

A pointwise weak-majorization inequality for linear maps over Euclidean Jordan algebras

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Abstract

Given a linear map T on a Euclidean Jordan algebra of rank n , we consider the set of all nonnegative vectors q in \mathcal{R}^n with decreasing components that satisfy the pointwise weak-majorization inequality $\lambda(|T(x)|) \prec_w q * \lambda(|x|)$, where λ is the eigenvalue map and $*$ denotes the componentwise product in \mathcal{R}^n . With respect to the weak-majorization ordering, we show the existence of the least vector in this set. When T is a positive map, the least vector is shown to be the join (in the weak-majorization order) of eigenvalue vectors of $T(e)$ and $T^*(e)$, where e is the unit element of the algebra. These results are analogous to the results of Bapat [4], proved in the setting of the space of all $n \times n$ complex matrices with singular value map in place of the eigenvalue map. We also extend two recent results of Tao, Jeong, and Gowda [22] proved for quadratic representations and Schur product induced transformations. As an application, we provide an estimate on the norm of a general linear map relative to spectral norms.

Key Words: Euclidean Jordan algebra, eigenvalue map, weak-majorization ordering, positive map, spectral norm

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1 Introduction

This paper deals with a pointwise weak-majorization inequality for an arbitrary linear map on a Euclidean Jordan algebra. Our motivation comes from several sources. In a 1991 paper, Bapat [4] proves the following result for a linear map T on the space \mathcal{M}_n of all $n \times n$ complex matrices: *There is a unique nonnegative vector $\eta(T)$ with decreasing components such that*

$$s(T(X)) \prec_w \eta(T) * s(X) \text{ for all } X \in \mathcal{M}_n,$$

with the additional property that if the above inequality holds for some q in place of $\eta(T)$, then $\eta(T) \prec_w q$. Here, $s(X)$ denotes the vector of singular values of X in \mathcal{M}_n written in the decreasing order, $*$ denotes the componentwise product in \mathcal{R}^n , and \prec_w is the weak-majorization preordering relation on \mathcal{R}^n . Specializing the above result, Bapat proves that if T is *positive* (meaning that it takes positive semidefinite matrices to positive semidefinite matrices), then, $\eta(T)$ is the join of the singular value vectors of $T(I)$ and $T^*(I)$ relative to the weak-majorization preordering, where I denotes the identity matrix.

Subsequently, in a 1999 paper, Niezgoda [20] studied a generalization in the setting of group majorization (Eaton triples) and described equivalent formulations for positive linear maps, see Theorem 3.1 and Examples 4.1 and 4.2 in [20].

Going in a different direction, in a recent paper, Tao, Jeong, and Gowda [22] proved, in the setting of Euclidean Jordan algebras, three weak-majorization inequalities. To elaborate, let $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ denote a Euclidean Jordan algebra of rank n with unit e and carrying the trace inner product. For $a \in \mathcal{V}$, consider the Lyapunov transformation L_a and the quadratic representation P_a defined, respectively, by

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x \quad (x \in \mathcal{V}).$$

Also, for any $n \times n$ real symmetric positive semidefinite matrix A and a fixed Jordan frame in \mathcal{V} , consider the corresponding Schur product induced linear transformation $D_A : \mathcal{V} \rightarrow \mathcal{V}$ defined by $D_A(x) = A \bullet x$. For any $x \in \mathcal{V}$, let $\lambda(x)$ denote the vector of eigenvalues of x written in the decreasing order. In [22], the following results were proved.

- $\lambda(|L_a(x)|) \prec_w \lambda(|a|) * \lambda(|x|)$ for all $x \in \mathcal{V}$.
- $\lambda(|P_a(x)|) \prec_w \lambda(a^2) * \lambda(|x|)$ for all $x \in \mathcal{V}$.
- $\lambda(|D_A(x)|) \prec_w \lambda(|\text{diag} A|) * \lambda(|x|)$ for all $x \in \mathcal{V}$.

In the above, the transformations L_a , P_a , and D_A are all self-adjoint and the last two are *positive*, that is, they keep the symmetric cone \mathcal{V}_+ of \mathcal{V} invariant. Writing T for any one of the these transformations, above statements could be written in a unified way:

$$\lambda(|T(x)|) \prec_w \lambda(|T(e)|) * \lambda(|x|) \text{ for all } x \in \mathcal{V}.$$

Motivated by the strong similarity between the results of Bapat and of Tao et al., we raise the question whether Bapat's results have analogs in the setting of Euclidean Jordan algebras. While there is a close connection between the C^* -algebra \mathcal{M}_n and the Euclidean Jordan algebra \mathcal{H}^n of all $n \times n$ complex Hermitian matrices, lack of matrix type multiplication and associative properties in a general Euclidean Jordan algebra hinder routine/obvious generalizations of results and proofs. Yet, with powerful and elegant Euclidean Jordan algebra machinery we show the following results: *Given any linear map $T : \mathcal{V} \rightarrow \mathcal{V}$, there exists a unique nonnegative vector $\eta(T)$ in \mathcal{R}^n with decreasing components such that*

$$\lambda(|T(x)|) \prec_w \eta(T) * \lambda(|x|) \text{ for all } x \in \mathcal{V}, \quad (1)$$

with the additional property that if the above inequality holds with q in place of $\eta(T)$, then $\eta(T) \prec q$. Furthermore, if T is positive, then $\eta(T)$ is the join of eigenvalue vectors of $T(e)$ and $T^(e)$ in the weak-majorization preordering on \mathcal{R}^n . In particular, when T is positive and self-adjoint, we have $\lambda(|T(x)|) \prec_w \lambda(T(e)) * \lambda(|x|)$ for all $x \in \mathcal{V}$. We note that this last statement recovers the two results of Tao et al., stated for P_a and D_A (proved in [22] by different techniques).*

Now, constructing some q that satisfies the pointwise inequality $\lambda(|T(x)|) \prec_w q * \lambda(|x|)$ is easy: One can take a large positive multiple of the vector of ones in \mathcal{R}^n . Demonstrating the existence of 'least' q (which implies uniqueness) and describing this q for a positive map requires more and nontrivial work. In our analysis, three key results from Euclidean Jordan algebras are used. The first one is the '*Fan-Theobald-von Neumann inequality*' [18, 1, 13]:

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \quad (x, y \in \mathcal{V}). \quad (2)$$

The second one is the '*variational principle*' [1]:

$$S_k(x) := \lambda_1(x) + \lambda_2(x) + \cdots + \lambda_k(x) = \max_{c \in \mathcal{I}^{(k)}} \langle x, c \rangle, \quad (3)$$

where $\mathcal{I}^{(k)}$ denotes the set of all idempotents of rank k in \mathcal{V} . The third key result is a weak-majorization inequality ([22], Lemma 4.2): If $\varepsilon \in \mathcal{V}$ with $\varepsilon^2 = e$ and $x \in \mathcal{V}$, then

$$\lambda(|x \circ \varepsilon|) \prec_w \lambda(|x|) * \lambda(|\varepsilon|). \quad (4)$$

Now, observing the commonality of results in both \mathcal{M}_n and Euclidean Jordan algebras, one might wonder if there is a general framework/setting where a unified result could be obtained. We mention Eaton triples [5] (equivalently, normal decomposition systems [17] which includes both \mathcal{M}_n and simple Euclidean Jordan algebras) or more generally, Fan-Theobald-von Neumann systems [9] (which include Eaton triples and all Euclidean Jordan algebras) as possible candidates for such a unified approach. The work of Niezgoda [20] mentioned earlier may be taken as a starting point in this regard.

Here is an outline of the paper. In Section 2, we cover some preliminaries. Section 3 deals with our main result (1) for linear maps and a consequence for positive ones. It also includes a result on majorization: *There exists $q \in \mathcal{R}^n$ with decreasing components such that $\lambda(T(x)) \prec q * \lambda(x)$*

for all $x \in \mathcal{V}$ if and only if T is a scalar multiple of a doubly stochastic map. In Section 4, we describe some properties of the nonlinear map η defined on the set of all linear maps on \mathcal{V} . As an application, we provide an estimate on the norm of a linear map relative to spectral norms. Examples are provided in Section 5 and a few open problems are mentioned in Section 6.

