A Riesz-Thorin type interpolation theorem in Euclidean Jordan algebras

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Abstract

In a Euclidean Jordan algebra $\mathcal{V}$ of rank n which carries the trace inner product, to each element a we associate the eigenvalue vector $\lambda(a)$ in $\mathcal{R}^n$ whose components are the eigenvalues of a written in the decreasing order. For any $p \in [1, \infty]$, we define the spectral p-norm of a to be the p-norm of $\lambda(a)$ in $\mathcal{R}^n$. In a recent paper, based on the K-method of real interpolation theory and a majorization technique, we described an interpolation theorem for a linear transformation on $\mathcal{V}$ relative to the same spectral p-norm. In this paper, using complex function theory methods, we describe a Riesz-Thorin type interpolation theorem relative to two spectral p-norms. We illustrate the result by estimating the norms of certain special linear transformations such as Lyapunov transformations, quadratic representations, and positive transformations.

Key Words: Euclidean Jordan algebra, Riesz-Thorin type interpolation theorem

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1 Introduction

Consider a Euclidean Jordan algebra $\mathcal{V}$ of rank n which carries the trace inner product. To each element a in $\mathcal{V}$, we associate the eigenvalue vector $\lambda(a)$ whose components are the eigenvalues of a written in the decreasing order. For any $p \in [1, \infty]$, we define the spectral p-norm on $\mathcal{V}$ by

$$||a||_p := ||\lambda(a)||_p,$$

where the right-hand side is the usual p-norm of the vector $\lambda(a)$ in $\mathcal{R}^n$. This is a special case of a spectral function on $\mathcal{V}$ which arises as the composition of a permutation invariant function on $\mathcal{R}^n$ and λ [2, 9]. The spectral p-norm is analogous to the Schatten p-norm of a complex square matrix (which is the p-norm of the vector of its singular values); In fact, the two norms coincide in the setting of the algebras of $n \times n$ real/complex Hermitian matrices. While there is a large body of literature on the singular values and Schatten norms [3], the non-associative nature of the Jordan product in a Jordan algebra prevents one from routinely stating and proving results for spectral p-norms. Yet, using novel techniques, several authors have studied spectral p-norms in general Euclidean Jordan algebras. In an early paper on interior point algorithms over a symmetric cone (which is the cone of squares in a Euclidean Jordan algebra), Schmieta and Alizadeh [12] describe some properties of the (spectral) 2-norm (also called the Frobenius norm) and the ∞-norm. Using majorization ideas, Tao et al. [13] show that $||\cdot||_p$ is a norm on $\mathcal{V}$; see [2, 9] for related results. In a recent paper [6], Gowda proves the Hölder type inequality $||x \circ y||_1 \leq ||x||_p \, ||y||_q$ (where q is the conjugate of p) and provides several references dealing with spectral p-norms.
The present paper deals with an interpolation theorem for a linear transformation on $\mathcal{V}$ relative to spectral p-norms. Given $r, s \in [1, \infty]$ and a linear transformation $T : \mathcal{V} \to \mathcal{V}$, we let

$$||T||_{r \to s} := \sup_{a \neq 0} \frac{||T(a)||_s}{||a||_r}.$$

In [6], based on the K-method of real interpolation theory [10], the following result was proved.

Theorem 1.1 *Suppose* $1 \leq r, s, p \leq \infty$, $0 \leq \theta \leq 1$, *and*

$$\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{s}. \tag{1}$$

Then, for any linear transformation $T : \mathcal{V} \to \mathcal{V}$,

$$||T||_{p \to p} \leq ||T||_{r \to r}^{1-\theta} \, ||T||_{s \to s}^{\theta}. \tag{2}$$

In particular,

$$||T||_{p \to p} \leq ||T||_{\infty \to \infty}^{1-\frac{1}{p}} \, ||T||_{1 \to 1}^{\frac{1}{p}}. \tag{3}$$

A key idea in the proof of the above result is the use of a majorization result that connects a K-functional defined on $\mathcal{V}$ with a K-functional on an L_p-space. In [6], the issue of proving an inequality of the type (2) that deals with the norm of T relative to two spectral norms (such as $||T||_{r\to s}$) was raised. In the present paper, based on standard complex function theory methods (especially, Hadamard's three lines theorem) we prove the following Riesz-Thorin type interpolation result.

