Weakly homogeneous variational inequalities and solvability of nonlinear equations over cones

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Abstract

Given a closed convex cone C in a finite dimensional real Hilbert space H, a weakly homogeneous map $f : C \to H$ is a sum of two continuous maps h and g, where h is positively homogeneous of degree γ (≥ 0) on C and $g(x) = o(||x||^\gamma)$ as $||x|| \to \infty$ in C. Given such a map f, a nonempty closed convex subset K of C, and a $q \in H$, we consider the variational inequality problem, $\mathrm{VI}(f, K, q)$, of finding an $x^* \in K$ such that $\langle f(x^*)+q, x-x^*\rangle \geq 0$ for all $x \in K$. In this paper, we establish some results connecting the variational inequality problem $\mathrm{VI}(f, K, q)$ and the cone complementarity problem $\mathrm{CP}(f^\infty, K^\infty, 0)$, where $f^\infty := h$ is the homogeneous part of f and K^∞ is the recession cone of K. We show, for example, that $\mathrm{VI}(f, K, q)$ has a nonempty compact solution set for every q when zero is the only solution of $\mathrm{CP}(f^\infty, K^\infty, 0)$ and the (topological) index of the map $x \mapsto x - \Pi_{K^\infty}(x - G(x))$ at the origin is nonzero, where G is a continuous extension of f^∞ to H. As a consequence, we generalize a complementarity result of Karamardian [13] formulated for homogeneous maps on proper cones to variational inequalities. The results above extend some similar results proved for affine variational inequalities and for polynomial complementarity problems over the nonnegative orthant in $\mathcal{R}^n$. As an application, we discuss the solvability of nonlinear equations corresponding to weakly homogeneous maps over closed convex cones. In particular, we extend a result of Hillar and Johnson [12] on the solvability of symmetric word equations to Euclidean Jordan algebras.

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1 Introduction

This paper is motivated by [8], where polynomial complementarity problems over the nonnegative orthant were considered. Given a polynomial map $f : \mathcal{R}^n \to \mathcal{R}^n$ and a vector $q \in \mathcal{R}^n$, the polynomial complementarity problem, PCP(f, q), is to find an $x \in \mathcal{R}^n$ such that

$$x \geq 0,\ y := f(x) + q \geq 0,\ \text{and } \langle x, y \rangle = 0.$$

By decomposing f as a sum of homogeneous polynomial maps with f^∞ denoting the 'leading term' of f, in [8], various results connecting PCP(f, q) and PCP$(f^\infty, 0)$ were established. In particular, it was shown that if PCP$(f^\infty, 0)$ has trivial solution (namely, zero) and the (topological) index of the map $x \mapsto \min\{x, f^\infty(x)\}$ at the origin is nonzero, then for all q, PCP(f, q) has a nonempty compact solution set. With the observation that every polynomial map $f : \mathcal{R}^n \to \mathcal{R}^n$ is the sum of a homogeneous map h (the leading term of f) and g (the sum of the rest of the terms of f) which grows much slower than an appropriate power of $||x||$, we extend these results to variational inequality problems over closed convex sets for (what we call) weakly homogeneous maps.

To elaborate, let H be a finite dimensional real Hilbert space and C be a closed convex cone in H. A continuous map $h : C \to H$ is said to be *positively homogeneous of degree* γ (≥ 0) if $h(\lambda x) = \lambda^\gamma h(x)$ for all $x \in C$ and $\lambda > 0$ in $\mathcal{R}$. A mapping $f : C \to H$ is said to be a *weakly homogeneous map of degree* γ if $f = h + g$, where $h : C \to H$ is positively homogeneous of degree γ and $g : C \to H$ is continuous and $g(x) = o(||x||^\gamma)$ as $||x|| \to \infty$ in C. Given such a map f on C, a closed convex set K contained in C, and a vector $q \in H$, we consider the variational inequality problem VI(f, K, q) of finding an $x^* \in K$ such that

$$\left\langle f(x^*) + q, x - x^* \right\rangle \geq 0 \text{ for all } x \in K.$$

When K is a closed convex cone, VI(f, K, q) becomes a complementarity problem, in which case, we denote it by CP(f, K, q). With numerous applications to various fields, (general) variational inequality problems and complementarity problems have been extensively studied in the literature, see, e.g., [4], [5], and [2].

