume (by going through a subsequence) that $V_i^k \to V_i \in \partial f(a)$ for each i; we may also assume that $\lambda_i^k \to \lambda_i$ for each i and $d^k \to d$. Then

$$\frac{f(a + t_k d^k) - f(a)}{t_k} \to \left(\sum_{i=1}^{n+1} \lambda_i V_i\right) d = Ad.$$

Since $A = \sum_1^{n+1} \lambda_i V_i \in \partial f(a)$, by Remark 3, the claim is justified.

Remark. (6) In [35–37], Warga introduces and studies the concepts of "local derivate container" for a locally Lipschitzian function on R^n and "unbounded derivate container" for a continuous function on R^n. He shows that the generalized Jacobian $\partial f(a)$ is a particular case of a "local derivate container" and establishes local and global homeomorphism results based on these concepts. In [34], Sussmann extends these notions in his study of "multidifferentials." It is not clear at this stage how these refined concepts are related to our H-differentials.

EXAMPLE 3. Consider an open set $\Omega \subseteq R^n$ and a locally Lipschitzian function $f: \Omega \to R^m$ that is *semismooth* at $a \in \Omega$. This means that for any $x^k \to a$, and $V_k \in \partial f(x^k)$,

$$f(x^k) - f(a) - V_k(x^k - a) = o(\|x^k - a\|).$$

The notion of semismoothness was introduced by Mifflin [14] for functionals and extended to vector functions by Qi [23]. It has attracted a lot of attention in the optimization community in recent times; see [25], etc. It has been shown by Qi and Sun [25] that f is directionally differentiable at a and by Shapiro [32] that f is Bouligand differentiable (B-differentiable, for short) which means that

$$f(x) - f(a) - f'(a; x - a) = o(\|x - a\|).$$

We claim that *for the given function f which is semismooth at a, the Bouligand (sub)differential $\partial_B f(a)$ is an H-differential of f.* To see the claim, we proceed as follows. Let $x^k \to a$. Then by the semismoothness property,

$$f(x^k) - f(a) - V_k(x^k - a) = o(\|x^k - a\|)$$

for any $V_k \in \partial f(x^k)$, and in particular, for any $V_k \in \partial_B f(x^k)$. Since the mapping $x \mapsto \partial_B f(x)$ is compact valued and upper semicontinuous, we may assume, by going through a subsequence that for some $V \in \partial_B f(a)$ and a subsequence $\{x^{k_j}\}$ that

$$f(x^{k_j}) - f(a) - V(x^{k_j} - a) = o(\|x^{k_j} - a\|).$$

This proves that $\partial_B f(a)$ is an H-differential f.

Piecewise smooth functions form an important class of functions satisfying the semismoothness property. Recall that a continuous function $f: R^n \to R^n$ is piecewise smooth if there exist continuously differentiable functions $f_j: R^n \to R^n$ such that

$$f(x) \in \{f_1(x), f_2(x), \dots, f_J(x)\} \qquad \forall x \in R^n.$$

(If each f_j is affine, f is said to be *piecewise affine*.) Such a function is semismooth [14] and its Bouligand differential is given by

$$\partial_B f(x^*) := \{Jf_i(x^*) : i \in I^e(f, x^*)\},$$

where $I^e(f, x^*)$ is the collection of (so-called essentially active) indexes i such that $x^* \in cl\, int\{z : f(z) = f_i(z)\}$ [31, Proposition 4.1.3].

EXAMPLE 4. In connection with generalized Newton methods for solving nonsmooth equations, Qi [24] introduces the concept of C-differentiability and studies its calculus rules. He calls a function $f : R^n \to R^n$, *C-differentiable* if for each $a \in R^n$, there exists a nonempty compact subset $T(a)$ (called a C-differential of f at a) such that

(i) the multivalued map $x \mapsto T(x)$ is upper semicontinuous at each point a,

(ii) for every $V \in T(x)$,

$$f(x) - f(a) - V(x - a) = o(\|x - a\|).$$

He notes that for a semismooth function f, the generalized Jacobian $\partial f(a)$ and the Bouligand differential $\partial_B(a)$ are examples of C-differentials. We observe here that a C-differentiable function is H-differentiable with the H-differential given by a C-differential.

