Noname manuscript No.
(will be inserted by the editor)

Weighted LCPs and interior point systems for copositive linear transformations on Euclidean Jordan algebras

M. Seetharama Gowda

January 9, 2019

Abstract In the setting of a Euclidean Jordan algebra V with symmetric cone V_+, corresponding to a linear transformation M, a 'weight vector' $w \in V_+$, and a $q \in V$, we consider the weighted linear complementarity problem wLCP(M, w, q) and (when w is in the interior of V_+) the interior point system IPS(M, w, q). When M is copositive on V_+ and q satisfies an interiority condition, we show that both the problems have solutions. A simple consequence, stated in the setting of $\mathbb{R}^n$ is that when M is a copositive plus matrix and q is strictly feasible for the linear complementarity problem LCP(M, q), the corresponding interior point system has a solution. This is analogous to a well-known result of Kojima et al. on $\mathbf{P}_*$-matrices and may lead to interior point methods for solving copositive LCPs.

Keywords Weighted LCPs · Interior point system · Euclidean Jordan algebra · Degree · Copositive linear transformation

Mathematics Subject Classification (2000) 90C30

1 Introduction

Motivated by the works of Potra [14,15], in a recent paper Chi et al. [1] studied the weighted horizontal linear complementarity problem which is defined as follows. Let $(V, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra with symmetric cone V_+ [4]. Given two linear transformations A and B on V, a 'weight vector' $w \in V_+$, and a $q \in V$, the *weighted horizontal linear complementarity problem*

M. Seetharama Gowda (✉)
Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland 21250, USA
E-mail: gowda@umbc.edu

wHLCP(A,B,w,q) is to find $(x,y)\in V\times V$ such that

$$\begin{array}{l} x\geq 0,\;\; y\geq 0,\\ x\circ y=w,\\ Ax+By=q, \end{array} \tag{1}$$

where $x\geq 0$ means that $x\in V_+$, etc. This problem reduces to a horizontal linear complementarity problem on V when $w=0$ and to the following 'interior point system' when $w>0$ (that is, when w is in the interior of V_+):

$$\begin{array}{l} x>0,\, y>0,\\ x\circ y=w,\\ Ax+By=q. \end{array} \tag{2}$$

For the algebra $V=\mathbb{R}^n$ (in which case, the Jordan product is the componentwise product), under the assumptions that $\{A,B\}$ is a sufficient pair (equivalently, a $\mathbf{P}_*$-pair) and q is strictly feasible, Potra [15] shows that wHLCP(A,B,w,q) has a solution for any $w\in V_+$. His proof consists in applying the Karush-Kuhn-Tucker conditions to an optimization problem over a bounded sublevel set. For a general V, using degree-theoretic tools and a weighted version of the Fischer-Burmeister map, Chi et al. ([1], Theorem 1) showed that when $\{A,B\}$ is an $\mathbf{R}_0$-pair and HLCP-deg(A,B) is nonzero, wHLCP(A,B,w,q) has a nonempty compact solution set for every $(w,q)\in V_+\times V$.

Specializing wHLCP(A,B,w,q) to $A=-M$ and $B=I$ (identity transformation), we get the *weighted linear complementarity problem* wLCP(M,w,q): Find $(x,y)\in V\times V$ such that

$$\begin{array}{l} x\geq 0,\, y\geq 0,\\ x\circ y=w,\\ y=Mx+q. \end{array} \tag{3}$$

When $w>0$, this leads to the *interior point system* IPS(M,w,q):

$$\begin{array}{l} x>0,\, y>0,\\ x\circ y=w,\\ y=Mx+q. \end{array} \tag{4}$$

While the study of wLCPs is new (initiated in [14,15] and continued in [1]), systems of the type (4) have appeared in interior point methods for solving linear complementarity problems and linear programming problems. In particular, the following result of Kojima et al. stated in the setting of $\mathbb{R}^n$ is well-known in the LCP literature:

Theorem 1 *([10], Lemmas 4.3 and 4.5) On $\mathbb{R}^n$, suppose the following conditions hold for M and q:*

(i) M is a $\mathbf{P}_0$-matrix (that is, all the principal minors of M are nonnegative).
(ii) q is strictly feasible, that is, there exists $\overline{x}>0$ such that $M\overline{x}+q>0$.