2 Preliminaries

Throughout this paper, we let \mathcal{R}^n denote the real Euclidean n -space with the usual inner product and \mathcal{R}_+^n denote the set of all nonnegative vectors in \mathcal{R}^n . We say that a vector $q = (q_1, q_2, \dots, q_n)$ in \mathcal{R}^n has decreasing components or that its components are written in the decreasing order if $q_1 \geq q_2 \geq \dots \geq q_n$. We let

$$(\mathcal{R}_+^n)^\downarrow = \{q = (q_1, q_2, \dots, q_n) \in \mathcal{R}_+^n : q_1 \geq q_2 \geq \dots \geq q_n\}.$$

For $p, q \in \mathcal{R}^n$, we write $p \geq q$ if $p - q \in \mathcal{R}_+^n$ and let $p * q$ denote their componentwise product. We write $\mathbf{1}_k$ for the vector in \mathcal{R}^n with 1s in the first k slots, and 0s elsewhere; $\mathbf{1}$ denotes the vector of ones in \mathcal{R}^n . Given $p = (p_1, p_2, \dots, p_n) \in \mathcal{R}^n$, we write $|p| := (|p_1|, |p_2|, \dots, |p_n|)$ for the vector of absolute values and $p^\downarrow := (p_1^\downarrow, p_2^\downarrow, \dots, p_n^\downarrow)$ for the decreasing rearrangement of p ; the latter is the vector obtained by rearranging the entries of p in a decreasing order. We note the *Hardy-Littlewood-Pólya rearrangement inequality* $\langle p, q \rangle \leq \langle p^\downarrow, q^\downarrow \rangle$ and, as a consequence,

$$\langle p, q \rangle \leq \langle |p|, |q| \rangle \leq \langle |p|^\downarrow, |q|^\downarrow \rangle \quad (p, q \in \mathcal{R}^n). \quad (5)$$

Given two vectors p and q in \mathcal{R}^n , we say that p is *weakly majorized* by q and write $p \prec_w q$ if $\sum_{i=1}^k p_i^\downarrow \leq \sum_{i=1}^k q_i^\downarrow$ for all indices k , $1 \leq k \leq n$. If, in addition, $\sum_{i=1}^n p_i^\downarrow = \sum_{i=1}^n q_i^\downarrow$, we say that p is *majorized* by q and write $p \prec q$. For any $p \in \mathcal{R}^n$ and index $k \in \{1, 2, \dots, n\}$, $S_k(p)$ denotes the sum of k largest components of p , that is, $S_k(p) := \sum_{i=1}^k p_i^\downarrow$. We will use the following result ([2], Problem II.5.16) in \mathcal{R}^n :

$$[r \geq 0 \text{ and } p \prec_w q] \Rightarrow r^\downarrow * p^\downarrow \prec_w r^\downarrow * q^\downarrow \Rightarrow \langle r^\downarrow, p^\downarrow \rangle \leq \langle r^\downarrow, q^\downarrow \rangle. \quad (6)$$

Consider the ordering relation on \mathcal{R}^n induced by weak-majorization. While this relation is merely reflexive and transitive on \mathcal{R}^n , it becomes antisymmetric on $(\mathcal{R}_+^n)^\downarrow$; thus, it is a *partial order* on $(\mathcal{R}_+^n)^\downarrow$. In this regard, the following result of Bapat is useful.

Proposition 2.1. (Bapat [4], Lemma 3 and Corollary 4)

- (a) Let Q be a nonempty subset of \mathcal{R}_+^n . Then, there is a unique $q^* \in (\mathcal{R}_+^n)^\downarrow$ such that $q^* \prec_w q$ for all $q \in Q$ and if $p \in \mathcal{R}_+^n$ with $p \prec_w q$ for all $q \in Q$, then $p \prec_w q^*$. We write $\text{w-inf}(Q) := q^*$.
- (b) Suppose S is a nonempty bounded subset of \mathcal{R}_+^n . Then there is a unique $p^* \in (\mathcal{R}_+^n)^\downarrow$ such that $s \prec_w p^*$ for all $s \in S$ and if $p \in \mathcal{R}_+^n$ with $s \prec_w p$ for all $s \in S$, then $p^* \prec_w p$. We write $\text{w-sup}(S) := p^*$. When $S = \{r, s\}$, the ‘join’ of r and s is defined/denoted by $r \vee_w s := \text{w-sup}(S)$.

In Item (a) above, q^* is constructed as follows. For $1 \leq k \leq n$, let

$$\beta_k := \inf_{q \in Q} S_k(q)$$

and

$$r_k := \beta_k - \beta_{k-1},$$

where $\beta_0 := 0$. Then $q^* := (r_1, r_2, \dots, r_n)$. Also, in Item (b), for a nonempty bounded subset S of \mathcal{R}_+^n , one defines $Q := \{q : s \prec_w q \text{ for all } s \in S\}$ and $p^* := \text{w-inf}(Q)$.

Note that when $r, s \in (\mathcal{R}_+^n)^\downarrow$,

$$r \vee_w s \prec_w \max\{r, s\},$$

where $\max\{r, s\}$ is the componentwise maximum of r and s .

Throughout, we let $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ denote a Euclidean Jordan algebra of rank n with unit element e [6, 11]; the Jordan product and inner product of elements x and y in \mathcal{V} are, respectively, denoted by $x \circ y$ and $\langle x, y \rangle$. We note (one of the defining properties of a Euclidean Jordan algebra):

$$\langle x \circ y, z \rangle = \langle x, y \circ z \rangle, \text{ for all } x, y, z \in \mathcal{V}.$$

It is well known [6] that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra. According to the *spectral decomposition theorem* [6], every element $x \in \mathcal{V}$ has a decomposition $x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$, where the real numbers x_1, x_2, \dots, x_n are (called) the eigenvalues of x and $\{e_1, e_2, \dots, e_n\}$ is a Jordan frame in \mathcal{V} . (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same.) The *trace* of x is defined by $\text{tr}(x) := x_1 + x_2 + \dots + x_n$. It is known that $(x, y) \mapsto \text{tr}(x \circ y)$ defines another inner product on \mathcal{V} that is compatible with the given Jordan product. *Throughout this paper, we assume that the inner product on \mathcal{V} is this trace inner product, that is, $\langle x, y \rangle = \text{tr}(x \circ y)$.*

The *rank* of an element x is the number of nonzero eigenvalues of x . We use the notation $x \geq 0$ ($x > 0$) when all the eigenvalues of x are nonnegative (respectively, positive) and $x \geq y$ (or, $y \leq x$) when $x - y \geq 0$, etc. We let

$$\mathcal{V}_+ := \{x \in \mathcal{V} : x \geq 0\}$$

denote the *symmetric cone* of \mathcal{V} . It is known that \mathcal{V}_+ is a self-dual (closed convex) cone.

Given a function $\phi : \mathcal{R} \rightarrow \mathcal{R}$, the corresponding *Löwner map* (still denoted by ϕ) is defined via spectral decomposition:

$$\text{If } x = \sum_{i=1}^n x_i e_i, \text{ then } \phi(x) := \sum_{i=1}^n \phi(x_i) e_i.$$

In particular, for $x = \sum_{i=1}^n x_i e_i$, we define $|x| := \sum_{i=1}^n |x_i| e_i$ and $x^+ := \sum_{i=1}^n x_i^+ e_i$. By writing $|x_i| = x_i \varepsilon_i$, where $\varepsilon_i = 1$ when $x_i \geq 0$ and $\varepsilon_i = -1$ when $x_i < 0$, we see that $|x| = x \circ \varepsilon$, where

$\varepsilon^2 = e$ (in fact, $\varepsilon := \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_n e_n$).

For any $x \in \mathcal{V}$, $\lambda(x)$ – called the *eigenvalue vector* of x – is the vector of eigenvalues of x written in the decreasing order. We write

$$\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$$

and note $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$. It is known that $\lambda : \mathcal{V} \rightarrow \mathcal{R}^n$ is continuous [1].

We define *weak-majorization* and *majorization* in \mathcal{V} by: $x \prec_w y$ in \mathcal{V} if and only if $\lambda(x) \prec_w \lambda(y)$ in \mathcal{R}^n and $x \prec y$ in \mathcal{V} if and only if $\lambda(x) \prec \lambda(y)$ in \mathcal{R}^n . The following implication is a consequence of the well-known Hirzebruch's max-min theorem [13]:

$$x \leq y \Rightarrow \lambda(x) \leq \lambda(y).$$

In particular, we have $\lambda(x) \leq \lambda(|x|)$ for all $x \in \mathcal{V}$.