4. SOME ELEMENTARY PROPERTIES OF H-DIFFERENTIALS

PROPOSITION 2. *Suppose Ω is an open subset of R^n and $f : \Omega \to R^n$ has an H-differential $T(a)$ at $a \in \Omega$ consisting of nonsingular matrices. Then a is an isolated point of the equation $f(x) = f(a)$.*

Proof. Assume the contrary that there is a sequence $x^k \to a$ such that $x^k \neq a$ and $f(x^k) = f(a)$ for all k. By the H-differentiability, there exists $A \in T(a)$ such that

$$A(x^{k_j} - a) = o(\|x^{k_j} - a\|)$$

and by a standard normalization argument, we produce a unit vector h such that $Ah = 0$. This contradicts the nonsingularity of matrices in $T(a)$. Hence the conclusion. ∎

PROPOSITION 3. *Let Ω be an open subset of R^n and suppose that the continuous function $f : \Omega \to R^n$ has an H-differential $T(a)$ at $a \in \Omega$ and there exists a matrix C such that*

$$tA + (1-t)C \text{ has positive determinant for all } A \in T(a), t \in [0,1]. \quad (5)$$

Then

$$\text{index}(f, a) = 1.$$

In particular, this conclusion holds under any one of the following conditions:

(a) *$T(a)$ consists of nonsingular* $\mathbf{P}_0$*-matrices*;

(b) *$T(a)$ is convex and consists of matrices with positive determinant*;

(c) *f is Fréchet differentiable at a with* $\det Jf(a) > 0$.

Proof. Since each $A \in T(a)$ is nonsingular, by the previous proposition, a is an isolated solution of $f(x) = f(a)$, hence the index of f at a is defined. We claim that for all small positive ε, the function

$$H(x,t) := t[f(x) - f(a)] + (1-t)C(x-a)$$

is never zero on the boundary of the ball $B(a; \varepsilon)$ for any $t \in [0,1]$. Assuming the contrary, we consider the sequences $x^k \to a$ and $t_k \to t^*$ such that

$$t_k[f(x^k) - f(a)] + (1-t_k)C(x^k - a) = 0$$

for all k. Using the H-differentiability of f at a and going through a subsequence of x^k, we see that there exist $M \in T(a)$ and a unit vector h such that $[t^*M + (1-t^*)C]h = 0$. We reach a contradiction since by assumption, $t^*M + (1-t^*)C$ is nonsingular, thus the claim about $H(x,t)$. Since for all small positive ε, $f(x) - f(a)$ is homotopic to $C(x-a)$ on the closure of $B(a; \varepsilon)$, by the homotopy invariance of the degree (cf. [13, Theorem 2.1.2]),

$$\begin{aligned} \text{index}(f,a) &= \deg(f, B(a;\varepsilon), f(a)) \\ &= \deg(C, B(a;\varepsilon), Ca) = \det C = 1. \end{aligned}$$

When $T(a)$ consists of nonsingular $\mathbf{P}_0$-matrices, condition (5) holds with $C = I$; when (b) holds, condition (5) holds for any $C \in T(a)$. When f is Fréchet differentiable with $\det Jf(a) > 0$, we can take $T(a) = \{Jf(a)\}$ and apply (b). Hence in these situations, we have the result. ∎

Remark. (7) A condition similar to (5) has been used by Kojima and Saigal [12].

Proofs of the following results are straightforward.

PROPOSITION 4 (Sum Rule for H-Differentiability). *Suppose that $\Omega \subseteq R^n$ is open, and f and g from Ω to R^m are H-differentiable at $a \in \Omega$ with H-differentials $T(a)$ and $S(a)$, respectively. Then $f + g$ is H-differentiable with an H-differential given by*

$$(T + S)(a) := \{A + B : A \in T(a), B \in S(a)\}.$$

PROPOSITION 5 (Chain Rule). *Suppose that $\Omega \subseteq R^n$ and $\Omega' \subseteq R^m$ are open, $f : \Omega \to R^m$ is H-differentiable at $a \in \Omega$ with H-differential $T(a)$, and $g : \Omega' \to R^l$ is H-differentiable at $b := f(a) \in \Omega'$ with an H-differential $S(b)$. Then $g \circ f$ is H-differentiable at a with an H-differential given by*

$$(S \circ T)(a) := \{BA : A \in T(a), B \in S(b)\}.$$

5. UNIVALENCE RESULTS

In this section, we describe various univalence results. We first define the concept of a restricted H-differential (like the one-sided derivative of a function at a point in R).