(*iii*) *For every real number* $c \geq 0$*, the sublevel set*

$$\{(x, y) : x \geq 0,\ y \geq 0,\ y = Mx + q,\ \text{and}\ \langle x, y\rangle \leq c\}$$

is bounded.

Then, for any $w > 0$*, the interior point system (4) has a unique solution. In particular, this conclusion holds when* M *is a* P_**-matrix and* q *is strictly feasible.*

The proof of this result given in [10] consists in (a) showing that the map $(x, y) \to x \circ y$ is one-to-one on the set $\{(x, y) : x > 0,\ y > 0,\ y = Mx + q\}$, ($b$) applying the invariance of domain theorem, and (c) showing that a $\mathbf{P}_*$-matrix satisfies conditions (i) and (iii).

Motivated by the above result of Kojima et al., we ask if a similar result could be stated for other types of matrices, particularly for copositive matrices which are matrices satisfying the condition $\langle Mx, x\rangle \geq 0$ for all $x \geq 0$. Since the $\mathbf{P}_0$-property is crucially used in Theorem 1 to prove the uniqueness part, we can no longer expect uniqueness in our formulation. Foregoing uniqueness, but working in a general Euclidean Jordan algebra setting, we show in Theorem 2 of this paper that (3) has a solution for any $w \geq 0$ and (4) has a solution for any $w > 0$ when M is a copositive linear transformation and q satisfies an interiority condition that is somewhat stronger than the bounded sublevel set condition (iii) above. A simple consequence of this result, stated in the setting of $\mathbb{R}^n$ is: *If* M *is a copositive plus matrix and* q *is strictly feasible, then for any* $w \geq 0$*, the weighted* LCP (3) *has a solution and for any* $w > 0$*, the interior point system (4) has a solution.* We anticipate that a result of this type may lead to interior point methods for solving copositive linear complementarity problems, see Remark 2.

The organization of the paper is as follows. In Section 2, we cover some preliminary material dealing with the min and Fischer-Burmeister maps, copositive transformations, and degree theory. Our main result, Theorem 2, is presented in Section 3. An important corollary related to copositive star transformations will also be presented in this section. We provide an example to show that Theorem 1 need not hold for semimonotone matrices.

2 Preliminaries

For most part, we follow the notation and basic results from [1]. For the sake of completeness, we recall some here. Throughout this paper, we let $(V, \circ, \langle\cdot,\cdot\rangle)$ denote a Euclidean Jordan algebra of rank n with symmetric cone V_+ [4,8]. Here, $x \circ y$ and $\langle x, y\rangle$ denote, respectively, the Jordan product and the inner product of elements x and y. We let e denote the unit element of V. For a subset S of V, the interior, closure, and boundary are respectively denoted by $int(S)$, $\overline{S}$, and $\partial(S)$. We write $\text{conv}(S)$ for the convex hull of S and $\text{cone}(S) :=$

$\{\lambda x : \lambda \geq 0,\, x \in S\}$ is the conic hull of S. We say that a nonempty set S is a cone if cone$(S) \subseteq S$. If $x \in V_+$ ($x \in int(V_+)$), we write $x \geq 0$ (respectively, $x > 0$). For a real number α, we write $\alpha^+ := \max\{\alpha, 0\}$. For $x \in V$, x^+ denotes the projection of x onto V_+, and we let $x^- := x^+ - x$, $|x| := x^+ + x^-$. These can also be described via the spectral decomposition of $x = \sum_{i=1}^n x_i e_i$ (where $x_1, x_2, \ldots, x_n$ are the eigenvalues of x and $\{e_1, e_2, \ldots, e_n\}$ is Jordan frame): $x^+ = \sum_{i=1}^n x_i^+ e_i$, $|x| = \sum_{i=1}^n |x_i| e_i$, etc. For $x, y \in V$, we define

$$x \sqcap y := x - (x - y)^+.$$

When $V = \mathbb{R}^n$ (with the usual componentwise product and the inner product), this reduces to $\min\{x, y\}$, the componentwise minimum of (vectors) x and y in $\mathbb{R}^n$. The maps $(x, y) \to x \sqcap y$ and

$$(x, y) \to x + y - \sqrt{x^2 + y^2}$$

will be called, respectively, the min map and the Fischer-Burmeister map. Below, we collect some properties of these maps.

Proposition 1 *([1], Propositions 1 and 2) The following statements hold in V:*

(*i*) $u + x \sqcap y = (u + x) \sqcap (u + y)$.