As $|\lambda(x)|^\downarrow = \lambda(|x|)$, combining (2) and (5), we get the following useful inequality:

$$\langle x, y \rangle \leq \langle \lambda(|x|), \lambda(|y|) \rangle \quad (x, y \in \mathcal{V}). \quad (7)$$

An element $c \in \mathcal{V}$ is an *idempotent* if $c^2 = c$; it is said to be a *primitive idempotent* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. By spectral decomposition theorem, corresponding to any nonzero idempotent c , there is a Jordan frame $\{e_1, e_2, \dots, e_n\}$ such that $c = e_1 + e_2 + \cdots + e_k$ for some k , $1 \leq k \leq n$. (Here, e_1, e_2, \dots, e_n are mutually orthogonal primitive idempotents); hence, the rank of such an idempotent is k . Let $\mathcal{I}^{(k)}$ denote the set of all idempotents of rank k , $1 \leq k \leq n$. We let

$$\mathcal{I} = \bigcup_{k=1}^n \mathcal{I}^{(k)} = \text{Set of all nonzero idempotents.}$$

As \mathcal{V} carries the trace inner product, the norm of any primitive idempotent is one and so $\text{tr}(c) = k$ for each $c \in \mathcal{I}^{(k)}$. As the set of all primitive idempotents is compact ([6], page 78), it follows easily that $\mathcal{I}^{(k)}$ is compact. Hence, \mathcal{I} is also compact.

In what follows, we use the letter ε for an element in \mathcal{V} with $\varepsilon^2 = e$. Such an element will have eigenvalues ± 1 . We let

$$\mathcal{E} := \{\varepsilon \in \mathcal{V} : \varepsilon^2 = e\} \quad \text{and} \quad \mathcal{I} \circ \mathcal{E} := \{c \circ \varepsilon : c \in \mathcal{I}, \varepsilon \in \mathcal{E}\}.$$

For any z in \mathcal{V} or in \mathcal{R}^n , $S_k(z)$ denotes the sum of the first k largest eigenvalues of z . (Note that \mathcal{R}^n is a Euclidean Jordan algebra in which the components of any vector are its eigenvalues, so our notation is consistent.)

We let $\mathcal{L}(\mathcal{V})$ denote the space of all (continuous) linear maps over \mathcal{V} . For $T \in \mathcal{L}(\mathcal{V})$, we say that T is

- (a) *positive* if $T(\mathcal{V}_+) \subseteq \mathcal{V}_+$;
- (b) *doubly substochastic* if T is positive, $T(e) \leq e$ and $T^*(e) \leq e$, where T^* denotes the adjoint

of T ;

- (c) *doubly stochastic* if T is positive, $T(e) = e$ and $T^*(e) = e$;
- (d) *an algebra automorphism* if T is invertible and $T(x \circ y) = T(x) \circ T(y)$ for all $x, y \in \mathcal{V}$;
- (e) *a cone automorphism*, if $T(\mathcal{V}_+) = \mathcal{V}_+$.

We respectively write $\text{DSS}(\mathcal{V})$, $\text{DS}(\mathcal{V})$, $\text{Aut}(\mathcal{V})$, and $\text{Aut}(\mathcal{V}_+)$ for the set of all doubly substochastic maps, doubly stochastic maps, algebra automorphisms, and cone automorphisms on \mathcal{V} . The following results are known:

- $\text{conv}(\text{Aut}(\mathcal{V})) \subseteq \text{DS}(\mathcal{V})$, where ‘conv’ stands for the convex hull, see [8].
- T is doubly substochastic if and only if $\lambda(T(x)) \prec_w \lambda(x)$ for all $x \geq 0$ in \mathcal{V} , see [16], Theorem 3.3.
- T is doubly stochastic if and only if $\lambda(T(x)) \prec \lambda(x)$ for all x in \mathcal{V} , see [15], Lemma 2.

3 Results

In this section, we establish our main result and state a consequence for positive maps. We will also consider a majorization result. *In what follows, \mathcal{V} is always assumed be to a (general) Euclidean Jordan algebra of rank n , unless otherwise stated.*

Theorem 3.1. *Let $T : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map. Then, there exists a unique $\eta(T)$ in $(\mathcal{R}_+^n)^\downarrow$ such that*

$$\lambda(|T(x)|) \prec_w \eta(T) * \lambda(|x|) \quad \text{for all } x \in \mathcal{V}, \quad (8)$$

with the additional property that if the above statement holds with q in place of $\eta(T)$, then $\eta(T) \prec_w q$.

Before the proof, we cover some preliminary results. Let $T : \mathcal{V} \rightarrow \mathcal{V}$ be an arbitrary (but fixed) linear map. Correspondingly, we define the following sets in $(\mathcal{R}_+^n)^\downarrow$:

$$Q := \{q \in (\mathcal{R}_+^n)^\downarrow : \lambda(|T(x)|) \prec_w q * \lambda(|x|) \quad \text{for all } x \in \mathcal{V}\}, \quad (9)$$

$$\Lambda := \{\lambda(|T(c \circ \varepsilon)|) : c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}\}, \quad (10)$$

$$\Lambda^* := \{\lambda(|T^*(c \circ \varepsilon)|) : c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}\}, \quad (11)$$

$$S := \Lambda \cup \Lambda^*. \quad (12)$$

We will show below that Q is nonempty. Hence, by Proposition 2.1, $\text{w-inf}(Q)$ is defined. Also, by the compactness of $\mathcal{I} \circ \mathcal{E}$ and the continuity of T and λ , it follows that Λ and Λ^* are both compact. Hence, $\text{w-sup}(\Lambda)$ and $\text{w-sup}(\Lambda^*)$ are defined. As S is the union of Λ and Λ^* , we have

$$\text{w-sup}(S) = \text{w-sup}(\Lambda) \vee_w \text{w-sup}(\Lambda^*).$$

In the following result, we describe the set Q in different, but equivalent ways.

Lemma 3.2. *Let T be a linear map on \mathcal{V} and $q \in (\mathcal{R}_+^n)^\downarrow$. Then, the following are equivalent:*

- (i) $\lambda(|T(x)|) \prec_w q * \lambda(|x|)$ for all $x \in \mathcal{V}$.
- (ii) $\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(|c \circ \varepsilon|)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$.
- (iii) $\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(|c|)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$.
- (iv) $\lambda(|T^*(x)|) \prec_w q * \lambda(|x|)$ for all $x \in \mathcal{V}$.
- (v) $\lambda(|T^*(c \circ \varepsilon)|) \prec_w q * \lambda(|c \circ \varepsilon|)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$.
- (vi) $\lambda(|T^*(c \circ \varepsilon)|) \prec_w q * \lambda(|c|)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$.

Proof. (i) \Rightarrow (ii): This follows by specializing x to $c \circ \varepsilon$.

(ii) \Rightarrow (iii): Suppose (ii) holds so that $\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(|c \circ \varepsilon|)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$. As $\lambda(|\varepsilon|) = \mathbf{1}$, in view of (4) and (6), this simplifies to $\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(|c|)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$, which is (iii).

(iii) \Rightarrow (iv): We assume (iii). To see (iv), we have to show that for each index k , $1 \leq k \leq n$, and $x \in \mathcal{V}$,

$$S_k(|T^*(x)|) \leq S_k(q * \lambda(|x|)).$$

Fix k and x , and let $c \in \mathcal{I}^{(k)}$ be arbitrary. Then, writing $|T^*(x)| = T^*(x) \circ \varepsilon$ for some $\varepsilon \in \mathcal{E}$, we have

$$\begin{aligned} \langle |T^*(x)|, c \rangle &= \langle T^*(x) \circ \varepsilon, c \rangle \\ &= \langle T^*(x), c \circ \varepsilon \rangle \\ &= \langle x, T(c \circ \varepsilon) \rangle \\ &\leq \langle \lambda(|x|), \lambda(|T(c \circ \varepsilon)|) \rangle \\ &\leq \langle \lambda(|x|), q * \lambda(|c|) \rangle \\ &= \langle q * \lambda(|x|), \lambda(|c|) \rangle, \end{aligned}$$

where the first inequality is due to (7), and the second one is due to condition (iii) coupled with (6). Then, as $\lambda(|c|) = \lambda(c) = \mathbf{1}_k$, we have

$$\langle |T^*(x)|, c \rangle \leq \langle q * \lambda(|x|), \lambda(c) \rangle = \sum_{i=1}^k (q * \lambda(|x|))_i = S_k(q * \lambda(|x|)).$$

Now, taking the maximum over $c \in \mathcal{I}^{(k)}$ and using (3), we get $S_k(|T^*(x)|) \leq S_k(q * \lambda(|x|))$. This proves that

$$\lambda(|T^*(x)|) \prec_w q * \lambda(|x|) \text{ for all } x \in \mathcal{V}.$$

(iv) \Rightarrow (v): This can be seen by specializing x to $c \circ \varepsilon$.