DEFINITION 2 (Restricted H-Differential). Let $a \in S \subseteq R^n$ with a being a limit point of S, and $f : S \to R^m$. We say that a (nonempty) subset $T_S(a)$ of $R^{m \times n}$ is an H_S-differential of f at a if for every sequence $\{x^k\}$ in S converging to a, there exists a subsequence $\{x^{k_j}\}$ and a matrix $A \in T_S(a)$ such that

$$f(x^{k_j}) - f(a) - A(x^{k_j} - a) = o(\|x^{k_j} - a\|). \tag{6}$$

We say that f is H_S-differentiable at a if f has an H_S-differential at a.

Note that in this definition, the sequences $\{x^k\}$ are taken only from S. Of course, when S is a neighborhood of a, this definition coincides with Definition 1. We also note that when f is H-differentiable at a, it is H_S-differentiable at a with $T(a)$ as an H_S-differential.

5.1. *Univalence over Compact Rectangles*

THEOREM 1. *Suppose K is a compact rectangle R^n, $f : K \to R^n$ is continuous, and at each $a \in$ bdy K, f is H_K-differentiable with an H_K-differential $T_K(a)$. Assume further that*

(i) *for all $a \in$ int K,* index(f, a) *is defined and positive, and*

(ii) *for all $a \in$ bdy K, $T_K(a)$ consists of* **P**-*matrices.*

Then f is one-to-one on K. Moreover, the normal map

$$F(x) := f(\Pi_K(x)) + x - \Pi_K(x) \tag{7}$$

is one-to-one on R^n.

Our proof of this theorem relies on degree theory and closely follows the arguments of Mas-Colell [15] and Kojima and Saigal [12]. The following lemma and its proof are basically in these cited references; we provide them here for completeness.

LEMMA 1. *Suppose K is a compact convex set in R^n, $f: K \to R^n$ is continuous, and*

$$F(x) := f(\Pi_K(x)) + x - \Pi_K(x). \tag{8}$$

If at each $a \in R^n$, index(F, a) *is defined and positive, then f is one-to-one on K.*

Proof. Suppose, if possible, there are vectors x_1 and x_2 in K such that

$$x_1 \neq x_2 \qquad \text{and} \qquad f(x_1) = f(x_2) = y^*.$$

Let $B(0; r)$ denote the open ball of radius r around the origin. We first claim that *for some $r > 0$, $K \subset B(0; r)$ and*

$$\deg(F, B(0; r), y^*) = 1. \tag{9}$$

To see this claim, let

$$\alpha := \max_{z \in K} \|z - f(z) + y^*\|$$

and take $r > \alpha$ large so that $K \subset B(0; r)$. Then for $\|x\| = r$, we have

$$\begin{aligned} \langle F(x) - y^*, x \rangle &= \|x\|^2 - \langle \Pi_K(x) - f(\Pi_K(x)) + y^*, x \rangle \\ &\geq r^2 - \alpha r \\ &> 0. \end{aligned}$$

This shows that the function $G(x, t) := t[F(x) - y^*] + (1 - t)x$ never vanishes on the boundary of $B(0; r)$ for any $t \in [0, 1]$. Hence by the homotopy invariance of degree [13, Theorem 2.1.2],

$$\deg(F, B(0; r), y^*) = \deg(\text{id}, B(0; r), 0) = 1,$$

where id denotes the identity map on R^n. Thus we have (9). We know at this particular stage that there are at least two solutions of the equation $F(x) = y^*$ in $B(0; r)$ and no solutions on the boundary of $B(0; r)$. Now, suppose that at each point a of R^n, index(F, a) is defined and positive. Then every solution of the equation $F(x) = y^*$ is isolated and so we may write

$$F^{-1}(y^*) \cap B(0; r) = \{x_1, x_2, x_3, \ldots, x_L\},$$

where $L \geq 2$. By (9) and the domain decomposition property of the degree [13, Theorem 2.2.1],

$$1 = \deg(F, B(0; r), y^*) = \sum_1^L \operatorname{index}(F, x_i).$$

Since by assumption, the index of F at each x_i is greater than or equal to one, we reach a contradiction. Hence f must be one-to-one on K. This completes the proof of the lemma. ∎