In this paper, we consider weakly homogeneous maps of positive degree. Let $f : C \to H$ be such a map and let $f(x) = h(x) + g(x)$, where h and g are as specified above. Then, (the "recession map") $f^\infty(x) := \lim_{\lambda\to\infty} \frac{f(\lambda x)}{\lambda^\gamma} = h(x)$, see Section 2. In this paper, we establish some results connecting VI(f, K, q) and the complementarity problem CP$(f^\infty, K^\infty, 0)$, where K^∞ denotes the recession cone of K. Among other things, we show the following:

- Assuming that zero is the only solution of CP$(f^\infty, K^\infty, 0)$ and the (topological) index of the map $x \mapsto x - \Pi_{K^\infty}(x - G(x))$ at the origin is nonzero, where G is a continuous extension of f^∞ to H, we show that for all q, VI(f, K, q) and CP(f, K^∞, q) have nonempty compact solution sets.

- Assuming that $\text{CP}(f^\infty, K^\infty, 0)$ and $\text{CP}(f^\infty, K^\infty, d)$ (or $\text{CP}(f, K^\infty, d)$) have (only) zero solutions for some $d \in \text{int}\left((K^\infty)^*\right)$, we show that for all q, $\text{VI}(f, K, q)$ and $\text{CP}(f, K^\infty, q)$ have nonempty compact solution sets.

These results demonstrate a close relationship between variational inequalities and complementarity problems for weakly homogeneous maps. The first result extends (similar) results proved in the setting of affine variational inequalities [7] (where f is affine and K is polyhedral) and polynomial complementarity problems [8]. The second result can be regarded as a generalization (and strengthening) of a result of Karamardian [13] formulated for homogeneous maps on proper cones.

By way of an application, we consider the solvability of equations of the form $f(x) = q$ over cones. Given a C and f (as above) and $q \in C$, we describe a method of proving the existence of an $x^* \in C$ such that $f(x^*) = q$. In particular, this method will be applied to extend the following result of Hillar and Johnson [12] to the setting of Euclidean Jordan algebras: Given positive definite matrices $A_1, A_2, \ldots, A_m$, a positive (semi)definite matrix Q, and positive exponents $r_1, r_2, \ldots, r_m$, the symmetric word equation

$$X^{r_m} A_m \cdots X^{r_2} A_2 X^{r_1} A_1 X^{r_1} A_2 X^{r_2} \cdots A_m X^{r_m} = Q$$

has a positive (semi)definite solution X.

2 Preliminaries

Throughout this paper, we fix a finite dimensional real Hilbert space H with inner product $\langle x, y\rangle$ and norm $||x||$. For any nonempty set S in H, we denote the interior, closure, and the boundary by $\text{int}(S)$, $\overline{S}$, and ∂S, respectively. We define the dual of S by $S^* := \{x \in H : \langle x, s\rangle \geq 0 \text{ for all } s \in S\}$. We frequently use the fact that

$$0 \neq x \in S,\, d \in \text{int}(S^*) \Rightarrow \langle d, x\rangle > 0. \tag{1}$$

(This can be seen by noting $d - \varepsilon\, x \in S^*$ for all small $\varepsilon > 0$.)

A set E in H is convex if $tx + (1-t)y \in E$ for all $t \in [0, 1]$ and $x, y \in E$; if in addition, $\lambda\, x \in E$ for all $\lambda > 0$ and $x \in E$, we say that E is a convex cone. A nonempty closed convex cone E is said to be pointed if $E \cap -E = \{0\}$, or equivalently, $\text{int}(E^*) \neq \emptyset$. A pointed closed convex cone with nonempty interior is called a proper cone. The recession cone of a closed convex set E is defined by $E^\infty := \{u \in H : u + E \subseteq E\}$, or alternatively, see [19, Theorem 8.2],

$$E^\infty = \left\{u \in H : \exists\, t_k \to \infty,\, \exists\, x_k \in E \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = u\right\}. \tag{2}$$

This is a closed convex cone.

2.1 Weakly homogeneous maps

Let K be a closed convex set and C be a closed convex cone in H such that $K \subseteq C$. Recall that a continuous map $f : C \to H$ is *weakly homogeneous of degree* γ (≥ 0) if $f = h + g$, where $h, g : C \to H$ are continuous and $h(\lambda\, x) = \lambda^\gamma\, h(x)$ for all $x \in C$ and $\lambda > 0$, and $g(x) = o(||x||^\gamma)$ (that is, $\frac{g(x)}{||x||^\gamma} \to 0$) as $||x|| \to \infty$ in C. Some elementary properties of such functions are stated below.