The implications (v) \Rightarrow (vi) and (vi) \Rightarrow (i) are seen by replacing T by T^* in the implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (iv). \square

Remark 1. Recall that a linear map T is *positive* if $T(\mathcal{V}_+) \subseteq \mathcal{V}_+$. For such a map, it is known ([22], Example 3.7) that

$$\lambda(|T(x)|) \prec_w \lambda(T(|x|)) \text{ for all } x \in \mathcal{V}. \quad (13)$$

Using this and with appropriate modifications, we can simplify Lemma 3.2 and its proof as follows (stated without proof):

When T is positive and $q \in (\mathcal{R}_+^n)^\downarrow$, the following are equivalent:

- (i) $\lambda(|T(x)|) \prec_w q * \lambda(|x|)$ for all $x \in \mathcal{V}$.
- (ii) $\lambda(T(x)) \prec_w q * \lambda(x)$ for all $x \geq 0$.
- (iii) $\lambda(T(c)) \prec_w q * \lambda(c)$ for all $c \in \mathcal{I}$.
- (iv) $\lambda(T^*(x)) \prec_w q * \lambda(x)$ for all $x \geq 0$.
- (v) $\lambda(T^*(c)) \prec_w q * \lambda(c)$ for all $c \in \mathcal{I}$.

In [20], Theorem 3.1, Niezgoda proves a result for certain types of linear maps in the setting of Eaton triples. When specialized, it will yield a result of the above type for positive linear maps on *simple* Euclidean Jordan algebras. (Note: So far, it is only known that every simple Euclidean Jordan algebra is a normal decomposition system, equivalently, an Eaton triple [18].)

Lemma 3.3. *Given a linear map $T : \mathcal{V} \rightarrow \mathcal{V}$, consider the sets Q and S defined in (9)-(12). Then the following hold:*

- (i) $s \prec_w q$ for all $s \in S$ and $q \in Q$.
- (ii) $\text{w-sup}(S) \in Q$.
- (iii) $\text{w-sup}(S) = \text{w-inf}(Q)$.

Proof. We first observe that Q is nonempty. This can be seen by taking $q = t \mathbf{1}$, where t is a large positive number, and using Item (iii) in Lemma 3.2 along with the compactness of Λ .

(i) Let $q \in Q$ so that (by the previous lemma), $\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(c)$ and $\lambda(|T^*(c \circ \varepsilon)|) \prec_w q * \lambda(c)$ for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$. Let $s \in S = \Lambda \cup \Lambda^*$. If $s \in \Lambda$, then $s = \lambda(|T(c \circ \varepsilon)|)$ for some $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$, in which case,

$$\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(c) \prec_w q,$$

where the second inequality follows from the nonnegativity of q and the fact that $\lambda(c) = \mathbf{1}_k$ for some index k . A similar statement ensues if $s \in \Lambda^*$. Hence, $s \prec_w q$. This proves (i). We also see, by Proposition 2.1 that

$$\text{w-sup}(S) \prec_w \text{w-inf}(Q).$$

(ii) Let $q := \text{w-sup}(S)$. To see $q \in Q$, it is enough to show by Lemma 3.2,

$$\lambda(|T(c \circ \varepsilon)|) \prec_w q * \lambda(c) \tag{14}$$

for all $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$. Consider an index k and $c \in \mathcal{I}^{(k)}$. Then (14) is equivalent to

$$S_l(|T(c \circ \varepsilon)|) \leq S_{\min\{k, l\}}(q) \quad \text{for all } 1 \leq l \leq n. \tag{15}$$

Now, fix $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$, $l \in \{1, 2, \dots, n\}$, and choose $\varepsilon' \in \mathcal{E}$ such that $|T(c \circ \varepsilon)| = T(c \circ \varepsilon) \circ \varepsilon'$, and let $c' \in \mathcal{I}^{(l)}$. If $l \leq k$, then we have

$$\begin{aligned} \langle |T(c \circ \varepsilon)|, c' \rangle &= \langle T(c \circ \varepsilon) \circ \varepsilon', c' \rangle \\ &\leq \langle \lambda(|T(c \circ \varepsilon) \circ \varepsilon'|), \lambda(c') \rangle \\ &\leq \langle \lambda(|T(c \circ \varepsilon)|), \lambda(c') \rangle \\ &\leq \langle q, \lambda(c') \rangle \\ &= S_l(q), \end{aligned}$$

where the first inequality is due to (7), and the second inequality is due to (4) coupled with (6). The last inequality is due to $\lambda(|T(c \circ \varepsilon)|) \prec_w q$ as $q = \text{w-sup}(S)$. By taking the supremum over $c' \in \mathcal{I}^{(l)}$, we get $S_l(|T(c \circ \varepsilon)|) \leq S_l(q)$.

On the other hand, if $l > k$, then

$$\begin{aligned} \langle |T(c \circ \varepsilon)|, c' \rangle &= \langle T(c \circ \varepsilon) \circ \varepsilon', c' \rangle \\ &= \langle T(c \circ \varepsilon), c' \circ \varepsilon' \rangle \\ &= \langle c \circ \varepsilon, T^*(c' \circ \varepsilon') \rangle \\ &= \langle c, [T^*(c' \circ \varepsilon')] \circ \varepsilon \rangle \\ &\leq \langle \lambda(c), \lambda([T^*(c' \circ \varepsilon')] \circ \varepsilon) \rangle \\ &\leq \langle \lambda(c), \lambda(|T^*(c' \circ \varepsilon')|) \rangle \\ &\leq \langle \lambda(c), q \rangle \\ &= S_k(q), \end{aligned}$$

where the first inequality is due to (7), second one due to (4), and the last one is to the inequality $\lambda(|T^*(c' \circ \varepsilon')|) \prec_w q = \text{w-sup}(S)$. Again, taking the supremum over $c' \in \mathcal{I}^{(l)}$ we get $S_l(|T(c \circ \varepsilon)|) \leq S_k(q)$. Hence, we have proved (15), so $q = \text{w-sup}(S) \in Q$. This proves (ii).

(iii) From (ii), $q := \text{w-sup}(S) \in Q$, hence, $\text{w-inf}(Q) \prec_w q = \text{w-sup}(S)$. From (i), the reverse inequality holds. Since the weak-majorization ordering is antisymmetric on $(\mathcal{R}_+^n)^\downarrow$, we have (iii). \square

We now come to the proof of our main theorem.

Proof of Theorem 3.1. Given T , we define Q and S as (9)-(12). Let $\eta(T) := \text{w-sup}(S)$. As this belongs to Q , we see statement (8) in Theorem 3.1. The additional item follows from the equality $\eta(T) = \text{w-sup}(S) = \text{w-inf}(Q)$. \square

Generally, finding/describing $\eta(T)$ may not be easy. However, when T is a positive map, we have a simple expression for $\eta(T)$. [We note that similar results have been obtained by Niezgoda \[21\] in the setting of Eaton triples.](#)

Corollary 3.4. *Let T be a positive linear map on \mathcal{V} . Then, $\text{w-sup}(\Lambda) = \lambda(T(e))$ and $\text{w-sup}(\Lambda^*) = \lambda(T^*(e))$. Hence,*

$$\eta(T) = \lambda(T(e)) \vee_w \lambda(T^*(e)).$$

In particular, if T is also self-adjoint, then $\eta(T) = \lambda(T(e))$.