Proof of the Theorem. Consider the continuous function F defined by (7). We show that the index of F at any point $a \in R^n$ is defined and positive. Then the above lemma gives the desired result. Since $F = f$ on K, at any $a \in \operatorname{int} K$, index $(F, a) = $ index$(f, a) \geq 1$ by assumption (i). Now suppose $a \notin \operatorname{int} K$. We show that F is H-differentiable at a with an H-differential consisting of **P**-matrices. To see this, we proceed as follows. Let, for ease of notation,

$$g(x) = \Pi_K(x).$$

Since K is a rectangle, g is piecewise smooth (in fact, piecewise affine) and hence (see Example 3), $\partial_B g(a)$ is an H-differential at every a. We claim that

$$R(a) := \{VA + I - A : V \in T_K(g(a)), A \in \partial_B g(a)\}$$

is an H-differential of F at a. To see this claim, consider a sequence $x^k \to a$. Then there is a subsequence of $\{x^k\}$ which we continue to write as $\{x^k\}$ for simplicity and a matrix $A \in \partial_B g(a)$ such that

$$g(x^k) - g(a) - A(x^k - a) = o(\|x^k - a\|). \tag{10}$$

Now let

$$y^k := g(x^k) \qquad \text{and} \qquad b := g(a)$$

so that $y^k \to b$. Since $y^k \in K$ for all k, from the definition of H_K-differentiability of f at b, there is a subsequence of $\{y^k\}$ which we continue to write as $\{y^k\}$ and a matrix $V \in T_K(b)$ such that

$$f(y^k) - f(b) - V(y^k - b) = o(\|y^k - b\|).$$

We easily verify that

$$F(x^k) - F(a) = [VA + I - A](x^k - a) + o(\|y^k - b\|) + o(\|x^k - a\|).$$

Since $y^k - b = A(x^k - a) + o(\|x^k - a\|)$ from (10), we see the H-differentiability of F with $R(a)$ as an H-differential. This proves the above claim. We now show that every matrix in $R(a)$ is a **P**-matrix. Let $M \in R(a)$ so that $M = VA + I - A$ where $V \in T_K(g(a))$ and $A \in \partial_B g(a)$. By our assumption, V is a **P**-matrix. Also, it is easily seen that A is a diagonal matrix whose diagonal consists of zeros and ones. By the form of M, it is easy to see that M is also a **P**-matrix. So we have proved that the H-differential $R(a)$ of F consists of **P**-matrices. By Proposition 3, F has index one at every point $a \notin \operatorname{int} K$. By the previous lemma, we have the one-to-oneness of f on K.

Now for the one-to-oneness of F, suppose, if possible, $u_1 \neq u_2$ and $F(u_1) = F(u_2)$. Let K^* be a bounded rectangle that contains u_1, u_2, and K in its interior. From the properties of F derived above, we see (via Proposition 3) that the conditions of the theorem are met for the rectangle K^* and the function F. We conclude that F is one-to-one on K^* leading to a contradiction. Hence F is one-to-one on R^n. ∎

Remarks. (8) In the proof of the above theorem, the **P**-matrix condition on the matrices of $T_K(a)$ (for $a \in \operatorname{bdy} K$) was used to show that every matrix $M \in R(a)$ is a **P**-matrix and hence F has index one at any $a \notin \operatorname{int} K$. *This condition on $T_K(a)$ can be weakened by demanding that the matrices in $R(a)$ be* **P**-*matrices for all $a \in R^n$*; *this means that for each $a \in \operatorname{bdy} K$, only certain minors of $T_K(a)$ need be positive*. Mas-Colell [15] exploits this idea in his univalence result on compact polyhedral sets. By considering a continuously differentiable function f on a compact polyhedral set K with nonempty interior, he shows, under appropriate conditions, that the index of the normal map F (see Lemma 1) is positive at any a, thereby getting the univalence of f on K. Kojima and Saigal [12] extend the validity of Mas-Colell's argument to a piecewise smooth function based on polyhedral subdivisions. Going in a different direction, Robinson [28] extends Mas-Colell's result to strongly Fréchet differentiable functions defined on noncompact polyhedral sets (and satisfying a certain properness condition). Robinson's analysis, unlike those of Mas-Colell and Kojima and Saigal, is based on strong approximation of functions and on homeomorphism properties of normal maps induced by linear transformations.