Proposition 2.1. *Suppose $f = h + g$ is weakly homogeneous of degree $\gamma > 0$. Then the following statements hold:*

- *(a)* $h(0) = 0$*; if* $f(0) = 0$*, then* $g(0) = 0$*.*
- *(b)* $\lim_{\lambda\to\infty} \frac{g(\lambda x)}{\lambda^\gamma} = 0$ *for all* $x \in C$*.*
- *(c)* $h(x) = \lim_{\lambda\to\infty} \frac{f(\lambda x)}{\lambda^\gamma}$ *for all* $x \in C$*.*
- *(d) In the representation* $f = h + g$*,* h *and* g *are unique on* C*.*

Proof. (a) $h(0) = h(2 \cdot 0) = 2^\gamma h(0)$. As $2^\gamma \neq 1$, we have $h(0) = 0$. Also, when $f(0) = 0$, we have $g(0) = f(0) - h(0) = 0$.
(b) Clearly this holds if $x = 0$. For $0 \neq x \in C$, $\frac{g(\lambda x)}{\lambda^\gamma} = \frac{g(\lambda x)}{||\lambda x||^\gamma} ||x||^\gamma \to 0$ as $\lambda \to \infty$ (since $||\lambda x|| \to \infty$).
(c) From (b),

$$\frac{f(\lambda x)}{\lambda^\gamma} = \frac{h(\lambda x)}{\lambda^\gamma} + \frac{g(\lambda x)}{\lambda^\gamma} = h(x) + \frac{g(\lambda x)}{\lambda^\gamma} \to h(x)$$

as $\lambda \to \infty$.
(d) Suppose f has two representations $f = h_1 + g_1 = h_2 + g_2$. Then from (c),

$$h_1(x) = \lim_{\lambda\to\infty} \frac{f(\lambda x)}{\lambda^\gamma} = h_2(x) \quad \text{for all } x \in C.$$

Hence, $g_1(x) = f(x) - h_1(x) = f(x) - h_2(x) = g_2(x)$ for all $x \in C$. □

Note: *Because of Item (c) above, we denote h by f^∞ and call it the 'leading term' of f.*

We now list some examples of weakly homogeneous maps.

1. Let C be any closed convex cone in H. Suppose f is a finite sum of homogeneous maps on C of the form

   $$f(x) = h_m(x) + h_{m-1}(x) + \cdots + h_1(x) + h_0(x),$$

   where $m > 0$, $h_j(x)$ is positively homogeneous of degree γ_j on C, and $\gamma_m > \gamma_{m-1} > \cdots > \gamma_1 > \gamma_0 = 0$. We claim that f is weakly homogeneous. To see this, let $||x|| \to \infty$ on C. Then, for $j < m$, $\frac{h_j(x)}{||x||^{\gamma_m}} = \frac{1}{||x||^{\gamma_m - \gamma_j}} h_j(\frac{x}{||x||})$. As $\frac{x}{||x||}$ varies over the intersection of the unit sphere and C, and h_j is continuous (resulting in the boundedness of $h_j(\frac{x}{||x||})$), we see that $\frac{h_j(x)}{||x||^{\gamma_m}} \to 0$. Letting $h = h_m$ and $g := f - h$, we see that f is weakly homogeneous.

2. This is a special case of the previous example. Let $H = C = \mathcal{R}^n$ and $f : \mathcal{R}^n \to \mathcal{R}^n$ be a polynomial map, that is, each component of f is a (real valued) polynomial function in n variables $x_1, x_2, \ldots, x_n$. We can decompose f as a sum of homogeneous polynomial maps of different (homogeneity) degrees; assuming that f is nonconstant, we let h denote the one that has the highest homogeneity degree and g denote the sum of the remaining terms. Then the equality $f = h + g$ shows that f is weakly homogeneous.

3. Let $H = \mathcal{R}^n$ and $C = \mathcal{R}^n_+$. A posynomial on $\mathcal{R}^n_+$ is of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_i \geq 0$ for all $i = 1, 2, \ldots, n$. A finite linear combination of posynomials is weakly homogeneous on $\mathcal{R}^n_+$.

(4) Let $H = \mathcal{S}^n$, the space of all $n \times n$ real symmetric matrices with $C = \mathcal{S}^n_+$, the cone of all positive semidefinite matrices in $\mathcal{S}^n$. Then, the following are weakly homogeneous on C:

(a) $f(X) := XAX + BX + XB^T$, where $A \in \mathcal{S}^n$ and $B \in \mathcal{R}^{n\times n}$.

(b) $f(X) := X - \sum_1^N A_i X^{\delta_i} A_i$, where $A_i \in \mathcal{S}^n$, $0 < \delta_i < 1$ for all i.

(c) $f(X) = X + \sin(X)$.

(d) $f(X) = XAXBXAX$, where $A, B \in \mathcal{S}^n$.