(*ii*) $\lambda(x \sqcap y) = \lambda x \sqcap \lambda y$ *for all* $\lambda \geq 0$.

(*iii*) *The following are equivalent; in each case, x and y operator commute.*
- (a) $x \sqcap y = 0$.
- (b) $x \geq 0$, $y \geq 0$, *and* $\langle x, y \rangle = 0$.
- (c) $x \geq 0$, $y \geq 0$, *and* $x \circ y = 0$.

(*iv*) *When $w \geq 0$, the following statements are equivalent:*
- (1) $x + y - \sqrt{x^2 + y^2 + 2w} = 0$.
- (2) $x \geq 0$, $y \geq 0$, *and* $x \circ y = w$.

Moreover, when $w = 0$ or $w = e$ (the unit element of V), above x and y operator commute.

(*v*) *Let $x, y \in V$ and $0 \leq t \leq 1$. Then, the following are equivalent:*
- (α) $t\left[x + y - \sqrt{x^2 + y^2}\right] + (1 - t)\,[x \sqcap y] = 0$.
- (β) $x \sqcap y = 0$.

Note: The following implication, observed in [1] (page 164) will be useful in transitioning from a weighted LCP to an interior point system:

$$\Big[x \geq 0,\, y \geq 0 \text{ and } x \circ y > 0\Big] \Rightarrow x > 0 \text{ and } y > 0.$$

2.1 Linear complementarity problems and copositive transformations

In this section, we recall definitions of various types of copositive (linear) transformations and state some known results. Let H be a finite dimensional real Hilbert space. For any set E in H, we define its dual by

$$E^* := \{x \in H : \langle x, y \rangle \geq 0 \text{ for all } y \in E\}.$$

Given a closed convex cone K in H, a linear transformation M on H, and a $q \in H$, we define the *linear complementarity problem*, $\mathrm{LCP}(M, K, q)$, as the problem of finding $x \in H$ such that

$$x \in K,\ Mx + q \in K^*,\ \text{and } \langle x, Mx + q\rangle = 0,$$

where K^* denotes the dual of K in H. We denote the solution set of $\mathrm{LCP}(M, K, q)$ by $\mathrm{SOL}(M, K, q)$. In addition, *when $H = V$ and $K = V_+$, we further abbreviate* $\mathrm{LCP}(M, V_+, q)$ and $\mathrm{SOL}(M, V_+, q)$ by $\mathrm{LCP}(M, q)$ and $\mathrm{SOL}(M, q)$ respectively. For the problem $\mathrm{LCP}(M, q)$, we say that q is *strictly feasible* if there exists $\overline{x} > 0$ in V such that $\overline{y} := M\overline{x} + q > 0$.

In what follows, we let S denote the solution set of $\mathrm{LCP}(M, K, 0)$, that is,

$$S := \{x \in H : x \in K, Mx \in K^*,\ \text{and } \langle x, Mx\rangle = 0\}. \tag{5}$$

Given H, K, and M as above, we say [5,3] that

(*a*) M is *copositive* on K if $\langle x, Mx\rangle \geq 0$ for all $x \in K$;

(*b*) M is *copositive star* on K if M is copositive on K and the implication

$$x \in S \Rightarrow -M^T x \in K^*$$

holds, where M^T denotes the transpose of M;

(*c*) M is *copositive plus* on K if M is copositive on K and the implication

$$x \in K,\ \langle x, Mx\rangle = 0 \Rightarrow \langle (M + M^T)x, u\rangle = 0$$

holds for all $u \in K$;

(*d*) M is *monotone* on H if $\langle x, Mx\rangle \geq 0$ for all $x \in H$.

It is easy to show that on K, every copositive plus transformation is copositive star and every monotone M is copositive plus. Also, when K has nonempty interior (e.g., when $K = V_+$ in $H = V$), M is copositive plus on K if and only if it is copositive and

$$x \in K,\ \langle x, Mx\rangle = 0 \Rightarrow (M + M^T)x = 0.$$

Copositive (plus, star) matrices are matrices that are copositive (respectively, plus, star) on $\mathbb{R}^n_+$.