(9) Because of the use of the normal map in Lemma 1 and the theorem above, the conditions imposed invariably involve positive determinants of matrices resulting in the indexes being positive. There are univalence results where negative determinants play a role. For example, a result of Inada (see [18, p. 20]) says that if f is a continuously differentiable function on a rectangle K such that the Jacobian matrix $Jf(x)$ is an N-matrix (it is one whose principle minors are all negative) at all $x \in K$, then f is one-to-one on K.

(10) In [38], Zangwill imposes conditions on a continuously differentiable function that are different from Mas-Colell's conditions to obtain univalence on a set that is diffeomorphic to the closed unit ball of R^n.

(11) For a piecewise affine function defined on R^n, univalence can be characterized by the "coherence orientation" of the collection of matrices defining the function and an index condition; see [7].

COROLLARY 1. *Let K be a compact rectangle in R^n and $f: K \to R^n$ be continuous. Then f is one-to-one on K under any one of the following conditions*:

(a) *f is H_K-differentiable at each point of K; for all $a \in \operatorname{int} K$, $T_K(a)$ consists of nonsingular $\mathbf{P}_0$-matrices and for all $a \in \operatorname{bdy} K$, $T_K(a)$ consists of $\mathbf{P}$-matrices.*

(b) *f is locally Lipschitzian at each point of K; for all $a \in \operatorname{int} K$ (the generalized Jacobian) $\partial f(a)$ consists of matrices with positive determinant and for all $a \in \operatorname{bdy} K$, $\partial f(a)$ consists of $\mathbf{P}$-matrices.*

(c) *f is semismooth at each point of K; for all $a \in \operatorname{int} K$ (the Bouligand differential) $\partial_B f(a)$ consists of nonsingular $\mathbf{P}_0$-matrices and for all $a \in \operatorname{bdy} K$, $\partial_B f(a)$ consists of $\mathbf{P}$-matrices.*

(d) *f is Fréchet differentiable at each point of K; at all $a \in \operatorname{int} K$, the Jacobian matrix $Jf(a)$ has positive determinant and at all points of $\operatorname{bdy} K$, the Jacobian matrix is a $\mathbf{P}$-matrix.*

Proof. When $a \in \operatorname{int} K$, under each of the conditions above, the H_K-differentiability coincides with the H-differentiability; we may write $T(a)$ for $T_K(a)$. Also, at such an a, under each of the above conditions, by Proposition 3, index (f, a) is one. The univalence follows from the previous theorem. ∎

Remark. (12) The special case of (d) above for a continuously differentiable f recovers a result of Garcia and Zangwill [6] and the rectangular version of a result of Mas-Colell [15]. Thus we have answered a question raised by Parthasarathy [18] whether Garcia and Zangwill's result is valid for differentiable functions.

5.2. *Univalence over Closed Rectangles*

THEOREM 2. *Suppose K is a closed rectangle in R^n, $f: K \to R^n$ is continuous, and at all $a \in K$, f is H_S-differentiable with $T_K(a)$ consisting of* **P**-*matrices. Then f is one-to-one on K.*

Proof. By part (a) of Corollary 1, f is one-to-one on each compact rectangle contained in K. Hence f is one-to-one on K. ∎

Remark. (13) As in Corollary 1, we can specialize f and deduce various consequences of the above theorem. For example, if f is Fréchet differentiable on the (closed) rectangle K and the Jacobian matrix $Jf(a)$ is a **P**-matrix at each point $a \in K$, then f is univalent on K. Thus we recover the Gale–Nikaido theorem.