2.2 Continuity of a projection map

Given a closed convex set E in H, let $\Pi_E(x)$ denote the orthogonal projection of an $x \in H$ onto E. We note that for any x, $\Pi_E(x)$ is the unique element x^* in E that satisfies the inequality $\langle x - x^*, z - x^*\rangle \leq 0$ for all $z \in E$ or equivalently,

$$||x - x^*|| \leq ||x - z|| \;\forall\; z \in E. \tag{3}$$

Also, the map $x \mapsto \Pi_E(x)$ is Lipschitz continuous with Lipschitz constant one and

$$0 \in E \text{ and } u \in E^* \Rightarrow \Pi_E(-u) = 0. \tag{4}$$

Now, let K be a closed convex set with its recession cone K^∞. Define the map

$$\mathcal{K}(t) = tK + K^\infty,\; 0 \leq t \leq 1.$$

Note that

$$\mathcal{K}(t) = t(K + K^\infty) = tK \;(0 < t \leq 1) \quad \text{and} \quad \mathcal{K}(0) = K^\infty,$$

where the first statement comes from the fact that K^∞ is a cone. In a key result to be presented in Section 4, we will require the joint continuity of the map $(t, x) \mapsto \Pi_{\mathcal{K}(t)}\Big(\theta(x,t)\Big)$, where $\theta(x,t) : H \times [0,1] \to H$ is continuous. This continuity property is shown in the result below, whose proof is essentially given in that of [4, Lemma 2.8.2].

Proposition 2.2. *Let $\mathcal{K}(t)$ be as above and $\theta(x,t) : H \times [0,1] \to H$ be continuous. Then, $(x,t) \mapsto \Pi_{\mathcal{K}(t)}\Big(\theta(x,t)\Big)$ is continuous.*

Proof. Fix $(x_0, t_0) \in H \times [0,1]$ and let $y_0 := \theta(x_0, t_0)$. Because

$$\Pi_{\mathcal{K}(t)}\Big(\theta(x,t)\Big) - \Pi_{\mathcal{K}(t_0)}\Big(\theta(x_0,t_0)\Big) = \Big[\Pi_{\mathcal{K}(t)}\Big(\theta(x,t)\Big) - \Pi_{\mathcal{K}(t)}(y_0)\Big] + \Big[\Pi_{\mathcal{K}(t)}(y_0) - \Pi_{\mathcal{K}(t_0)}(y_0)\Big]$$

and $||\Pi_{\mathcal{K}(t)}\Big(\theta(x,t)\Big) - \Pi_{\mathcal{K}(t)}(y_0)|| \leq ||\theta(x,t) - y_0||$, to prove continuity, we need only show that $\Pi_{\mathcal{K}(t)}(y_0)$ is continuous at t_0. If $t_0 > 0$, this follows from the (easily verifiable) equality

$$\Pi_{\mathcal{K}(t)}(y_0) = \Pi_{tK}(y_0) = t\,\Pi_K\left(\frac{y_0}{t}\right) \;\forall\, t \in (0,1].$$

So, we assume that $t_0 = 0$ and verify the following:

$$\lim_{t\to 0, t>0} \Pi_{\mathcal{K}(t)}(y_0) = \Pi_{K^\infty}(y_0). \tag{5}$$

To see this, let $z_0 := \Pi_{K^\infty}(y_0) \in K^\infty$ and $z_0(t) := \frac{1}{t}\Pi_{tK}(y_0) \in K$ for all $0 < t \leq 1$. By (3), for any fixed $u_0 \in K$, we have

$$||y_0 - tz_0(t)|| \leq ||y_0 - tu_0|| \;\forall\; 0 < t \leq 1.$$

This shows that the set $\{tz_0(t) : 0 < t \leq 1\}$ is bounded. We show that $tz_0(t) \to z_0$ as $t \to 0$ in $(0, 1]$ by showing that z_0 is the only accumulation point of the set $\{tz_0(t) : 0 < t \leq 1\}$. Suppose there is a sequence $t_n z_0(t_n) \to w \in H$ as $t_n \to 0$. From (2), $w \in K^\infty$. Also, letting $n \to \infty$ in

$$||y_0 - t_n z_0(t_n)|| \leq ||y_0 - t_n\Big[u_0 + \frac{u}{t_n}\Big]|| \;\forall\, u \in K^\infty,$$

we get

$$||y_0 - w|| \leq ||y_0 - u|| \;\forall\; u \in K^\infty.$$

It follows that $w = z_0 = \Pi_{K^\infty}(y_0)$. This proves (5). $\square$

2.3 Variational inequalities

Throughout this paper, we will study variational inequalities via their equation formulations. To set up notation, let E be a nonempty closed convex set in H. Recall that for a continuous map $\phi : E \to H$ and $q \in H$, the *variational inequality problem*, $\mathrm{VI}(\phi, E, q)$, is the problem of finding an x^* such that $x^* \in E$ and $\langle \phi(x^*) + q, x - x^* \rangle \geq 0$ for all $x \in E$. We write $\mathrm{SOL}(\phi, E, q)$ for the solution set of $\mathrm{VI}(\phi, E, q)$. When E is a closed convex cone, the above variational inequality becomes the *complementarity problem* $\mathrm{CP}(\phi, E, q)$ [4]: Find $x^* \in H$ such that

$$x^* \in E,\ \phi(x^*) + q \in E^*, \text{ and } \langle \phi(x^*) + q, x^* \rangle = 0.$$