We make a few remarks regarding relevance and importance of copositive matrices/transformations in the LCP theory. Since their introduction by Motzkin in 1952, copositive matrices (and their generalizations) have appeared in numerous areas such as game theory, optimization, engineering, statistics, etc., see e.g., [2,3,9] for theory, methods, and applications. They first appeared in the LCP literature in the 1960s in the works of Lemke (who was credited for describing a simplex-like algorithm for solving certain copositive LCPs), Cottle, and others, see [2]. Linear transformations that are copositive over cones have useful and interesting LCP related properties, see Proposition 3 below; see [3] for their relevance and applications in variational inequality problems. Now-a-days, certain cones related to copositive matrices (such as the copositive cone and completely positive cone) have become important in conic programming.

In the result below, we consider a bounded sublevel set condition (like the one that appears in Theorem 1) and relate it to an interiority condition.

Proposition 2 *Let H, K, M, and q be as above. Let S be given by (5). Consider the following statements.*

(a) $q \in \text{int}(S^*)$.
(b) $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$.
(c) *For every real number $c \geq 0$, the sublevel set*

$$\{(x, y) : x \in K, y \in K^*,\ y = Mx + q,\ \text{and}\ \langle x, y\rangle \leq c\}$$

is bounded in $H \times H$.

Then, $(a) \Leftrightarrow (b)$. When M is copositive on K, $(b) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$: Suppose $q \in \text{int}(S^*)$ and $0 \neq x \in S$. Then, for all small positive ε, $q - \varepsilon\, x \in S^*$. This implies that $\langle q - \varepsilon\, x, x\rangle \geq 0$, that is, $\langle q, x\rangle \geq \varepsilon\, ||x||^2 > 0$.
$(b) \Rightarrow (a)$: Assume that $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$. If $S = \{0\}$, then $S^* = H$ and (a) holds trivially. So, assuming $S \neq \{0\}$, we let $E := \{x \in S : ||x|| = 1\}$. As S is closed, E is compact. Hence, conv(E), the convex hull of E, is compact and convex. As S is a cone, from $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$ we see that $\langle q, y\rangle > 0$ for all $y \in E$ and so $\langle q, y\rangle > 0$ for all $y \in \text{conv}(E)$. This implies that $0 \notin \text{conv}(E)$. Then, the convex cone $C := \text{cone}(\text{conv}(E))$ is *closed.* It is easy to see that $S^* = C^*$ and the condition $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$ is equivalent to the condition $0 \neq x \in C \Rightarrow \langle q, x\rangle > 0$. A simple application of the supporting hyperplane theorem shows that the latter condition is equivalent to $q \in \text{int}(C^*) = \text{int}(S^*)$.
$(b) \Rightarrow (c)$: Assume that M is copositive on K. Suppose, for some real number $c \geq 0$, the sublevel set in (c) is unbounded. Without loss of generality, consider an unbounded sequence x_k such that $x_k \in K$, $y_k = Mx_k + q \in K^*$, and $\langle x_k, Mx_k + q\rangle \leq c$ for all $k = 1, 2, \ldots$. We normalize x_k, let $k \to \infty$, and assume (without loss of generality) that $\overline{x} = \lim \frac{x_k}{||x_k||}$ to get

$$\overline{x} \in K,\ M\overline{x} \in K^*, \text{ and } \langle \overline{x}, M\overline{x}\rangle = 0,$$

where the last equality follows from the copositivity of M. This means that $0 \neq \overline{x} \in S$. Also, because M is copositive, the inequality $\langle x_k, Mx_k + q\rangle \leq c$ implies that $\langle x_k, q\rangle \leq \langle x_k, Mx_k + q\rangle \leq c$ for all $k = 1, 2, \ldots$. This leads to $\langle \overline{x}, q\rangle \leq 0$ which is the negation of (b). Hence, $(b) \Rightarrow (c)$ when M is copositive.
□

Proposition 3 *The following statements hold:*

(a) *If M is copositive on K and $q \in \text{int}(S^*)$, then* LCP(M, K, q) *has a solution.*

(b) *If M is copositive star on K, then $S^* = \overline{K^* - M(K)}$.*
(c) *Let $H = V$ and $K = V_+$. Suppose M be copositive star on V_+. Then, q is strictly feasible if and only if $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$.*