5.3. *Univalence over Open Rectangles*

THEOREM 3. *Suppose D is an open rectangle in R^n, L be a compact subset of D, and $f: D \to R^n$ with the following properties.*

(a) *f is continuously differentiable at each point of L and determinant $Jf(a) > 0$ for all $a \in L$,*

(b) *at each $a \in D \setminus L$, f is H-differentiable with $T(a)$ consisting of* $\mathbf{P}_0$-*matrices,*

(c) *the set*

$$E := \{a \in D \setminus L : T(a) \text{ contains a singular matrix}\}$$

is discrete (*i.e.*, *every point of E is isolated*). *Then f is univalent on D.*

Proof. Since (continuous or H-) differentiability implies continuity, f is continuous on Ω. By item (a), for all small positive ε, the determinant $Jf(a) + \varepsilon I$ is positive at every point of L. For all such ε, $f(x) + \varepsilon x$ is H-differentiable on $D \setminus L$ with an H-differential ($= T(a) + \varepsilon I$) consisting of **P**-matrices. By Proposition 3, the index of $f(x) + \varepsilon x$ is positive at any point of L. By application of Theorem 1, we see that for all small positive ε, $f(x) + \varepsilon x$ is univalent on any compact rectangle between L and D and hence on D. It follows from the definition that f is weakly univalent on D. Now for a fixed $a \in D$, consider the equation $f(x) = f(a)$. If there is a solution u of this equation that is not in E, then by Proposition 2, u is an isolated solution and hence by Proposition 1, u is the only solution in which case $u = a$. Now suppose that every solution of this equation is in E. Since every point of E is isolated, a (being an element of E) is an isolated solution of the equation $f(x) = f(a)$. Once again, we apply Proposition 1 to conclude that $f(x) = f(a)$ has a unique solution, namely, a. Since a is arbitrary, we conclude that f is one-to-one. This completes the proof of the theorem. ∎

By taking L to be the empty set, we obtain the following corollary.

COROLLARY 2. *Suppose D is an open rectangle in R^n and $f : D \to R^n$ is H-differentiable at every $a \in D$ with $T(a)$ consisting of $\mathbf{P}_0$-matrices. If the set of points a in D for which the corresponding $T(a)$ contains a singular matrix is discrete, then f is univalent on D. In particular, if $T(a)$ consists of nonsingular $\mathbf{P}_0$-matrices for all $a \in D$, then f is univalent on D.*

Remark. (14) The above theorem and its corollary are similar to the following results.

Chua and Lam [3, Theorem 2.1]. Let $n \neq 2$. Suppose $f : R^n \to R^n$ is a continuously differentiable norm-coercive function with the property that the determinant of $Jf(x)$ is zero on a set of discrete points and positive elsewhere. Then f is a homeomorphism of R^n.

The function $f : R^2 \to R^2$ defined by $f(x, y) = (x^2 - y^2, 2xy)$ shows that the above result is false for $n = 2$. We also note that for this function, the index(f, a) is positive at any a.

Parthasarathy [18, Theorem 2′, p. 48]. Suppose $f : R^n \to R^n$ is continuously differentiable with det $Jf(x) > 0$ everywhere. If $Jf(x)$ is a $\mathbf{P}_0$-matrix at every point outside of a bounded rectangle, then f is one-to-one.

Parthasarathy [18, Theorem 2″, p. 48]. Let $n \neq 2$. Suppose $f : R^n \to R^n$ is continuously differentiable with det $Jf(x) > 0$ at all x except on a set of isolated points where det $Jf(x) = 0$. Suppose further that $Jf(x)$ is a $\mathbf{P}_0$-matrix at every point outside of a bounded rectangle. Then f is one-to-one.

Remark. (15) It is not clear if condition (a) in Theorem 3 can be replaced by

(a′) At each point $a \in L$, index(f, a) is defined and positive.

6. A SUFFICIENT CONDITION FOR MONOTONICITY

In this section, we give an interesting application of Theorem 2. Recall that a function $f : S \subseteq R^n \to R^n$ is said to be *monotone on S* if for all x, $y \in S$,

$$\langle f(x) - f(y), x - y \rangle \geq 0. \tag{11}$$

We say that f is *strictly monotone* if strict inequality holds in (11) for all $x \neq y$ in S.

THEOREM 4. *Let S be a convex set in R^n and $f: S \to R^n$ be H_S-differentiable at each point a of S with an H_S-differential $T_S(a)$ consisting of positive definite (positive semidefinite) matrices. Then f is strictly monotone (respectively, monotone) on S.*

We recall that a (square) matrix M is positive definite (semidefinite) if the function $f(x) = Mx$ is strictly monotone (monotone) on R^n.