Let $\Phi : H \to H$ be a continuous extension of ϕ to all of H. Such an extension exists due to the ($\mathcal{R}^n$ version, hence the H-version of) Tietze extension theorem; since E is a closed convex set, $\phi \circ \Pi_E$ is one such extension. Now consider the 'natural map' [4]

$$\Phi_{(E,q)}(x) := x - \Pi_E\Big(x - \Phi(x) - q\Big)$$

which is defined on all of H. It is well-known, see [4], Prop. 1.5.8, that $\Phi_{(E,q)}(x^*) = 0$ if and only if x^* solves $\mathrm{VI}(\phi, E, q)$.

We now come to variational inequalities corresponding to weakly homogeneous maps. We make the following blanket assumption.

Throughout this paper, K denotes a nonempty closed convex set and C denotes a closed convex cone such that $K \subseteq C \subseteq H$. We let $f : C \to H$ denote a weakly homogeneous map with degree $\gamma > 0$ and let f^∞ denote its leading term. We assume that $f(0) = 0$.

(The assumption $f(0) = 0$ is due to the equivalence of $\mathrm{VI}(f, K, q)$ and $\mathrm{VI}(f - f(0), K, q + f(0))$.)

In order to simplify the notation and avoid repetition, we let

$$\Gamma := \Big\{(K, C, f, f^\infty) : K,\ C,\ f, \text{ and } f^\infty \text{ as above}\Big\}.$$

Given $(K, C, f, f^\infty) \in \Gamma$, *we let* F *and* G *denote, respectively, (some) continuous extensions of* f *and* f^∞ *to* H. We define the corresponding 'natural maps' [4]:

$$F_{(K,q)}(x) := x - \Pi_K\Big(x - F(x) - q\Big) \quad \text{and} \quad G_{K^\infty}(x) := x - \Pi_{K^\infty}\Big(x - G(x)\Big).$$

We observe that

$$F_{(K,q)}(x^*) = 0 \Leftrightarrow x^* \text{ solves } \text{VI}(f, K, q) \text{ and } G_{K^\infty}(x^*) = 0 \Leftrightarrow x^* \text{ solves } \text{CP}(f^\infty, K^\infty, 0).$$

The solution set $\text{SOL}(f, K, q)$ of $\text{VI}(f, K, q)$ is closed. Note that the solution set of $\text{CP}(f^\infty, K^\infty, 0)$, namely, $\text{SOL}(f^\infty, K^\infty, 0)$ always contains zero (as $f^\infty(0) = 0$ from Proposition 2.1) and is invariant under multiplication by positive numbers. Moreover,

$$\text{SOL}(f^\infty, K^\infty, 0) = \{0\} \quad \text{if and only if} \quad [G_{K^\infty}(x) = 0 \Leftrightarrow x = 0]\,. \tag{6}$$

As $\gamma > 0$, we note that if $||x_k|| \to \infty$ and $\bar{x} := \lim\limits_{k\to\infty} \frac{x_k}{||x_k||}$, then

$$\lim_{k\to\infty} \frac{f(x_k)}{||x_k||^\gamma} = \lim_{k\to\infty} \Big[f^\infty\Big(\frac{x_k}{||x_k||}\Big) + \frac{g(x_k)}{||x_k||^\gamma}\Big] = f^\infty(\bar{x}). \tag{7}$$

2.4 Degree Theory

To study the solvability of VIs, in this paper, we employ degree theory. While [17], [18] are very good sources, all necessary concepts/results concerning degree theory that we use can be found in [4], specifically Prop. 2.1.3. Here is a short review. Suppose Ω is a bounded open set in H, $\phi : \overline{\Omega} \to H$ is continuous, and $p \in H$ with $p \notin \phi(\partial\,\Omega)$. Then the topological degree of ϕ over Ω with respect to p is defined; it is an integer and will be denoted by $\deg(\phi, \Omega, p)$. When this degree is nonzero, the equation $\phi(x) = p$ has a solution in Ω.
Now let Ω and ϕ be as above; suppose $x^* \in \Omega$ and the equation $\phi(x) = \phi(x^*)$ has a unique solution in $\overline{\Omega}$, namely x^*. Then, $\deg(\phi, \Omega', \phi(x^*))$ is constant over all bounded open sets Ω' containing x^* and contained in Ω. This common degree is called the *(topological) index* of ϕ at x^*; it will be denoted by $\text{ind}\,(\phi, x^*)$. So, in this setting,