Remarks. (1) Items (a) and (b) are (essentially) proved in [6]; see also Propositions 2.5.11 and Lemma 2.5.2 in [3]. The 'only if' part of Item (c) can be seen as follows: If $0 \neq x \in S$, then $-M^T x \geq 0$. Hence, writing $q = \overline{y} - M(\overline{x})$ where $\overline{x} > 0$ and $\overline{y} > 0$, we have

$$\langle q, x\rangle = \langle \overline{y}, x\rangle - \langle M(\overline{x}), x\rangle = \langle \overline{y}, x\rangle + \langle \overline{x}, -M^T x\rangle > 0.$$

The 'if' part (which is not needed in the paper) can be proved by an application of the separation theorem.
When $H = \mathbb{R}^n_+$ and $K = \mathbb{R}^n_+$, Item (a) can be improved: If M is copositive on $\mathbb{R}^n_+$ and $q \in S^*$, then LCP(M, K, q) has a solution, see [2], Theorems 3.8.6, 4.4.13, and Remark 4.4.14. In the same setting, (b) can be improved: When M is copositive star on $\mathbb{R}^n_+$, $S^* = \mathbb{R}^n_+ - M(\mathbb{R}^n_+)$, see [2], Theorem 3.8.13.

2.2 Degree theory

We employ degree theoretic arguments to prove our main result. All necessary results concerning degree theory are given in [3] (specifically, Proposition 2.1.3); see also, [12,13]. For completeness, we recall some standard notation and properties.

Let Ω be a bounded open set in $\mathbb{R}^n$, $g : \overline{\Omega} \to \mathbb{R}^n$ be continuous, and $p \notin g(\partial\Omega)$. Then the degree of g over Ω with respect to p is defined; it is an integer and will be denoted by $\deg(g, \Omega, p)$. One crucial property is:

When $deg(g, \Omega, p) \neq 0$, the equation $g(x) = p$ has a solution in Ω.

Suppose $g(x) = p$ has a unique solution, say, x^* in Ω. Then $\deg(g, \Omega', p)$, which is the same as $\deg(g, \Omega', g(x^*))$, is constant over all bounded open sets Ω' containing x^* and contained in Ω. This common degree, denoted by $\mathrm{ind}(g, x^*)$, is called the (topological) index of g at x^*. If g is also differentiable at x^* with nonsingular derivative, then ([7], page 869)

$$\mathrm{ind}(g, x^*) = sgn \det g'(x^*). \tag{6}$$

In particular, if $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map such that $g(x) = 0 \Leftrightarrow x = 0$, then for any bounded open set containing 0, we have

$$\mathrm{ind}(g, 0) = \deg(g, \Omega, 0);$$

moreover, when g is the identity map, $\mathrm{ind}(g, 0) = 1$.

A continuous map $H(x, t) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ is called a homotopy. Given such an H, suppose that for some bounded open set Ω in $\mathbb{R}^n$, $0 \notin H(\partial\Omega, t)$

for all $t \in [0,1]$. Then, the *homotopy invariance property of degree* says that $\deg\Big(H(\cdot,t),\Omega,0\Big)$ is independent of t. In particular, if the zero set

$$\Big\{x : H(x,t) = 0 \text{ for some } t \in [0,1]\Big\}$$

is bounded, then for any bounded open set Ω in $\mathbb{R}^n$ containing this zero set, we have

$$\deg\Big(H(\cdot,1),\Omega,0\Big) = \deg\Big(H(\cdot,0),\Omega,0\Big).$$

Note: All of the above concepts and results are also valid over any finite dimensional real Hilbert space (such as V or $V \times V$) in place of $\mathbb{R}^n$. When it is required to show that the zero set of a map or a system of equations is bounded, we frequently employ the so-called *normalization argument.* This requires normalizing a sequence of vectors (with their norms going to infinity) to produce a unit vector that violates a given criteria.

3 The main existence result

In what follows, we consider a Euclidean Jordan algebra V with its symmetric cone V_+.