Theorem 1.2 *Let $r_0, r_1, s_0, s_1 \in [1,\infty]$ and $\theta \in [0,1]$. Consider r_θ and s_θ in $[1,\infty]$ defined by*

$$\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and} \quad \frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}.$$

Then, for any linear transformation T on $\mathcal{V}$,

$$||T||_{r_\theta \to s_\theta} \leq C\, ||T||_{r_0\to s_0}^{1-\theta}\, ||T||_{r_1\to s_1}^{\theta}, \tag{4}$$

where C is a constant, $1 \leq C \leq 4$, that depends only on r_0, r_1, s_0, s_1.

Illustrating this result, we estimate the norms of some special linear transformations on $\mathcal{V}$ such as Lyapunov transformations, quadratic representations, and positive transformations.

2 Preliminaries

Throughout this paper $(\mathcal{V}, \circ, \langle \cdot,\cdot \rangle)$ denotes a Euclidean Jordan algebra of rank n with unit element e [4], [8]. We let letters a, b, c, d, and v denote elements of $\mathcal{V}$, x and y denote elements of $\mathcal{R}^n$, and write z for a complex variable. For $a, b \in \mathcal{V}$, we denote their Jordan product and inner product by $a \circ b$ and $\langle a, b\rangle$, respectively. It is known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra.

According to the *spectral decomposition theorem* [4], any element $a \in \mathcal{V}$ has a decomposition

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

where the real numbers $a_1, a_2, \ldots, a_n$ are (called) the eigenvalues of a and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same.) Then, $\lambda(a)$, called the *eigenvalue vector* of a, is the vector of eigenvalues of a written in the decreasing order. The *trace and spectral p-norm* of a are defined by

$$tr(a) := a_1 + a_2 + \cdots + a_n \quad \text{and} \quad ||a||_p := ||\lambda(a)||_p,$$

where $||\lambda(a)||_p$ denotes the usual p-norm of a vector in $\mathcal{R}^n$. An element a is said to be *invertible* if all its eigenvalues are nonzero. We note that the set of invertible elements is dense in $\mathcal{V}$. *Throughout* *this paper, we assume that the inner product is the trace inner product, that is,* $\langle a,b\rangle = tr(a\circ b)$.

Given a spectral decomposition $a=\sum_{j=1}^n a_je_j$ and a real number $\gamma>0$, we write

$$|a| := \sum_{j=1}^n |a_j|e_j,\ \ |a|^\gamma := \sum_{j=1}^n |a_j|^\gamma e_j \quad \text{and} \quad ||a||_1 = \sum_{j=1}^n |a_j| = tr(|a|). \tag{5}$$

In what follows, we say that q is the conjugate of $p\in[1,\infty]$ if $\frac{1}{p}+\frac{1}{q}=1$ and denote the conjugate of $r\in[1,\infty]$ by r'. Also, we use the standard convention that $1/\infty=0$.

Based on the Fan-Theobald-von Neumann type inequality [2]

$$\langle a,b\rangle \le \langle \lambda(a),\lambda(b)\rangle \quad (a,b\in\mathcal{V})$$

and majorization techniques, the following result was proved in [6].

Theorem 2.1 *Let* $p\in[1,\infty]$ *with conjugate* q*. Then the following statements hold in* $\mathcal{V}$*:*

(*i*) $|\langle a,b\rangle| \le ||a\circ b||_1 \le ||a||_p\,||b||_q$.

(*ii*) $\sup_{b\neq 0}\frac{|\langle a,b\rangle|}{||b||_q} = ||a||_p$.

3 The proof of the interpolation theorem

The Riesz-Thorin interpolation theorem, stated in the setting of L_p-spaces, is well-known in classical analysis. There is also a Riesz-Thorin type result available for linear transformations on the space of complex $n\times n$ matrices with respect to Schatten p-norms, see the interpolation theorem of Calderón-Lions ([11], Theorem IX.20). Our Theorem 1.2 is stated in the setting of Euclidean Jordan algebras relative to spectral p-norms. In the absence of an isomorphism type argument that immediately gives our result, we offer a proof that mimics the classical proof based on the Hadamard's three lines theorem of complex function theory ([5], Theorem 6.27). In the proof given below, we complexify the real inner product space $\mathcal{V}$ and define norms on this complexification in such a way to have a Hölder type inequality. This procedure results in a constant C in the Riesz-Thorin type inequality (4) that is different from 1. Possibly, a different argument may show that this constant can be replaced by 1.