$$\text{ind}\,(\phi, x^*) := \deg\Big(\phi, \Omega', \phi(x^*)\Big)$$

for any bounded open set Ω' containing x^* and contained in Ω. In particular, *if* $\phi : H \to H$ *is a continuous map such that* $\phi(x) = 0 \Leftrightarrow x = 0$, *then, for any bounded open set containing* 0, *we have*

$$\text{ind}\,(\phi, 0) = \deg\Big(\phi, \Omega, \phi(0)\Big) = \deg\Big(\phi, \Omega, 0\Big);$$

moreover, when ϕ *is the identity map,* $\text{ind}\,(\phi, 0) = 1$.
Let $\mathcal{H}(x, t) : H \times [0, 1] \to H$ be continuous (in which case, we say that $\mathcal{H}$ is a homotopy). Suppose that for some bounded open set Ω in H, $0 \notin \mathcal{H}(\partial\Omega, t)$ for all $t \in [0, 1]$. Then, the *homotopy invariance property of degree* says that $\deg\Big(\mathcal{H}(\cdot, t), \Omega, 0\Big)$ is independent of t. In particular, if the zero set

$$\{x : \mathcal{H}(x, t) = 0 \text{ for some } t \in [0, 1]\}$$

is bounded, then for any bounded open set Ω in H that contains this zero set, we have

$$\deg\Big(\mathcal{H}(\cdot,1),\Omega,0\Big)=\deg\Big(\mathcal{H}(\cdot,0),\Omega,0\Big).$$

To discuss the solvability of VI(f,K,q), we consider the equation $F_{(K,q)}(x)=0$ and $\deg\big(F_{(K,q)},\Omega,0\big)$, where F is a continuous extension of f to H. The following result shows that this degree is independent of the extension F. A similar statement holds for CP$(f^\infty,K^\infty,0)$.

Proposition 2.3. *Let $(K,C,f,f^\infty)\in\Gamma$ and $q\in H$. Suppose F and F' are two continuous extensions of f. Let Ω be a bounded open set in H such that $0\notin F_{(K,q)}\,(\partial\Omega)$. Then $0\notin F'_{(K,q)}\,(\partial\Omega)$ and* $\deg\big(F_{(K,q)},\Omega,0\big)=\deg\Big(F'_{(K,q)},\Omega,0\Big)$.

Proof. Suppose, if possible, $0\in F'_{(K,q)}\,(\partial\Omega)$ so that there exists $x\in\partial\Omega$ such that $F'_{(K,q)}(x)=0$. This means that $x\in \text{SOL}(F',K,q)=\text{SOL}(f,K,q)=\text{SOL}(F,K,q)$. Then, $0\in F_{(K,q)}\,(\partial\Omega)$, a contradiction. Thus, $0\notin F'_{(K,q)}\,(\partial\Omega)$. Now, consider the homotopy

$$\mathcal{H}(x,t):=x-\Pi_K\Big(x-\Big[tF(x)+(1-t)F'(x)+q\Big]\Big)\ \text{for}\ x\in\overline{\Omega},\ t\in[0,1].$$

Note that for any $t\in[0,1]$, $tF'+(1-t)F$ is a continuous extension of f. Then, from what has been proved, $0\notin H\,(\partial\Omega,t)$ for all $t\in[0,1]$. Hence, from the homotopy invariance property of the degree,

$$\deg\big(F_{(K,q)},\Omega,0\big)=\deg\big(\mathcal{H}(\cdot,1),\Omega,0\big)=\deg\big(\mathcal{H}(\cdot,0),\Omega,0\big)=\deg\Big(F'_{(K,q)},\Omega,0\Big).$$

□

3 Boundedness of solution sets

In the proofs of many of our results, we are required to show that a certain collection of sets are uniformly bounded. To do this, we employ the so-called *normalization argument*, where a certain sequence of vectors (with their norms going to infinity) is normalized to yield a unit vector that violates a given criteria. We illustrate this in the following proposition.

Proposition 3.1. *Let $(K,C,f,f^\infty)\in\Gamma$. If $\text{SOL}(f^\infty,K^\infty,0)=\{0\}$, then for any bounded set B in H, $\bigcup_{q\in B}\text{SOL}(f,K,q)$ is bounded.*

Proof. Assume that $\text{SOL}(f^\infty,K^\infty,0)=\{0\}$ and let B be a bounded set in H. Suppose if possible, $\bigcup_{q\in B}\text{SOL}(f,K,q)$ is unbounded. Then, there exist sequences q_k in B and $x_k\in\text{SOL}(f,K,q_k)$ such that $||x_k||\to\infty$. Now, for all k, $x_k\in K$, and

$$\langle f(x_k)+q_k,x-x_k\rangle\geq 0\ \forall\, x\in K.$$

By dividing the above relation by $\|x_k\|^{\gamma+1}$ and noting that $x_0+\|x_k\|u\in K$ for every $u\in K^\infty$ with (fixed) $x_0\in K$, we obtain,

$$\left\langle\frac{f(x_k)+q_k}{\|x_k\|^\gamma},u+\frac{x_0-x_k}{\|x_k\|}\right\rangle\geq 0\ \forall\, u\in K^\infty.$$