Theorem 2 *Let M be a linear transformation on V that is copositive on V_+. We let $S := \mathrm{SOL(M,0)}$ and $q \in V$ satisfy the implication*

$$0 \neq x \in S \Rightarrow \langle q, x\rangle > 0.$$

Then for any $w \geq 0$, $\mathrm{wLCP}(M,w,q)$ *has a nonempty compact solution set. In particular, if $w > 0$, then* $\mathrm{IPS}(M,w,q)$ *has a nonempty compact solution set.*

Proof We fix $w \geq 0$ and assume that the implication $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$ holds. We show, by degree theoretic arguments that the system

$$\begin{gathered} x + y - \sqrt{x^2 + y^2 + 2w} = 0, \\ y - (Mx + q) = 0 \end{gathered}$$

has a nonempty compact solution set. This, in view of Item (iv) of Proposition 1, shows that $\mathrm{wLCP}(M,w,q)$ has a nonempty compact solution set.
We fix a $d > 0$. With $z = (x,y) \in V \times V$ and $t \in [0,1]$, we define the following map:

$$F(z,t) := \begin{bmatrix} x + y - \sqrt{x^2 + y^2 + 2tw} \\ y - \Big[Mx + (1-t)d + tq\Big] \end{bmatrix}.$$

Clearly, F is continuous in (z,t),

$$F(z,0) := \begin{bmatrix} x + y - \sqrt{x^2 + y^2} \\ y - \Big[Mx + d\Big] \end{bmatrix},$$

and

$$F(z,1) := \begin{bmatrix} x + y - \sqrt{x^2 + y^2 + 2w} \\ y - \Big[Mx + q\Big] \end{bmatrix}.$$

Let

$$Z := \{z : F(z,t) = 0 \text{ for some } t \in [0,1]\}.$$

We show by a *normalization argument* that Z is bounded. Suppose, if possible, Z is unbounded. Let $z_k := (x_k, y_k)$, $t_k \in [0,1]$ with $||z_k|| \to \infty$, and $F(z_k, t_k) = 0$ for all $k = 1, 2, \ldots$. Without loss of generality, we assume that $||x_k|| \to \infty$. We also suppose that $t_k \to \overline{t}$ and

$$\overline{x} := \lim_{k\to\infty} \frac{x_k}{||x_k||}.$$

Now, using Item (iv) in Proposition 1 and $F(z_k, t_k) = 0$, we get the inequalities $x_k \geq 0$, $Mx_k + (1 - t_k)d + t_k q \geq 0$ for all k. These lead to

$$\overline{x} \geq 0 \text{ and } M\overline{x} \geq 0. \tag{7}$$

Item (iv) in Proposition 1 also gives $x_k \circ \Big[Mx_k + (1 - t_k)d + t_k q\Big] = t_k w$. By taking the inner product of this with e, we get, for all k,

$$\langle x_k, Mx_k\rangle + (1 - t_k)\langle x_k, d\rangle + t_k\langle x_k, q\rangle = t_k\,\langle w, e\rangle. \tag{8}$$

Dividing this by $||x_k||^2$ and taking the limit as $k \to \infty$, we get

$$\langle \overline{x}, M\overline{x}\rangle = 0. \tag{9}$$

Combining this with (7), we see that $0 \neq \overline{x} \in S$. Now, M is copositive on V_+. Hence, from (8),

$$(1 - t_k)\langle x_k, d\rangle + t_k\langle x_k, q\rangle \leq t_k\,\langle w, e\rangle$$

for all k. Dividing this by $||x_k||$ and taking limit, we get

$$(1 - \overline{t})\langle \overline{x}, d\rangle + \overline{t}\langle \overline{x}, q\rangle \leq 0.$$

However, this is not possible, as $0 \neq \overline{x} \in S$ and $d > 0$ imply that $\langle \overline{x}, q\rangle > 0$ and $\langle \overline{x}, d\rangle > 0$. Thus, the set Z is bounded.

Now consider the map

$$H(z,t) := \begin{bmatrix} t\Big[x + y - \sqrt{x^2 + y^2}\Big] + (1 - t)\,[x \sqcap y] \\ y - \Big[Mx + d\Big] \end{bmatrix}.$$

Clearly, H is continuous in (z,t),

$$H(z,0) := \begin{bmatrix} x \sqcap y \\ y - \Big[Mx + d\Big] \end{bmatrix}.$$

and

$$H(z,1) := \begin{bmatrix} x + y - \sqrt{x^2 + y^2} \\ y - \Big[Mx + d\Big] \end{bmatrix} = F(z,0).$$