Proof. Consider $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$. Since $\lambda(|c \circ \varepsilon|) \prec_w \lambda(c) \leq \lambda(e) = \mathbf{1}$, the first component in $\lambda(|c \circ \varepsilon|)$ is less than or equal to 1. Hence all components in $\lambda(|c \circ \varepsilon|)$ are less than or equal to one. It follows (by considering the spectral decomposition) that $|c \circ \varepsilon| \leq e$. As T is positive, $T(|c \circ \varepsilon|) \leq T(e)$ and so $\lambda(T(|c \circ \varepsilon|)) \leq \lambda(T(e))$. However, by (13),

$$\lambda(|T(c \circ \varepsilon)|) \prec_w \lambda(T(|c \circ \varepsilon|)).$$

Hence, $\lambda(|T(c \circ \varepsilon)|) \prec_w \lambda(T(e))$. As $c \circ \varepsilon$ is arbitrary in $\mathcal{I} \circ \mathcal{E}$, we see that

$$\text{w-sup}(\Lambda) \prec_w \lambda(T(e)).$$

But $e \in \mathcal{I} \circ \mathcal{E}$, and so, $\lambda(T(e)) \prec_w \text{w-sup}(\Lambda)$. Thus,

$$\text{w-sup}(\Lambda) = \lambda(T(e)).$$

Now, T^* is also positive (this is due to \mathcal{V}_+ being a self-dual cone); hence, by above, $\text{w-sup}(\Lambda^*) = \lambda(T^*(e))$. So,

$$\eta(T) = \text{w-sup}(\Lambda \cup \Lambda^*) = \lambda(T(e)) \vee_w \lambda(T^*(e)),$$

which completes the proof. □

We note a simple bound when T is positive:

$$\eta(T) = \lambda(T(e)) \vee_w \lambda(T^*(e)) \prec_w \max\{\lambda(T(e)), \lambda(T^*(e))\}.$$

Remark 2. In the last statement of the above corollary, the requirement that T be self-adjoint can be slightly relaxed. Suppose $T = P\Phi$, where Φ is an algebra automorphism of \mathcal{V} and P is positive and self-adjoint; see Example 4 for such a map. As \mathcal{V} carries the trace inner product, Φ^* is also an algebra automorphism of \mathcal{V} , hence preserves eigenvalues. So, $\lambda(T^*(e)) = \lambda(\Phi^*P(e)) = \lambda(P(e)) = \lambda(P(\Phi(e))) = \lambda(T(e))$. Thus, when $T = P\Phi$,

$$\eta(T) = \lambda(T(e)).$$

We note that if $T = \Phi P$, where Φ and P are as above, then, $\eta(T) = \lambda(T^*(e))$.

Motivated by our main theorem, we ask if (8) has a majorization analog. The following result provides an answer.

In what follows, we use the fact that $(-\mathbf{1} * \lambda(x))^\perp = \lambda(-x)$ for all $x \in \mathcal{V}$, and note that $p \prec q$ in \mathcal{R}^n is, by definition, equivalent to $p^\perp \prec q^\perp$.

Theorem 3.5. *Let T be a linear map on \mathcal{V} . Then, the following are equivalent:*

- (i) *There exists a vector q in \mathcal{R}^n with decreasing components such that $\lambda(T(x)) \prec q * \lambda(x)$ for all $x \in \mathcal{V}$.*
- (ii) *T is a scalar multiple of a doubly stochastic map.*

Proof. (i) \Rightarrow (ii): We assume that q in (i) is given by $q = (q_1, q_2, \dots, q_n)$. We fix a Jordan frame $\{f_1, f_2, \dots, f_n\}$ in \mathcal{V} and let $a := \sum_{i=1}^n q_i f_i$ so that $q = \lambda(a)$. Then (i) reads

$$\lambda(T(x)) \prec \lambda(a) * \lambda(x) \text{ for all } x \in \mathcal{V}.$$

This implies that $\sum_{i=1}^n \lambda_i(T(x)) = \sum_{i=1}^n \lambda_i(a) \lambda_i(x)$ for all x . Since \mathcal{V} carries the trace inner product, $\sum_{i=1}^n \lambda_i(T(x)) = \langle T(x), e \rangle = \langle x, T^*(e) \rangle$ and so

$$\langle x, T^*(e) \rangle = \langle \lambda(x), \lambda(a) \rangle.$$

Let $b := T^*(e)$ so that

$$\langle x, b \rangle = \langle \lambda(x), \lambda(a) \rangle \text{ for all } x \in \mathcal{V}. \quad (16)$$

We claim that

$$\langle \lambda(x), \lambda(b) \rangle = \langle \lambda(x), \lambda(a) \rangle \text{ for all } x \in \mathcal{V}. \quad (17)$$

To see this, fix any $x \in \mathcal{V}$ and consider the spectral decomposition $b = \sum_{i=1}^n \lambda_i(b) e_i$, where $\{e_1, e_2, \dots, e_n\}$ is a Jordan frame. Corresponding to this Jordan frame and the given x , define

$$y := \sum_{i=1}^n \lambda_i(x) e_i.$$

Then, applying (16) to y , we have

$$\langle y, b \rangle = \langle \lambda(y), \lambda(a) \rangle.$$

As $\lambda(x) = \lambda(y)$ and $\langle y, b \rangle = \sum_{i=1}^n \lambda_i(x) \lambda_i(b) = \langle \lambda(x), \lambda(b) \rangle$, we see that

$$\langle \lambda(x), \lambda(b) \rangle = \langle \lambda(x), \lambda(a) \rangle.$$

This proves our claim. By specializing $\lambda(x)$ in (17) (for example, to $\mathbf{1}_k$ for any index k , $1 \leq k \leq n$), we get $\lambda(b) = \lambda(a)$. But then, (16) leads to

$$\langle x, b \rangle = \langle \lambda(x), \lambda(b) \rangle \text{ for all } x \in \mathcal{V}.$$

This means that every x in \mathcal{V} ‘strongly operator commutes’ [9] with b . As shown in the Remark below, this can happen if and only if b is a scalar multiple of e . Since $\lambda(b) = \lambda(a)$, a must also be a scalar multiple of e (this can be seen via the spectral decomposition). Let $a = te$ for some $t \in \mathcal{R}$ so that $q = t\mathbf{1}$ and

$$\lambda(T(x)) \prec t\mathbf{1} * \lambda(x) \text{ for all } x \in \mathcal{V}. \quad (18)$$

If $t = 0$, then $\lambda(T(x)) \prec 0$ for all x . Hence, $T = 0$, that is, T is a multiple of the Identity map (which is doubly stochastic). For nonzero t , we complete the proof by showing that $D := \frac{1}{t}T$ is doubly stochastic. When $t > 0$, because of (18), $\lambda(D(x)) \prec \mathbf{1} * \lambda(x)$ for all x ; so D is doubly stochastic. On the other hand, when $t < 0$, say $t = -1$, $\lambda(-1T(x)) = \lambda(T(-x)) \prec q * \lambda(-x)$ implies

$$\lambda(D(x)) = \lambda(-1T(x)) = \lambda(T(-x)) \prec [(-1)\mathbf{1} * \lambda(-x)]^\dagger = \lambda(x),$$

that is, $\lambda(D(x)) \prec \lambda(x)$ for all $x \in \mathcal{V}$. Now, $T = tD$ says that T is a scalar multiple of D .

(ii) \Rightarrow (i). Suppose $T = tD$, where $t \in \mathcal{R}$ and D is a doubly stochastic map. Since $\lambda(D(x)) \prec \mathbf{1} * \lambda(x)$ for all x , by scaling, $\lambda(T(x)) \prec q * \lambda(x)$ for all x . (This scaling is obvious when $t \geq 0$. When $t < 0$, say $t = -1$, $\lambda(T(x)) = \lambda(-D(x)) = \lambda(D(-x)) \prec \lambda(-x) = [-\mathbf{1} * \lambda(x)]^\downarrow$.) Putting $q = t\mathbf{1}$, we see that $\lambda(T(x)) \prec q * \lambda(x)$ for all $x \in \mathcal{V}$. \square

Remark 3. We show that if b *strongly operator commutes* [9] with every $x \in \mathcal{V}$, that is, if $\langle x, b \rangle = \langle \lambda(x), \lambda(b) \rangle$ for all $x \in \mathcal{V}$, then b is a scalar multiple of e . Suppose this condition holds. Then, putting $x = -b$, we have $-||b||^2 = \langle -b, b \rangle = \langle \lambda(-b), \lambda(b) \rangle$. As \mathcal{V} carries the trace inner product, $||\lambda(y)||^2 = ||y||^2$ for all $y \in \mathcal{V}$. Hence,

$$||\lambda(-b) + \lambda(b)||^2 = \langle \lambda(-b) + \lambda(b), \lambda(-b) + \lambda(b) \rangle = ||b||^2 - 2||b||^2 + ||b||^2 = 0.$$

This implies that $\lambda(-b) = -\lambda(b)$. As $\lambda(-b)$ has components in the decreasing order and $-\lambda(b)$ has components in the increasing order, we see that $\lambda_1(b) = \lambda_n(b)$. This proves that all components in $\lambda(b)$ are equal. Thus, b is a multiple of e .