Fixing x_0, letting $k \to \infty$ and assuming (without loss of generality) $\lim_{k\to\infty} \frac{x_k}{||x_k||} = \bar{x}$, we get (from (7)),

$$\langle f^\infty(\bar{x}), u - \bar{x} \rangle \geq 0, \ \forall\, u \in K^\infty.$$

Because of (2), $\bar{x} \in K^\infty$. Hence, $\bar{x} \in \mathrm{SOL}(f^\infty, K^\infty, 0)$. As $||\bar{x}|| = 1$, we reach a contradiction to our assumption. The conclusion follows. ☐

Remarks 1. *When* $\mathrm{SOL}(f^\infty, K^\infty, 0) = \{0\}$*, the solution set* $\mathrm{SOL}(f, K, q)$ *is compact for any* q *(but may be empty).*

4 The main result

We now present our main result. Here, the solvability of $\mathrm{VI}(f, K, q)$ is studied via a related recession map/cone complementarity problem. Results of this type have been previously described for affine variational inequalities and polynomial complementarity problems, see below for more explanation.

Theorem 4.1. *Let* $(K, C, f, f^\infty) \in \Gamma$*. Suppose the following conditions hold:*

(a) $\mathrm{SOL}(f^\infty, K^\infty, 0) = \{0\}$ *and*

(b) $\mathrm{ind}(G_{K^\infty}, 0) \neq 0$.

Then, for all $q \in H$*,* $\mathrm{VI}(f, K, q)$ *and* $\mathrm{CP}(f, K^\infty, q)$ *have nonempty compact solution sets.*

Note: From (6), condition (a) implies that $x = 0$ is the only solution of the equation $G_{K^\infty}(x) = 0$. Hence $\mathrm{ind}(G_{K^\infty}, 0)$ is well defined. Condition (b) stipulates that this number is nonzero. Also, from Proposition 2.3, $\mathrm{ind}\,(G_{K^\infty}, 0)$ is independent of the extension G of f^∞.

Proof. We fix a q and show that $\mathrm{VI}(f, K, q)$ has a nonempty compact solution set. By replacing K by K^∞ we see that $\mathrm{CP}(f, K^\infty, q)$ has a nonempty compact solution set.
For $x \in H$ and $t \in [0, 1]$, let

$$\mathcal{H}(x, t) := x - \Pi_{\mathcal{K}(t)}\Big(x - \Big\{(1-t)G(x) + t[F(x) + q]\Big\}\Big),$$

where $\mathcal{K}(t) := tK + K^\infty$. By an application of Proposition 2.2, we see that $\mathcal{H}(x, t)$ is jointly continuous. Thus, $\mathcal{H}(x, t)$ defines a homotopy between $\mathcal{H}(x, 0) = G_{K^\infty}(x)$ and $\mathcal{H}(x, 1) = F_{(K,q)}(x)$. We now show by a normalization argument (as in the previous proposition) that the set

$$Z := \{x : \mathcal{H}(x, t) = 0 \ \text{ for some } t \in [0, 1]\},$$

is bounded. Assume if possible that Z is unbounded. Then, there exist sequences t_k in $[0, 1]$, x_k in H such that $\mathcal{H}(x_k, t_k) = 0$ for all k and $||x_k|| \to \infty$. We have $x_k \in \mathcal{K}(t_k) \subseteq C$, $G(x_k) = f^\infty(x_k)$, $F(x_k) = f(x_k)$, and

$$\Big\langle (1 - t_k) f^\infty(x_k) + t_k[f(x_k) + q], x - x_k \Big\rangle \geq 0 \ \forall\, x \in \mathcal{K}(t_k).$$

By dividing the above relation by $\|x_k\|^{\gamma+1}$ and noting that $t_k x_0 + \|x_k\| u \in \mathcal{K}(t_k)$ for every $u \in K^\infty$ with (fixed) $x_0 \in K$, we obtain,

$$\Big\langle \frac{(1 - t_k) f^\infty(x_k) + t_k[f(x_k) + q]}{\|x_k\|^\gamma}, u + \frac{t_k x_0 - x_k}{\|x_k\|} \Big\rangle \geq 0 \ \forall\, u \in K^\infty.$$

Letting $k \to \infty$ and assuming (without loss of generality) $\lim_{k\to\infty} t_k = \bar{t}$ and $\lim_{k\to\infty} \frac{x_k}{||x_k||} = \bar{x}$, we get (from (7)),

$$\langle f^\infty(\bar{x}), u - \bar{x} \rangle \geq 0, \ \forall\, u \in K^\infty. \tag{8}$$