Suppose for some (z,t), $H(z,t) = (0,0)$. Then,

$$t\left[x + y - \sqrt{x^2+y^2}\right] + (1-t)x \sqcap y = 0,$$

which, by Item (v) of Proposition 1, implies that $x \sqcap y = 0$. We also have $y = Mx + d$. Thus, $x \geq 0, y \geq 0,$ and $\langle x, Mx + d\rangle = 0$. Since M is copositive, we get $\langle x, d\rangle = 0$. We conclude that $x = 0$ (as $d > 0$). Hence, $z = (0, M0+d) = (0, d)$. Thus, the set

$$\{z : H(z,t) = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Let Ω be a bounded open set in $V \times V$ that contains the zero sets of F and H. Then, by the homotopy invariance property of degree, we have

$$\begin{aligned} &\deg\Big(F(\cdot,1), \Omega, 0\Big) = \deg\Big(F(\cdot,0), \Omega, 0\Big) \\ &= \deg\Big(H(\cdot,1), \Omega, 0\Big) = \deg\Big(H(\cdot,0), \Omega, 0\Big). \end{aligned}$$

We now show that $\deg\Big(H(\cdot,0), \Omega, 0\Big)$ is nonzero. First, we note (as observed previously) that $(0,d)$ is the only solution of $H(x,t) = 0$ in Ω. When $z = (x,y)$ is close to $(0,d)$, x is close to 0 and $y = Mx+d$ is close to d. Hence $x-(Mx+d)$ is close to $-d$. Then, $(x-y)^+ = 0$ and so $x \sqcap y = x$. Thus, for $z = (x,y)$ close to $(0,d)$,

$$H(z,0) = \begin{bmatrix} x \sqcap y \\ y - \Big[Mx + d\Big] \end{bmatrix} = \begin{bmatrix} x \\ y - \Big[Mx + d\Big] \end{bmatrix}.$$

The derivative of this at $(0,d)$ is

$$\begin{bmatrix} I & 0 \\ -M & I \end{bmatrix}$$

which is clearly nonsingular. So, $\deg\Big(H(\cdot,0), \Omega, 0\Big)$ (which, by (6), is the sign of the determinant of the above derivative at $(0,d)$) is nonzero. Thus, we have shown that $\deg\Big(F(\cdot,1), \Omega, 0\Big)$ is nonzero. This implies that the equation $F(z,1) = 0$ has a solution and all solutions lie in the bounded open set Ω. This proves our first assertion about wLCP(M,w,q). The second assertion about IPS(M,w,q) follows immediately (see the note following Proposition 1). □

Remarks. (2) The proof of the above theorem can be modified to get the following.

Suppose the conditions of the theorem hold and let B be a nonempty bounded subset of V_+. Then, as w varies over B, the solution sets of wLCP(M, w, q) *are uniformly bounded. Also, as w varies over elements of B that satisfy $w > 0$, the solution sets of* IPS(M, w, q) *are uniformly bounded.* This can be seen by modifying the definitions of F and Z to include w_k in place of w and showing that the ensuing zero sets are uniformly bounded.
To see an application of this, suppose $w > 0$ and let $\varepsilon_k \downarrow 0$ in R. When the conditions of the theorem are in place, the sequence (x_k, y_k) satisfying the conditions

$$\begin{aligned} &x_k > 0,\ y_k > 0,\\ &x_k \circ y_k = \varepsilon_k\, w,\\ &y_k = Mx_k + q, \end{aligned} \tag{10}$$

will have a subsequence that converges to a solution of LCP(M, q).

(3) We now compare the above theorem with a result in [1]. Suppose M satisfies the $\mathbf{R}_0$-property on V_+, that is, $S = \mathrm{SOL(M, 0)} = \{0\}$. Then, the map $g : x \mapsto x \sqcap Mx$ vanishes only at zero. When ind$(g, 0)$ is nonzero, Corollary 2 in [1] asserts that wLCP(M, w, q) has a solution for any $w \geq 0$ and $q \in V$ and IPS(M, w, q) has a solution for any $w > 0$ and $q \in V$. (In particular, this conclusion holds for linear transformations satisfying the **R**-property, **P**-property, or strictly copositive property.) Note that this is a 'universal/global' result as it holds for all q, while our Theorem 2 is an 'individual/local' result tailored to a particular q. When M is copositive on V_+ and satisfies the $\mathbf{R}_0$-property, it is easy to show that ind$(g, 0)$ is nonzero by considering the homotopy $H(x, t) := x \sqcap \Big(tMx + (1-t)x\Big)$. Thus, when the copositive transformation has the $\mathbf{R}_0$-property, the two results give the same conclusion.

We now state an important consequence of Theorem 2.