4 Properties of η

In this section, we describe some properties of the map $\eta : T \mapsto \eta(T)$ from $\mathcal{L}(\mathcal{V})$ to $(\mathcal{R}_+^n)^\downarrow$, where $\mathcal{L}(\mathcal{V})$ denotes the set of all (continuous) linear maps on \mathcal{V} . First, some notation. For any $x \in \mathcal{V}$, we let

$$||x||_\infty := ||\lambda(x)||_\infty = \max_{1 \leq i \leq n} |\lambda_i(x)| = \lambda_1(|x|)$$

and for $T \in \mathcal{L}(\mathcal{V})$,

$$||T||_\infty := \sup_{0 \neq x \in \mathcal{V}} \frac{||T(x)||_\infty}{||x||_\infty}.$$

Theorem 4.1. *The following statements hold for $T, T_1, T_2 \in \mathcal{L}(\mathcal{V})$ and $\alpha \in \mathcal{R}$:*

- (a) $||\eta(T)||_\infty = \max\{||T||_\infty, ||T^*||_\infty\}$.
- (b) $\eta(\alpha T) = |\alpha| \eta(T)$.
- (c) $\eta(T_1 T_2) \prec_w \eta(T_1) * \eta(T_2)$.
- (d) $\eta(T_1 + T_2) \prec_w \eta(T_1) + \eta(T_2)$.
- (e) η is continuous.
- (f) η is ‘isotonic’: If $T_1(x) \prec T_2(x)$ for all $x \in \mathcal{V}$, then $\eta(T_1) \prec_w \eta(T_2)$.

Proof. (a) Fix T . From the pointwise inequality, $\lambda(|T(x)|) \prec_w \eta(T) * \lambda(|x|)$, we see that

$$||T(x)||_\infty = \lambda_1(|T(x)|) \leq (\eta(T))_1 \lambda_1(|x|) = ||\eta(T)||_\infty ||x||_\infty.$$

It follows that $\|T\|_\infty \leq \|\eta(T)\|_\infty$. By Lemma 3.2, $\lambda(|T^*(x)|) \prec_w \eta(T) * \lambda(|x|)$. This yields, $\|T^*\|_\infty \leq \|\eta(T)\|_\infty$. Hence,

$$\max\{\|T\|_\infty, \|T^*\|_\infty\} \leq \|\eta(T)\|_\infty.$$

We now prove the reverse inequality. Let, for $1 \leq k \leq n$,

$$\theta_k := \sup_{c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}} \lambda_k(|T(c \circ \varepsilon)|) \quad \text{and} \quad \theta_k^* := \sup_{c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}} \lambda_k(|T^*(c \circ \varepsilon)|).$$

Let $\theta := (\theta_1, \theta_2, \dots, \theta_n)$ and $\theta^* := (\theta_1^*, \theta_2^*, \dots, \theta_n^*)$. As the components in any $\lambda(x)$ are decreasing, we see that $\theta, \theta^* \in (\mathcal{R}_+^n)^\downarrow$. Let

$$\bar{q} := \max\{\theta, \theta^*\}.$$

For the given T , we define the sets Q , Λ , Λ^* and S as in (9)-(12). Suppose $s \in \Lambda$ so that $s = \lambda(|T(c \circ \varepsilon)|)$ for some $c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}$. Then, for any k , $1 \leq k \leq n$,

$$S_k(s) = \sum_{i=1}^k \lambda_i(|T(c \circ \varepsilon)|) \leq \sum_{i=1}^k \theta_i \leq S_k(\bar{q}).$$

We have a similar statement when $s \in \Lambda^*$. Hence, $s \prec_w \bar{q}$ for all $s \in S$. This implies that $\text{w-sup}(S) \prec_w \bar{q}$. However, $\text{w-sup}(S) = \text{w-inf}(Q) = \eta(T)$ and so, $\eta(T) \prec_w \bar{q}$. Thus,

$$\eta(T) \prec_w \max\{\theta, \theta^*\}.$$

This implies that

$$\|\eta(T)\|_\infty = (\eta(T))_1 \leq \max\{\theta_1, \theta_1^*\}.$$

However,

$$\theta_1 = \sup_{c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}} \lambda_1(|T(c \circ \varepsilon)|) = \sup_{c \circ \varepsilon \in \mathcal{I} \circ \mathcal{E}} \|T(c \circ \varepsilon)\|_\infty \leq \|T\|_\infty,$$

where the inequality is due to

$$\|T(c \circ \varepsilon)\|_\infty \leq \|T\|_\infty \|c \circ \varepsilon\|_\infty = \|T\|_\infty \lambda_1(|c \circ \varepsilon|) \leq \|T\|_\infty \lambda_1(|c|) = \|T\|_\infty.$$

Similarly, $\theta_1^* \leq \|T^*\|_\infty$. Hence,

$$\|\eta(T)\|_\infty \leq \max\{\|T\|_\infty, \|T^*\|_\infty\}.$$

Since the reverse inequality has already been proved, we have Item (a).

(b) This is easy to see from the uniqueness part in the main theorem.

(c) Let $T_1, T_2 \in \mathcal{L}(\mathcal{V})$. Then,

$$\lambda(|T_1 T_2(x)|) \prec_w \eta(T_1) * \lambda(|T_2(x)|) \prec_w \eta(T_1) * \eta(T_2) * \lambda(|x|) \text{ for all } x \in \mathcal{V},$$

where we have used (6) in the second inequality. From the main theorem, we have $\eta(T_1 T_2) \prec_w \eta(T_1) * \eta(T_2)$.

(d) Suppose $a, b \in \mathcal{V}$. Writing $|a + b| = (a + b) \circ \varepsilon$ for some $\varepsilon \in \mathcal{E}$, we have

$$\begin{aligned}\lambda(|a + b|) &= \lambda((a + b) \circ \varepsilon) \\ &= \lambda(a \circ \varepsilon + b \circ \varepsilon) \\ &\prec \lambda(a \circ \varepsilon) + \lambda(b \circ \varepsilon) \\ &\leq \lambda(|a \circ \varepsilon|) + \lambda(|b \circ \varepsilon|) \\ &\prec_w \lambda(|a|) + \lambda(|b|),\end{aligned}$$

where the first inequality follows from Lidskii type inequality $\lambda(x + y) \prec \lambda(x) + \lambda(y)$ in \mathcal{V} (which easily follows from a result on simple Euclidean Jordan algebras [19] or from a general result on hyperbolic polynomials [14]) and the last inequality follows from (4). Now, we put $a = T_1(x)$, $b = T_2(x)$ and use the main theorem to get, for any $x \in \mathcal{V}$,

$$\lambda(|T_1(x) + T_2(x)|) \prec_w (\eta(T_1) + \eta(T_2)) * \lambda(|x|).$$

This gives the stated conclusion in (d).

(e) From Items (b) and (d), we see that for each index k , the function $T \mapsto S_k(\eta(T))$ is positively homogeneous and subadditive, hence convex. As convex functions (with domain $\mathcal{L}(\mathcal{V})$), these functions are continuous. Hence the k th component of $\eta(T)$, being the difference of $S_k(\eta(T))$ and $S_{k-1}(\eta(T))$, is also continuous in T . This means that $\eta(T)$ is continuous in T .