Recall that a and b denote elements of $\mathcal{V}$ and z denotes a complex variable. For $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ in $\mathcal{R}^n$, we write $x+iy=(x_1+iy_1,x_2+iy_2,\ldots,x_n+iy_n)\in\mathbb{C}^n$. Let T be a linear transformation on $\mathcal{V}$. We consider complexifications of $\mathcal{V}$ and T:

$$\widetilde{\mathcal{V}} := \mathcal{V}+i\mathcal{V} \quad \text{and} \quad \widetilde{T}(a+ib) := T(a)+i\,T(b) \quad (a,b\in\mathcal{V}).$$

We define the inner product and spectral p-norm on $\widetilde{\mathcal{V}}$ as follows. For $a, b, c, d \in \mathcal{V}$,

$$\langle a+ib, c+id\rangle := \Big[\langle a,c\rangle + \langle b,d\rangle\Big] + i\Big[\langle b,c\rangle - \langle a,d\rangle\Big] \quad \text{and} \quad ||a+ib||_p := ||a||_p + ||b||_p.$$

It is easily seen that $\widetilde{\mathcal{V}}$ is a complex inner product space, $\widetilde{T}$ is a (complex) linear transformation on $\widetilde{\mathcal{V}}$. We state the following simple lemma.

Lemma 3.1 Consider $\widetilde{\mathcal{V}}$ and $\widetilde{T}$ as above. Let $p \in [1,\infty]$ with conjugate q, and $r, s \in [1,\infty]$. Then,

(*i*) $|\langle a+ib, c+id\rangle| \leq ||a+ib||_p\, ||c+id||_q$ for all $a, b, c, d \in \mathcal{V}$, and

(*ii*) $||\widetilde{T}||_{r\to s} = ||T||_{r\to s}$.

Proof. (*i*) By the definition of inner product in $\widetilde{\mathcal{V}}$ and Theorem 2.1,

$$|\langle a+ib, c+id\rangle| \leq |\langle a,c\rangle| + |\langle a,d\rangle| + |\langle b,c\rangle| + |\langle b,d\rangle| \leq ||a||_p\,||c||_q + ||a||_p\,||d||_q + ||b||_p\,||c||_q + ||b||_p\,||d||_q.$$

Since the right-hand side is $||a+ib||_p\,||c+id||_q$, the stated inequality follows.
(*ii*) For $a, b \in \mathcal{V}$,

$$||\widetilde{T}(a+ib)||_s = ||T(a) + i\,T(b)||_s = ||T(a)||_s + ||T(b)||_s \leq ||T||_{r\to s}(||a||_r + ||b||_r) = ||T||_{r\to s}\,||a+ib||_r.$$

This implies that $||\widetilde{T}||_{r\to s} \leq ||T||_{r\to s}$. The reverse inequality holds as $\widetilde{T}$ is an extension of T to $\widetilde{\mathcal{V}}$. Hence we have (*ii*). $\square$

We now come to the proof of Theorem 1.2. In what follows, *for any $p \in [1,\infty]$ with conjugate q, we let*

$$C_p = \begin{cases} \sqrt{2} & \text{if } 1 \leq p \leq 2, \\ 2^{\frac{1}{q}} & \text{if } 2 \leq p \leq \infty\,. \end{cases}$$

Proof. Let the assumptions of the theorem be in place. Recalling that s' denotes the conjugate of (any) $s \in [1,\infty]$, we define

$$C := \max\{C_{r_0}C_{s_0'}, C_{r_1}C_{s_1'}\} \tag{6}$$

which is a number between 1 and 4, and depends only on r_0, r_1, s_0, s_1. We show that (4) holds for this C. Since (4) clearly holds when $\theta = 0$ or $\theta = 1$, from now on, we assume that $0 < \theta < 1$.
Let

$$\alpha_j := \frac{1}{r_j},\ \beta_j := \frac{1}{s_j},\ \text{and } M_j := ||T||_{r_j\to s_j} \quad (j = 0, 1),$$

$$\alpha := \frac{1}{r_\theta},\ \beta := \frac{1}{s_\theta},\ \text{and } M_\theta := ||T||_{r_\theta\to s_\theta},$$

and for a complex variable z,

$$\alpha(z) := (1-z)\alpha_0 + z\alpha_1 \quad \text{and} \quad \beta(z) := (1-z)\beta_0 + z\beta_1.$$