Corollary 1 *Suppose M is copositive star on V_+ (for example, M is copositive plus on V_+ or monotone on V) and q is strictly feasible. Then, the conclusions of the theorem hold.*

Proof. Suppose M is copositive star on V_+ and q is strictly feasible. Then, by Item (c) in Proposition 3, $0 \neq x \in S \Rightarrow \langle q, x\rangle > 0$. Since M is copositive on V_+, Theorem 2 holds. ☐

The following example shows that in the above theorem, the interiority condition on q cannot be weakened to a bounded sublevel set condition (such as the one imposed in Item (c) of Theorem 1).

Example 1: In $V = \mathbb{R}^2$ with $V_+ = \mathbb{R}^2_+$, let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Then, M is copositive on $\mathbb{R}^2_+$ (as it is a nonnegative matrix), q is strictly feasible (that is for some $\overline{x} > 0$, $M\overline{x}+q > 0$), and for every $c \geq 0$, the sublevel set

$$\{(x,y) : x \geq 0,\, y \geq 0,\, y = Mx + q,\, \langle x, y\rangle \leq c\}$$

is bounded. However, corresponding to $w = 0$, the problem wLCP(M, w, q) over $\mathbb{R}^2_+$ does not have a solution. Also, the IPS(M, w, q) with $w = e$ (the vector of ones) does not have a solution. We note that the above M is an E_0-matrix (also called a semimonotone matrix [2]): For every nonzero x in $\mathbb{R}^2_+$, there is an index i such that

$$x_i(Mx)_i \geq 0.$$

Hence, this example also shows that Theorem 1 fails to hold when the P_0-matrix condition is replaced by the E_0-matrix condition.

The following example shows that even in the monotone case, mere solvability of a linear complementarity problem does not imply the solvability of the corresponding interior point system.

Example 2: In $V = \mathbb{R}^2$ with $V_+ = \mathbb{R}^2_+$, let

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, M (being positive semidefinite) is monotone on $\mathbb{R}^2$, hence copositive plus on $\mathbb{R}^2_+$. Also, LCP(M, q) solvable (as zero is a solution). Yet, q is not strictly feasible and so IPS(M, w, q) with w being the vector of ones cannot have a solution.

References

1. Chi, X., Gowda, M.S., Tao, J.: The weighted horizontal linear complementarity problem on a Euclidean Jordan algebra, Jour. Global Optim., 73 (2019) 153-169.
2. Cottle, R.W., Pang, J.-S., Stone, R.: The Linear Complementarity Problem. Academic Press, Boston (1992).
3. Facchinei, F., Pang, J.S.: Finite–dimensional variational inequalities and complementarity problems. Vol. I, Springer, New York (2003).
4. Faraut, J., Koranyi, A.: Analysis on Symmetric Cones. New York, Oxford University Press, Oxford Mathematical Monographs (1994).
5. Gowda, M.S.: Pseudomonotone and copositive star matrices, Linear Alg. Appl., 113 (1989) 107-118.
6. Gowda, M.S.: Complementarity problems over locally compact cones, SIAM J. Control and Optim., 27 (1989) 836-841.
7. Gowda, M.S.: Applications of degree theory to linear complementarity problems. Math. Oper. Res. 18 (1993) 868-879.
8. Gowda, M.S., Sznajder, R., Tao, J.: Some P-properties for linear transformations on Euclidean Jordan algebras. Special issue on Positivity, Linear Algebra Appl. 393 (2004) 203-232.
9. Hirriart-Urruty, J.-B., Seeger, A.: A variational approach to copositive matrices. SIAM Rev. 52 (2010) 593-629.

10. Kojima, M., Megiddo, N., Noma, T., Yoshise, A.: A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems. Springer Verlag, Berlin (1991).
11. Lemke, C.E.: Bimatrix equilibrium points and mathematical programming. Management Sci., 11 (1965) 681-689.
12. Lloyd, N.G.: Degree Theory. Cambridge University Press, London (1978).
13. Ortega, J.M., Rheinboldt, W.C.: Iterative Solutions of Nonlinear Equations in Several Variables, Academic Press, New York (1970).
14. Potra, F.A.: Weighted complementarity problems – a new paradigm for computing equilibria. SIAM J. Optim. 22(4) (2012) 1634-1654.
15. Potra, F.A.: Sufficient weighted complementarity problems. Comput. Optim. Appl. 64 (2016) 467-488.