(f) Suppose $T_1(x) \prec T_2(x)$ for all $x \in \mathcal{V}$. Then, by definition $\lambda(T_1(x)) \prec \lambda(T_2(x))$ for all $x \in \mathcal{V}$. Hence, $|\lambda(T_1(x))| \prec_w |\lambda(T_2(x))|$, or equivalently, $\lambda(|T_1(x)|) \prec_w \lambda(|T_2(x)|)$ for all x . Then, the pointwise inequality

$$\lambda(|T_1(x)|) \prec_w \lambda(|T_2(x)|) \prec \eta(T_2) * \lambda(|x|)$$

implies that $\eta(T_1) \prec_w \eta(T_2)$. □

Remark 4. In this remark, it is convenient to let p, r, s denote real numbers. For $r \in [1, \infty]$ and $u \in \mathcal{R}^n$, $\|u\|_r$ is the usual r -norm of u in \mathcal{R}^n . For any $x \in \mathcal{V}$, we define the corresponding *spectral norm*

$$\|x\|_r := \|\lambda(x)\|_r,$$

which is $[\sum_{i=1}^n |\lambda_i(x)|^r]^{1/r}$ when $1 \leq r < \infty$ and $\max_{1 \leq i \leq n} |\lambda_i(x)|$ when $r = \infty$.

Given $r, s \in [1, \infty]$ and $T \in \mathcal{L}(\mathcal{V})$, we define the norm of T from $(\mathcal{V}, \|\cdot\|_r)$ to $(\mathcal{V}, \|\cdot\|_s)$ by

$$\|T\|_{r \rightarrow s} := \sup_{0 \neq x \in \mathcal{V}} \frac{\|T(x)\|_s}{\|x\|_r}.$$

Now, suppose $p, r, s \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Then, the following inequality holds for any $T \in \mathcal{L}(\mathcal{V})$ and $x \in \mathcal{V}$:

$$\|T(x)\|_p \leq \|\eta(T)\|_r \|x\|_s. \tag{19}$$

To see this, we follow the argument given in Theorem 5.1 of [22]. Starting from the inequality $\lambda(|T(x)|) \prec_w \eta(T) * \lambda(|x|)$, (19) is easily seen when $p = \infty$ and $p = 1$. For $1 < p < \infty$, we use the fact that the function $\phi : t \mapsto t^p$ is an increasing convex function on $[0, \infty)$ to get ([2], Exercise

II.3.2)

$$\|T(x)\|_p \leq \|\eta(T) * \lambda(|x|)\|_p.$$

An application of (classical) generalized Hölder's inequality gives (19). Additionally, as in Theorem 5.1 of [22], we can prove the following:

$$\|T\|_{r \rightarrow s} \leq \begin{cases} \|\eta(T)\|_\infty & \text{if } r \leq s, \\ \|\eta(T)\|_{\frac{rs}{r-s}} & \text{if } s < r. \end{cases}$$

To see a special case, suppose T is positive and self-adjoint. In this case, $\eta(T) = \lambda(T(e))$ and so,

$$\|T\|_{r \rightarrow s} \leq \begin{cases} \|T(e)\|_\infty & \text{if } r \leq s, \\ \|T(e)\|_{\frac{rs}{r-s}} & \text{if } s < r. \end{cases}$$

In some special cases, equality holds, see e.g., Theorem 5.1 in [22].

5 Examples

In this section, we present some examples.

Example 1. (*Lyapunov transformation L_a and the quadratic representation P_a*)

Recall that for any $a \in \mathcal{V}$, L_a is defined by $L_a(x) = a \circ x$. As mentioned in the Introduction, we have $\lambda(|L_a(x)|) \prec_w \lambda(|a|) * \lambda(|x|)$ for all $x \in \mathcal{V}$. (Note that this result does not come from any of our results above.) By our main theorem, we see that $\eta(L_a) \prec_w \lambda(|a|)$. However, by putting $x = e$ in the inequality $\lambda(|L_a(x)|) \prec_w \eta(L_a) * \lambda(|x|)$ we see that $\lambda(|a|) \prec_w \eta(L_a)$. Hence, $\eta(L_a) = \lambda(|a|)$. Now, P_a , defined by $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$, is positive and self-adjoint; hence, from Corollary 3.4, we have $\eta(P_a) = \lambda(P_a(e)) = \lambda(a^2)$.

Example 2. (*Doubly stochastic maps*)

Suppose T is doubly stochastic so that T is positive and $T(e) = T^*(e) = e$. In this case, from Corollary 3.4, $\lambda(|T(x)|) \prec_w \lambda(|x|)$ for all $x \in \mathcal{V}$ and $\eta(T) = \mathbf{1}$. As noted earlier, $\lambda(T(x)) \prec_w \lambda(x)$ for all $x \in \mathcal{V}$. These apply when T is a convex combination of algebra automorphisms [8].

Example 3. (*Doubly substochastic maps*)

When T is doubly substochastic, T is positive with $T(e) \leq e$ and $T^*(e) \leq e$. So, $\eta(T) = \lambda(T(e)) \vee_w \lambda(T^*(e)) \prec_w \lambda(e) = \mathbf{1}$ and $\lambda(|T(x)|) \prec_w \lambda(|x|)$ for all $x \in \mathcal{V}$, or equivalently, $\lambda(T(x)) \prec_w \lambda(x)$ for all $x \geq 0$.

Example 4. (*Cone automorphisms*)

Suppose \mathcal{V} is a *simple* Euclidean Jordan algebra and $T \in \overline{\text{Aut}(\mathcal{V}_+)}$ (the closure of $\text{Aut}(\mathcal{V}_+)$ in $\mathcal{L}(\mathcal{V})$). We claim that

$$\eta(T) = \lambda(T(e)).$$

To see this, first suppose $T \in \text{Aut}(\mathcal{V}_+)$. Because \mathcal{V} is assumed to be simple, we can write $T = P_a \Phi$, where Φ is an algebra automorphism and P_a is the quadratic representation of (some)

$a > 0$, see [6], Page 56. So, by Remark 2, $\eta(T) = \lambda(T(e))$. Now the result for any $T \in \overline{\text{Aut}(\mathcal{V}_+)}$ follows from the continuity of η and λ . In the same setting, consider a linear map $S : \mathcal{V} \rightarrow \mathcal{V}$ which is a sum of a finite number of maps in $\overline{\text{Aut}(\mathcal{V}_+)}$. Then, S is a positive map and Corollary 3.4 can be applied. To illustrate these results, let $\mathcal{V} = \mathcal{H}^n$. Then, any $T \in \overline{\text{Aut}(\mathcal{H}_+^n)}$ is of the form $T(X) = AXA^*$ for some $n \times n$ complex square matrix A . In this setting,

$$\eta(T) = \lambda(T(I)) = \lambda(AA^*).$$

Now consider a *completely positive map* S on \mathcal{H}^n , which is, by definition, a finite sum of the form $S(X) := \sum_{k=1}^N A_k X A_k^*$ with $A_k \in \mathcal{M}_n$ for all k . Letting $C := S(I) := \sum_{k=1}^N A_k A_k^*$ and $D := S^*(I) := \sum_{k=1}^N A_k^* A_k$, we have

$$\eta(S) \prec_w \lambda(C) \vee_w \lambda(D) \prec_w \max\{\lambda(C), \lambda(D)\}.$$

Example 5. (*\mathbf{Z} and Lyapunov-like transformations*)

We say that a linear map $L : \mathcal{V} \rightarrow \mathcal{V}$ is a \mathbf{Z} -transformation [12] if

$$[x, y \geq 0 \text{ and } \langle x, y \rangle = 0] \Rightarrow \langle L(x), y \rangle \leq 0.$$

It is said to be *Lyapunov-like* if the inequality on the right becomes an equality.

Such maps appear in dynamical systems theory. If a \mathbf{Z} -transformation L is also positive stable (meaning that all eigenvalues of L have positive real parts), then it is known that L^{-1} is a positive map on \mathcal{V} [12]. In this case, Corollary 3.4 is applicable to L^{-1} . To see an important special case, suppose A is an $n \times n$ positive stable complex matrix. Then, L_A , defined on \mathcal{H}^n by $L_A(X) := AX + XA^*$ is Lyapunov-like and positive stable. Let

$$C := L_A^{-1}(I) \quad \text{and} \quad D := L_{A^*}^{-1}(I).$$

Then,

$$\eta(L_A^{-1}) = \lambda(C) \vee_w \lambda(D) \prec_w \max\{\lambda(C), \lambda(D)\}.$$

We note that for any $X \in \mathcal{H}^n$, $L_A^{-1}(X)$ has an integral representation

$$L^{-1}(X) = \int_0^\infty e^{tA} X e^{-tA^*} dt,$$

with a similar representation for $L_{A^*}^{-1}(X)$. These will give us integral representations for C and D , but it is unclear how to represent $\eta(T)$ either in an integral form or in a closed form.