We show that

$$M_\theta \le C\, M_0^{1-\theta}\, M_1^{\theta}. \tag{7}$$

Now, using Theorem 2.1, Item (ii),

$$M_\theta = ||T||_{r_\theta \to s_\theta} = \sup_{0\neq a\in\mathcal{V}} \frac{||T(a)||_{s_\theta}}{||a||_{r_\theta}} = \sup_{0\neq a,b\in\mathcal{V}} \frac{|\langle T(a), b\rangle|}{||a||_{r_\theta}||b||_{s'_\theta}} = \sup_{||a||_{r_\theta}=1=||b||_{s'_\theta}} |\langle Ta, b\rangle|.$$

To prove (7), it is enough to show that for any a and b in $\mathcal{V}$ with $||a||_{r_\theta} = 1 = ||b||_{s'_\theta}$,

$$|\langle Ta, b\rangle| \le C\, M_0^{1-\theta}\, M_1^{\theta}. \tag{8}$$

By continuity, it is enough to prove this for a and b invertible (that is, with all their eigenvalues nonzero). We fix such a and b and write their spectral decompositions:

$$a = \sum_{j=1}^{n} |a_j| \varepsilon_j e_j \quad \text{and} \quad b = \sum_{j=1}^{n} |b_j| \delta_j f_j,$$

where $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ are Jordan frames, $\varepsilon_j, \delta_j \in \{-1, 1\}$ for all j, and a_js are the eigenvalues of a, etc. Now, with the observation that $0 < \alpha, \beta < 1$, we define two elements in $\widetilde{\mathcal{V}}$:

$$a_z := \sum_{j=1}^{n} |a_j|^{\frac{\alpha(z)}{\alpha}} \varepsilon_j e_j \quad \text{and} \quad b_z := \sum_{j=1}^{n} |b_j|^{\frac{1-\beta(z)}{1-\beta}} \delta_j f_j,$$

where we consider only the principal values while defining the exponentials. Then the function

$$\phi(z) := \langle \widetilde{T}(a_z), b_z\rangle$$

is continuous on the strip $\{z : 0 \le Re(z) \le 1\}$ and analytic in its interior.
We estimate $|\phi(z)|$ on the lines $Re(z) = 0$ and $Re(z) = 1$ and then apply Hadamard's three lines theorem ([5], Theorem 6.27). First, suppose $Re(z) = 0$. Let

$$|a_j|^{\frac{\alpha(z)}{\alpha}} = x_j + i\, y_j,\ x := (x_1, x_2, \dots, x_n) \in \mathcal{R}^n,\ \text{and } y := (y_1, y_2, \dots, y_n) \in \mathcal{R}^n.$$

Then, $|x_j + iy_j| = \left| |a_j|^{\frac{\alpha(z)}{\alpha}} \right| = |a_j|^{\frac{\alpha_0}{\alpha}}$. When $r_0 = \infty$, that is, when $\alpha_0 = 0$, $|x_j + iy_j| = 1$ for all j and hence (in $\mathbb{C}^n$), $||x + iy||_{r_0} = 1$. When, $r_0 < \infty$, $|x_j + iy_j|^{r_0} = |a_j|^{r_\theta}$. So, because $||a||_{r_\theta} = 1$, we have $||x + iy||_{r_0}^{r_0} = \sum_{j=1}^n |x_j + iy_j|^{r_0} = \sum_{j=1}^n |a_j|^{r_\theta} = 1$. Thus, in both cases,

$$||x + iy||_{r_0} = 1. \tag{9}$$

Now, $a_z = \sum_{j=1}^n (x_j + iy_j)\varepsilon_j e_j = (\sum_{j=1}^n x_j \varepsilon_j e_j) + i(\sum_{j=1}^n y_j \varepsilon_j e_j)$ and so,

$$||a_z||_{r_0} = ||\sum_{j=1}^{n} x_j \varepsilon_j e_j||_{r_0} + ||\sum_{j=1}^{n} y_j \varepsilon_j e_j||_{r_0} = ||x||_{r_0} + ||y||_{r_0}.$$