Proof. We first prove the strict monotonicity of f under the assumption that $T_S(a)$ consists of positive definite matrices at every $a \in S$. Suppose, if possible, that there are distinct points x and y in S such that

$$\langle f(x) - f(y), x - y \rangle \leq 0.$$

Define the function $\phi : [0, 1] \to R$ by

$$\phi(t) := \langle f(y + t(x - y)) - f(y), x - y \rangle.$$

It is easily seen that ϕ is $H_{[0,1]}$-differentiable at any point $t^* \in [0, 1]$ with an $H_{[0,1]}$-differential given by

$$T_{[0,1]}(t^*) = \{\langle A(x - y), x - y \rangle : A \in T_S(y + t^*(x - y))\}.$$

Since every A is assumed to be positive definite and $x \neq y$, we see that $T_{[0,1]}(t^*)$ consists only of positive numbers. By Theorem 2, ϕ is one-to-one on $[0, 1]$. But the continuity of ϕ along with $\phi(0) = 0$ and $\phi(1) \leq 0$ clearly contradicts this one-to-oneness. Thus f is strictly monotone on S. Finally when $T_S(a)$ consists of positive semidefinite matrices, we can apply the previous argument to the function $f(x) + \varepsilon x$ for any small positive ε and let ε go to zero to conclude that f is monotone. ∎

Remark. (16) The above result is well known for Fréchet differentiable functions. For locally Lipschitz functions with the generalized Jacobian as an H-differential, the result appears in [10].

7. CONCLUDING REMARKS

In this paper, by introducing the concepts of H-differentiability and the H-differential of a function, we have given sufficient conditions for a function to be one-to-one on a (compact/closed/open) rectangle thus generalizing the results of Gale and Nikaido, Garcia and Zangwill, Mas-Colell, and Robinson to nonsmooth functions. It is worth pointing out that in their paper [5], Gale and Nikaido prove much more than univalence. They actually show that if the Jacobian matrix of a Fréchet differentiable function f is a **P**-matrix at every point of a rectangle, then f is a **P**-function on that rectangle; that is, for all $x \neq y$ in the rectangle,

$$\max_{\{i : x_i \neq y_i\}} (x_i - y_i)[f_i(x) - f_i(y)] > 0.$$

This raises the question of whether the **P**-property can be established in the nonsmooth situation as well. For an affirmative answer, we refer to Song and Gowda [33].

REFERENCES

1. S. Banach and S. Mazur, Uber mehrdeutige stetige Abildungen, *Studia Math.* **5** (1934), 174–178.
2. M. S. Berger, "Nonlinearity and Functional Analysis," Academic Press, New York, 1977.
3. L. O. Chua and Y. F. Lam, Global homeomorphisms of vector valued functions, *J. Math. Anal. Appl.* **39** (1972), 600–624.
4. F. H. Clarke, "Optimization and Nonsmooth Analysis," Les Publications CRM, Montreal, 1989.
5. D. Gale and H. Nikaido, The Jacobian matrix and global univalence of mappings, *Math. Ann.* **159** (1965), 81–93.
6. C. B. Garcia and W. I. Zangwill, On univalence and P-matrices, *Linear Algebra Appl.* **24** (1979), 239–250.
7. M. S. Gowda, An analysis of zero set and global error bound properties of a piecewise affine function via its recession function, *SIAM J. Matrix Anal. Appl.* **17** (1996), 594–609.
8. M. S. Gowda and R. Sznajder, Weak univalence and connectedness of inverse images of continuous functions, *Math. Oper. Res.* **24** (1999), 255–261.
9. J. Hadamard, Sur les transformations ponctuelles, *Bull. Soc. Math. France* **34** (1906), 71–84.
10. H. Jiang and L. Qi, Local uniqueness and convergence of iterative methods for nonsmooth variational inequalities, *J. Math. Anal. Appl.* **196** (1995), 314–331.
11. Y. Kawamura, Invertibility of Lipschitz continuous mappings and its application to electrical network equations, *SIAM J. Math. Anal.* **10** (1979), 253–265.
12. M. Kojima and R. Saigal, A study of PC^1 homeomorphism on subdivided polyhedrons, *SIAM J. Math. Anal.* **10** (1979), 1299–1312.
13. N. G. Lloyd, "Degree Theory," Cambridge Univ. Press, Cambridge, UK, 1978.
14. R. Mifflin, Semismooth and semiconvex functions in constrained optimization, *SIAM J. Control Optim.* **15** (1977), 952–972.
15. A. Mas-Colell, Homeomorphisms of compact convex sets and the Jacobian matrix, *SIAM J. Math. Anal.* **10** (1979), 1105–1109.
16. G. H. Meisters, "Annotated Bibliography on Polynomial Maps," Department of Mathematics and Statistics, University of Nebraska, Lincoln, NE, July 1997.
17. J.-S. Pang and L. Qi, Nonsmooth equations: Motivation and algorithms, *SIAM J. Optim.* **3** (1993), 443–465.
18. T. Parthasarathy, "Global Univalence Theorems," Lecture Notes in Math., Vol. 977, Springer-Verlag, New York, 1983.
19. R. Plastock, Homeomorphisms between Banach spaces, *Trans. Amer. Math. Soc.* **200** (1974), 169–183.