Example 6. (*Löwner maps*)

Given a function $\phi : \mathcal{R} \rightarrow \mathcal{R}$, consider the corresponding Löwner map ϕ defined on \mathcal{V} (see Section 2). Motivated by the inequality (8), we ask if the absolute value function can be replaced by ϕ . Keeping close to the properties of the absolute value function, we say that a function $\phi : \mathcal{R} \rightarrow \mathcal{R}$ is *sublinear* if

1. $\phi(\mu t) = \mu \phi(t)$ for all $\mu \geq 0$ and $t \in \mathcal{R}$;

2. $\phi(t+s) \leq \phi(t) + \phi(s)$ for all $t, s \in \mathcal{R}$.

It is easy to see that sublinear functions on \mathcal{R} are of the form $\phi(t) = \alpha t$ for $t \geq 0$ and $\phi(t) = \beta t$ for $t \leq 0$, where (constants) $\alpha, \beta \in \mathcal{R}$ satisfy $\beta \leq \alpha$. Among these, we consider ones that are nonnegative (that is, $\phi(t) \geq 0$ for all t). Examples include

$$\phi(t) = |t|, \quad \phi(t) = \max\{t, 0\}, \quad \text{and} \quad \phi(t) = \max\{-t, 0\}.$$

Now suppose ϕ is sublinear and nonnegative, and consider the corresponding Löwner map. Then for any positive linear map T we have

$$\lambda(\phi(T(x))) \prec_w \lambda(T(\phi(x))) \prec_w \eta(T) * \lambda(\phi(x)) \text{ for all } x \in \mathcal{V}.$$

Here, the first inequality follows from Lemma 3.6 in [22] and the second inequality follows from Remark 1, Item (ii).

Example 7. For any $a, b \in \mathcal{V}$, consider the map

$$P_{a,b} := L_a L_b + L_b L_a - L_{a \circ b}.$$

Clearly, $P_{a,a} = P_a$. Now, for any $a > 0$, $0 \leq t \leq 1$, it has been shown in [10] that

$$P_{\sqrt{a}}(x) \prec P_{a^t, a^{1-t}}(x) \prec L_a(x) \text{ for all } x \in \mathcal{V}.$$

Hence, by the ‘isotonicity’ property of η (Item (f) in Theorem 4.1), we have

$$\eta(P_{\sqrt{a}}) \prec_w \eta(P_{a^t, a^{1-t}}) \prec_w \eta(L_a).$$

Since $\eta(P_{\sqrt{a}}) = \lambda(a) = \eta(L_a)$ (note $a > 0$), the above inequality reduces to $\lambda(a) \prec_w \eta(P_{a^t, a^{1-t}}) \prec_w \lambda(a)$. It follows that

$$\eta(P_{a^t, a^{1-t}}) = \lambda(a) \quad (a > 0, 0 \leq t \leq 1).$$

It would be interesting to compute $\eta(P_{a,b})$ for general $a, b \in \mathcal{V}$.

Example 8. (*Schur product induced maps*)

Consider a fixed Jordan frame $\{e_1, e_2, \dots, e_n\}$ in \mathcal{V} . This induces the Peirce decomposition ([6], Theorem IV.2.1) $\mathcal{V} = \sum_{1 \leq i \leq j \leq n} \mathcal{V}_{ij}$, and for any $x \in \mathcal{V}$, $x = \sum_{i \leq j} x_{ij}$, where $x_{ij} \in \mathcal{V}_{ij}$. Now, for any given $A = [a_{ij}] \in \mathcal{S}^n$, we define the Schur product $A \bullet x := \sum_{i \leq j} a_{ij} x_{ij}$. Properties of the Schur product and the induced transformation $D_A : x \mapsto A \bullet x$ are studied in [11]. The Lyapunov transformation L_a and the quadratic representation P_a are special cases. Further special cases are described below.

(i) Suppose $A \in \mathcal{S}^n$ is positive semidefinite and consider the map $D_A : \mathcal{V} \rightarrow \mathcal{V}$ defined by $D_A(x) := A \bullet x$. Then, D_A is a self-adjoint positive linear map. So,

$$\eta(D_A) = \lambda(D_A(e)) = (\text{diag } A)^\downarrow.$$

In what follows, we provide an estimate for $\eta(D_A)$, where A is not necessarily positive semidefinite. We write $A = A^+ - A^-$, where A^+ and A^- are positive semidefinite. By Theorem 4.1, we have

$$\begin{aligned}\eta(D_A) &= \eta(D_{A^+} - D_{A^-}) \\ &\prec_w \eta(D_{A^+}) + \eta(D_{A^-}) \\ &= (\text{diag}(A^+))^\downarrow + (\text{diag}(A^-))^\downarrow.\end{aligned}$$

On the other hand, $|\text{diag}(A)|^\downarrow = \lambda(|D_A(e)|) \prec_w \eta(D_A) * \lambda(e) = \eta(D_A)$. Hence, we have

$$|\text{diag}(A)|^\downarrow \prec_w \eta(D_A) \prec_w (\text{diag}(A^+))^\downarrow + (\text{diag}(A^-))^\downarrow$$

for any symmetric matrix A .

(ii) Now suppose $A = [a_{ij}]$, $B = [b_{ij}] \in \mathcal{S}^n$ with $b_{ij} \neq 0$ for all i, j . Define the matrix $C := \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix} \in \mathcal{S}^n$. Then, $A \bullet x = C \bullet (B \bullet x) = D_C(B \bullet x)$. Hence,

$$\lambda(|A \bullet x|) \prec_w \eta(D_C) * \lambda(|B \bullet x|) \quad (x \in \mathcal{V}).$$

In particular, when C is positive semidefinite (in which case, D_C is a self-adjoint, positive map),

$$\lambda(|A \bullet x|) \prec_w (\text{diag } C)^\downarrow * \lambda(|B \bullet x|) \quad (x \in \mathcal{V}).$$

(iii) Suppose $\lambda(A \bullet x) \prec \lambda(B \bullet x)$ for all $x \in \mathcal{V}$. Pointwise majorization results of this type have been recently studied in [10]. Thanks to the isotonicity of η , the pointwise inequality $\lambda(A \bullet x) \prec \lambda(B \bullet x)$ implies that $\eta(D_A) \prec_w \eta(D_B)$. Example 7 above is an illustration of this.

6 Some open problems

Motivated by our results and examples, we raise some open problems.

Problem 1. Consider a linear map $T : \mathcal{H}^n \rightarrow \mathcal{H}^n$. This can be extended to a (complex) linear map $\tilde{T} : \mathcal{M}_n \rightarrow \mathcal{M}_n$ by

$$\tilde{T}(X) := T(A) + iT(B),$$

where $A := \frac{X+X^*}{2}$ and $B := \frac{X-X^*}{2i}$ are in \mathcal{H}^n . Then, we have $\eta(T)$ coming from Theorem 3.1 and $\eta(\tilde{T})$ coming from the result of Bapat (mentioned in the Introduction). To emphasize the algebras involved and to differentiate them, let us write $\eta(T, \mathcal{H}^n)$ and $\eta(\tilde{T}, \mathcal{M}_n)$. Then, $s(\tilde{T}(X)) \prec_w \eta(\tilde{T}, \mathcal{M}_n) * s(X)$ for all $X \in \mathcal{M}_n$ and $\lambda(|T(X)|) \prec_w \eta(T, \mathcal{H}^n) * \lambda(|X|)$ for all $X \in \mathcal{H}^n$. From Theorem 3.1 we see that

$$\eta(T, \mathcal{H}^n) \prec_w \eta(\tilde{T}, \mathcal{M}_n).$$

We now ask if (or under what conditions) the equality holds in the above.

Problem 2. For a matrix $A \in \mathcal{M}_n$, consider the Lyapunov transformation L_A on \mathcal{H}^n . Computing the norms of L_A and its inverse (whenever defined) relative to spectral p -norms is still an open problem [7, 3]. A related problem could be the description of $\eta(L_A)$; see Example 1 (for Hermitian A).

Problem 3. Given $A \in \mathcal{S}^n$ a fixed Jordan frame in \mathcal{V} , consider the Schur product induced map D_A (as in Example 8). Is there a description of $\eta(D_A)$?

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