In view of (9), from Proposition 4.1 in the Appendix, we have,

$$||a_z||_{r_0} \leq C_{r_0}.$$

Similarly, $||b_z||_{s_0'} \leq C_{s_0'}$. Hence, when $Re(z) = 0$, Lemma 3.1 gives

$$|\phi(z)| \leq ||\widetilde{T}(a_z)||_{s_0}\, ||b_z||_{s_0'} \leq ||\widetilde{T}||_{r_0 \to s_0}\, ||a_z||_{r_0} ||b_z||_{s_0'} \leq ||T||_{r_0 \to s_0}\, C_{r_0}\, C_{s_0'} = C_{r_0} C_{s_0'}\, M_0.$$

A similar computation shows that

$$Re(z) = 1 \Rightarrow |\phi(z)| \leq C_{r_1} C_{s_1'} M_1.$$

By Hadamard's three lines theorem,

$$|\phi(\theta)| \leq \left(C_{r_0} C_{s_0'}\, M_0\right)^{1-\theta} \left(C_{r_1} C_{s_1'}\, M_1\right)^{\theta}.$$

We recall that $C = \max\{C_{r_0} C_{s_0'}, C_{r_1} C_{s_1'}\}$. Now, $a_\theta = a$ and $b_\theta = b$, and so, $\phi(\theta) = \langle T(a), b\rangle$. Hence,

$$|\langle T(a), b\rangle| \leq C\, M_0^{1-\theta}\, M_1^{\theta}.$$

This gives (8) and the proof is complete. □

Remarks. Instead of the constant C defined in (6), one may consider a slightly better constant, namely, $\max\{(C_{r_0} C_{s_0'})^{1-\theta}, (C_{r_1} C_{s_1'})^{\theta}\}$. However, this constant depends on θ.

We now consider the problem of estimating the norms of certain special linear transformations on $\mathcal{V}$ relative to spectral p-norms. First, we make two observations. Writing T^* for the adjoint of a linear transformation T on $\mathcal{V}$, we note, thanks to Theorem 2.1, that

$$||T^*||_{r \to s} = ||T||_{s' \to r'},$$

where r' denotes the conjugate of r, etc. Also, knowing the norms $||T||_{1 \to 1}$, $||T||_{\infty \to \infty}$, $||T||_{1 \to p}$, and $||T||_{p \to 1}$, etc., one can estimate $||T||_{r \to s}$ for various r and s. When $r = s$, (3) gives such an estimate. In the result below, we consider the case $r \neq s$.

Corollary 3.2 *Let $1 \leq r \neq s \leq \infty$. Then, for any linear transformation $T : \mathcal{V} \to \mathcal{V}$,*

$$||T||_{r \to s} \leq \begin{cases} 2\sqrt{2}\, ||T||_{\infty \to \infty}^{1-\frac{1}{r}}\, ||T||_{1 \to \frac{s}{r}}^{\frac{1}{r}} & \text{if } r < s, \\ 2\sqrt{2}\, ||T||_{\infty \to \infty}^{1-\frac{1}{s}}\, ||T||_{\frac{r}{s} \to 1}^{\frac{1}{s}} & \text{if } r > s\,. \end{cases}$$

Proof. The stated inequalities are obtained by specializing Theorem 1.2. When $r < s$, we let

$$r_0 = \infty,\ s_0 = \infty,\ r_1 = 1,\ s_1 = \frac{s}{r},\ r_\theta = r,\ s_\theta = s, \text{ and } \theta = \frac{1}{r}.$$

In this case, $C = \max\{C_{r_0} C_{s_0'}, C_{r_1} C_{s_1'}\} = 2\sqrt{2}$. When $r > s$, we let

$$r_0 = \infty,\ s_0 = \infty,\ r_1 = \frac{r}{s},\ s_1 = 1,\ r_\theta = r,\ s_\theta = s, \text{ and } \theta = \frac{1}{s}.$$