20. B. H. Pourciau, Hadamard's theorem for locally Lipschitzian maps, *J. Math. Anal. Appl.* **85** (1982), 279–285.
21. B. H. Pourciau, Global properties of proper Lipschitzian maps, *SIAM J. Math. Anal.* **14** (1983), 796–799.
22. B. H. Pourciau, Global invertibility of nonsmooth mappings, *J. Math. Anal. Appl.* **131** (1988), 170–179.
23. L. Qi, Convergence analysis of some algorithms for solving nonsmooth equations, *Math. Oper. Res.* **18** (1993), 227–244.
24. L. Qi, C-differentiability, "C-Differential Operators and Generalized Newton Methods," Research Report, School of Mathematics, The University of New South Wales, Sydney, New South Wales 2052, Australia, January 1996.
25. L. Qi and J. Sun, A nonsmooth version of Newton's method, *Math. Programming* **58** (1993), 353–367.
26. M. Radulescu and S. Radulescu, Global inversion theorems and applications to differential equations, *Nonlinear Anal.* **4** (1980), 951–965.
27. S. Radulescu and M. Radulescu, Global univalence and global inversion theorems in Banach spaces, *Nonlinear Anal.* **13** (1989), 539–553.
28. S. M. Robinson, Homeomorphism conditions for normal maps of polyhedra, *in* "Optimization and Nonlinear Analysis" (A. Ioffe, M. Marcus, and S. Reich, Eds.), Pitman Research Notes in Mathematics, Vol. 244, pp. 240–248, Essex, England, 1992.
29. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
30. R. T. Rockafellar and R. J.-B. Wets, "Variational Analysis," Grundlehren Math. Wiss., Vol. 317, Springer-Verlag, Berlin, 1998.
31. S. Scholtes, Introduction to piecewise differentiable functions, preprint, 53/1994, Institute für Statistik und Mathematische Wirtschaftstheorie, Univeristät Karlsruhe, 7500 Karlsruhe, Germany, May 1994.
32. A. Shapiro, On concepts of directional differentiability, *J. Optim. Theory Appl.* **66** (1990), 477–487.
33. Y. Song and M. S. Gowda, On characterizations of $\mathbf{P}$ and $\mathbf{P}_0$ properties in nonsmooth functions, *Math. Oper. Res.*, in press.
34. H. Sussmann, Multidifferential calculus: Chain rule, open mapping and transversal intersection theorems, *in* "Optimal Control: Theory, Algorithms and Applications" (W. W. Hager and P. M. Pardalos, Eds.), Kluwer Academic, Dordrecht, 1997.
35. J. Warga, Derivate containers, inverse functions, and controllability, *in* "Calculus of Variations and Control Theory" (D. L. Russell, Ed.), Academic Press, New York, 1976.
36. J. Warga, Fat homeomorphisms and unbounded derivate containers, *J. Math. Anal. Appl.* **81** (1981), 545–560.
37. J. Warga, Homeomorphisms and local C^1 approximations, *Nonlinear Anal.* **12** (1988), 593–597.
38. W. I. Zangwill, Nonconvexities in univalence, *Math. Oper. Res.* **8** (1983), 196–204.