In this case also, $C = 2\sqrt{2}$. $\square$

Remarks. In the result above, by considering $\max\{(C_{r_0}C_{s_0'})^{1-\theta}, (C_{r_1}C_{s_1'})^{\theta}\}$, one can replace the constant $2\sqrt{2}$ by the following:

$$(2\sqrt{2})^{\max\{1-\frac{1}{r},\frac{1}{r}\}} \text{ when } r < s \text{ and } (2\sqrt{2})^{\max\{1-\frac{1}{s},\frac{1}{s}\}} \text{ when } r > s.$$

We now illustrate our results via some examples. For any $a \in \mathcal{V}$, consider the *Lyapunov transformation* L_a and the *quadratic representation* P_a defined by

$$L_a(v) := a \circ v \quad \text{and} \quad P_a(v) := 2a \circ (a \circ v) - a^2 \circ v \quad (v \in \mathcal{V}).$$

These self-adjoint linear transformations appear prominently in the study of Euclidean Jordan algebras. The norms of these transformations relative to some spectral p-norms have been described in [6]. For $r, s \in [1, \infty]$, we have

$$||a||_\infty \leq ||L_a||_{r\to s} \quad \text{and} \quad ||a^2||_\infty = ||a||_\infty^2 \leq ||P_a||_{r\to s}.$$

Additionally, for any $p \in [1, \infty]$ with conjugate q,

- $||L_a||_{p\to p} = ||L_a||_{p\to\infty} = ||L_a||_{1\to q} = ||a||_\infty$ and $||L_a||_{p\to 1} = ||L_a||_{\infty\to q} = ||a||_q$,
- $||P_a||_{p\to p} = ||P_a||_{p\to\infty} = ||P_a||_{1\to q} = ||a||_\infty^2$ and $||P_a||_{p\to 1} = ||P_a||_{\infty\to q} = ||a^2||_q$.

We now come to the estimation of $||L_a||_{r\to s}$ and $||P_a||_{r\to s}$ for $r \neq s$. First suppose $1 \leq r < s \leq \infty$. Then, using the above properties and the fact that for any $x \in \mathcal{R}^n$, $||x||_p$ is a decreasing function of p over $[1, \infty]$, we have

$$||a||_\infty \leq ||L_a||_{r\to s} = \sup_{0\neq v\in\mathcal{V}} \frac{||L_a(v)||_s}{||v||_r} \leq \sup_{0\neq v\in\mathcal{V}} \frac{||L_a(v)||_r}{||v||_r} = ||L_a||_{r\to r} = ||a||_\infty.$$

Thus,

$$||L_a||_{r\to s} = ||a||_\infty \;\; (1 \leq r < s \leq \infty).$$

A similar argument shows that

$$||P_a||_{r\to s} = ||a||_\infty^2 \;\; (1 \leq r < s \leq \infty).$$

When $1 \leq s < r \leq \infty$, Corollary 3.2 yields the following estimate:

$$||L_a||_{r\to s} \leq 2\sqrt{2}\, ||a||_{(\frac{r}{s})'}.$$

For the same s and r, we can get a different estimate

$$||L_a||_{r\to s} \leq 2\, C_q\, ||a||_p, \tag{10}$$

where $\frac{1}{p} = \frac{1}{s} - \frac{1}{r}$ (so that $p = s(\frac{r}{s})'$) and q is the conjugate of p. To see this, we apply Theorem 1.2

with

$$r_0 = \infty,\ s_0 = p,\ r_1 = q,\ s_1 = 1,\ r_\theta = r,\ s_\theta = s, \text{ and } \theta = \frac{q}{r}.$$

Then,

$$||L_a||_{r\to s} \le C\,||L_a||_{\infty\to p}^{1-\theta}\,||L_a||_{q\to 1}^{\theta} = C\,||a||_p,$$

where $C = \max\{C_{r_0}C_{s_0'},\, C_{r_1}C_{s_1'}\} = 2\,C_q$. To see an interesting consequence of (10), let $1 \le r, s, p \le \infty$ with $r \ne s$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{r}$. Then, using the inequality $||a \circ b||_s \le ||L_a||_{r\to s}\,||b||_r$, the estimate (10) leads to

$$||a \circ b||_s \le 2\,C_q\,||a||_p\,||b||_r \quad (a, b \in \mathcal{V}),$$

which can be regarded as a *generalized Hölder type inequality*. We remark that the special case $s = 1$ was already covered in Theorem 2.1 with 1 in place of $2C_q$. It is very likely that the inequality $||a \circ b||_s \le ||a||_p\,||b||_r$ holds in the general case as well.

Analogous to the above norm estimates of L_a, we can estimate $||P_a||_{r\to s}$ when $r > s$ (with p and q defined above):

$$||P_a||_{r\to s} \le 2\sqrt{2}\,||a^2||_{(\frac{r}{s})'} \quad \text{and} \quad ||P_a||_{r\to s} \le 2\,C_q\,||a^2||_p.$$

We now consider a *positive linear transformation* P on $\mathcal{V}$, which is a linear transformation on $\mathcal{V}$ satisfying the condition

$$a \ge 0 \Rightarrow P(a) \ge 0,$$

where $a \ge 0$ means that a belongs to the symmetric cone of $\mathcal{V}$ (or, equivalently, it is the square of some element of $\mathcal{V}$). Examples of such transformations include:

- Any nonnegative matrix on the algebra $\mathcal{R}^n$.
- Any quadratic representation P_a on $\mathcal{V}$ [4].
- The transformation P_A defined on $\mathcal{S}^n$ (the algebra of $n \times n$ real symmetric matrices) by $P_A(X) = AXA^T$, where $A \in \mathcal{R}^{n\times n}$.
- The transformation $P = L^{-1}$ on $\mathcal{V}$, where $L : \mathcal{V} \to \mathcal{V}$ is linear, positive stable (which means that all eigenvalues of L have positive real parts) and satisfies the Z-property:

  $$a \ge 0, b \ge 0,\ \langle a, b\rangle = 0 \Rightarrow \langle L(a), b\rangle \le 0.$$

  In particular, on the algebra $\mathcal{H}^n$ (of $n \times n$ complex Hermitian matrices), $P = L_A^{-1}$, where A is a complex $n \times n$ positive stable matrix and $L_A(X) := AX + XA^*$.
- Any *doubly stochastic transformation* on $\mathcal{V}$ [7]: It is a positive linear transformation P with $P(e) = e = P^*(e)$.

For any positive linear transformation P on $\mathcal{V}$, and $p \in [1,\infty]$ with conjugate q, we have the following from [6]:

(i) $||P||_{\infty\to p} = ||P(e)||_p$ and $||P||_{p\to 1} = ||P^*(e)||_q$.

(ii) $||P||_{p\to\infty} \leq ||P(e)||_\infty$ and $||P||_{1\to p} \leq ||P^*(e)||_\infty$.

(iii) $||P||_{p\to p} \leq ||P(e)||_\infty^{1-\frac{1}{p}}\, ||P^*(e)||_\infty^{\frac{1}{p}}$.

So, for a positive P, an application of Corollary 3.2 gives the following inequalities:

(i) $||P||_{r\to s} \leq 2\sqrt{2}||P(e)||_\infty^{1-\frac{1}{r}}\, ||P^*(e)||_\infty^{\frac{1}{r}}$ when $r < s$.

(ii) $||P||_{r\to s} \leq 2\sqrt{2}||P(e)||_\infty^{1-\frac{1}{s}}\, ||P^*(e)||_{(\frac{r}{s})'}^{\frac{1}{s}}$ when $r > s$.

Additionally, when P is also self-adjoint and $r > s$, analogous to (10), one can get the following estimate:

$$||P||_{r\to s} \leq 2\, C_q\, ||P(e)||_p.$$

4 Appendix

Proposition 4.1 *Given $p \in [1,\infty]$ with conjugate q, consider the following real valued functions defined over $\mathcal{R}^n \times \mathcal{R}^n$, $n \geq 2$:*

$$f(x,y) = ||x||_p + ||y||_p \quad \text{and} \quad g(x,y) = ||x+iy||_p.$$

Then,

$$\max\Big\{ f(x,y) :\ g(x,y) = 1 \Big\} = C_p, \tag{11}$$

where

$$C_p = \begin{cases} \sqrt{2} & \text{if } 1 \leq p < 2, \\ 2^{\frac{1}{q}} & \text{if } 2 \leq p \leq \infty\,. \end{cases}$$

Proof. By the continuity of f and g, and the compactness of the constraint set, the maximum in (11) is attained.

It is easy to see that the pair $(\overline{x}, \overline{y})$ with $\overline{x} = 2^{-\frac{1}{p}}(1,0,0\ldots,0)$ and $\overline{y} = 2^{-\frac{1}{p}}(0,1,0\ldots,0)$ satisfies the (constraint) equation $g(x,y) = 1$. Hence

$$C_p \geq f(\overline{x},\overline{y}) = 2^{\frac{1}{q}}. \tag{12}$$

Consider any pair $(x,y) \in \mathcal{R}^n \times \mathcal{R}^n$ with $g(x,y) = 1$. Writing $x = (x_1, x_2, \ldots, x_n)$, etc., by Hölder's inequality, we have

$$||x||_p + ||y||_p \leq 2^{\frac{1}{q}} \Big(||x||_p^p + ||y||_p^p \Big)^{\frac{1}{p}} = 2^{\frac{1}{q}} \Big(\sum_{j=1}^{n} (|x_j|^p + |y_j|^p) \Big)^{\frac{1}{p}}. \tag{13}$$

We consider three cases.

Case 1: $p = \infty$. By (12), $C_\infty \geq 2^{\frac{1}{q}} = 2$ (as $q = 1$). Since $|x_j + iy_j| \leq 1$ for all j (from our constraint), we get $||x||_\infty, ||y||_\infty \leq 1$; hence $C_\infty \leq 2$. We conclude that $C_\infty = 2$.

Case 2: $2 \leq p < \infty$.
In this case, we use the well-known Clarkson inequality for complex numbers z and w (see [1], page 163):

$$2(|z|^p + |w|^p) \leq |z + w|^p + |z - w|^p.$$

Then, for each j, with $z = x_j$ and $w = iy_j$, we have

$$2\big(|x_j|^p + |y_j|^p\big) \leq |x_j + iy_j|^p + |x_j - iy_j|^p.$$

Summing over j and noting $|x_j + iy_j| = |x_j - iy_j|$, we get

$$\sum_{j=1}^{n} \big(|x_j|^p + |y_j|^p\big) \leq \sum_{j=1}^{n} |x_j + iy_j|^p = g(x, y)^p = 1.$$

It follows from (13) that $||x||_p + ||y||_p \leq 2^{\frac{1}{q}}$. As this holds for all (x, y) with $g(x, y) = 1$, we have $C_p \leq 2^{\frac{1}{q}}$. From (12) we conclude that $C_p = 2^{\frac{1}{q}}$.

Case 3: $1 \leq p < 2$.
Let $\delta := n^{-\frac{1}{p}} 2^{-\frac{1}{2}}$. It is easy to see that the pair $(\overline{x}, \overline{y})$ with $\overline{x} = \delta(1, 1, \ldots, 1) = \overline{y}$ satisfy the constraint equation $g(x, y) = 1$. As $f(\overline{x}, \overline{y}) = \sqrt{2}$ we have, $C_p \geq \sqrt{2}$.
Now, as $1 \leq p < 2$, we use a refined version of Clarkson inequality presented in [1], Theorem 2.3:

$$2^{p-1}(|z|^p + |w|^p) + (2 - 2^{\frac{p}{2}}) \min\{|z + w|^p, |z - w|^p\} \leq |z + w|^p + |z - w|^p.$$

Then, for each j, with $z = x_j$ and $w = iy_j$, we have

$$2^{p-1}\big(|x_j|^p + |y_j|^p\big) + (2 - 2^{\frac{p}{2}}) \min\{|x_j + iy_j|^p, |x_j - iy_j|^p\} \leq |x_j + iy_j|^p + |x_j - iy_j|^p.$$

Simplifying this expression and summing over j, we get

$$\sum_{j=1}^{n} \big(|x_j|^p + |y_j|^p\big) \leq 2^{1-\frac{p}{2}} \Big(\sum_{j=1}^{n} |x_j + iy_j|^p\Big) = 2^{1-\frac{p}{2}} g(x, y)^p = 2^{1-\frac{p}{2}}.$$

This leads, via (13), to

$$||x||_p + ||y||_p \leq 2^{\frac{1}{q}} \, (2^{1-\frac{p}{2}})^{\frac{1}{p}} = \sqrt{2}.$$

Now, taking the maximum of $||x||_p + ||y||_p$ over (x, y), we get $C_p \leq \sqrt{2}$. Thus, when $1 \leq p < 2$,

$$C_p = \sqrt{2}.$$

This completes our proof. □

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