Now, $x_k \in \mathcal{K}(t_k) = t_k\, K + K^\infty$ for all k. If $t_k = 0$ for infinitely many k, then $x_k \in K^\infty$ for all such k, in which case, $\bar{x} = \lim_{k\to\infty} \frac{x_k}{||x_k||} \in K^\infty$ (as K^∞ is a closed cone). On the other hand, if t_k is positive for infinitely many k, then $x_k \in \mathcal{K}(t_k) = t_k\, K + K^\infty = t_k\, K$ for all such k. Writing $x_k = t_k y_k$ for $y_k \in K$, we see that $||y_k|| = \frac{||x_k||}{t_k} \to \infty$ (whether $\bar{t}$ is zero or not). From (2),

$$\bar{x} = \lim_{k\to\infty} \frac{x_k}{||x_k||} = \lim_{k\to\infty} \frac{y_k}{||y_k||} \in K^\infty$$

and from (8),

$$\bar{x} \in \mathrm{SOL}(f^\infty, K^\infty, 0).$$

Since $\|\bar{x}\| = 1$, this contradicts (a). We conclude Z is bounded. Now, let Ω be a bounded open set in H that contains Z. Then, by the homotopy invariance property of degree and (6), we have

$$\deg\Big(\mathcal{H}(\cdot, 1), \Omega, 0\Big) = \deg\Big(\mathcal{H}(\cdot, 0), \Omega, 0\Big) = \mathrm{ind}\Big(G_{K^\infty}, 0\Big) \neq 0.$$

This means that the zero set of $\mathcal{H}(\cdot, 1)$ (being a subset of Z) is nonempty and bounded; hence $\mathrm{VI}(f, K, q)$ has a nonempty bounded solution set. From the closedness of $\mathrm{SOL}(f, K, q)$, we see that the solution set is also compact. □

We now consider two special cases. First consider $\mathrm{VI}(f, K, q)$, where $H = \mathcal{R}^n$, f is linear, and K is polyhedral. We let $f(x) = Mx$, where M is a matrix. Then, $\mathrm{VI}(f, K, q)$, denoted by $\mathrm{AVI}(M, K, q)$, is called an *affine variational inequality.* In this setting,

$$F(x) = G(x) = f(x) = f^\infty(x) = Mx,$$

and, using our previous notation,

$$F_{(K,q)}(x) = x - \Pi_K(x - Mx - q) \quad \text{and} \quad G_{K^\infty}(x) = x - \Pi_{K^\infty}(x - Mx).$$

Because $F_{(K,q)}$ is a piecewise affine map [7], its recession part is ([7], Example 6)

$$\lim_{\lambda\to\infty} \frac{F_{(K,q)}(\lambda x)}{\lambda} := x - \Pi_{K^\infty}(x - Mx) = G_{K^\infty}(x).$$

It has been shown in [7], Theorem 4.2, that a piecewise affine map is surjective if its recession part vanishes only at the origin and has nonzero topological index. In our setting, this yields: If $G_{K^\infty}(x) = 0 \Rightarrow x = 0$ and $\mathrm{ind}(G_{K^\infty}, 0) \neq 0$, then the equation $F_{(K,q)}(x) = 0$ has a nonempty compact solution set, i.e., $\mathrm{AVI}(M, K, q)$ has a nonempty compact solution set. Thus, our result above recovers the AVI result.

Next, consider a polynomial complementarity problem $\mathrm{PCP}(f, q)$, which is $\mathrm{VI}(f, K, q)$ where $H = \mathcal{R}^n$, f is a polynomial map, and $K = \mathcal{R}^n_+$. In this setting, f^∞ is the leading term of f, $K^\infty = \mathcal{R}^n_+$, and $G_{K^\infty}(x) = x - \Pi_{\mathcal{R}^n_+}(x - f^\infty(x)) = \min\{x, f^\infty(x)\}$. In this special case, the above theorem reduces to Theorem 3.1 in [8].

Remarks 2. One may ask if the above result is of 'coercive type' and hence could be deduced from known results. Recall that a continuous map $\phi : K \to H$ is said to be "coercive" on K [4] if there is an $x_0 \in K$ and $\zeta \geq 0$ and a $c > 0$ such that

$$\langle \phi(x), x - x_0 \rangle \geq c||x||^{\zeta}$$

for all sufficiently large x in K. For such a map, it is known that $\deg(\Phi_K, \Omega, 0) = 1$ for some bounded open set Ω and $\mathrm{VI}(\phi, K, 0)$ has a nonempty compact solution set, see [4], Propositions 2.2.3 and 2.2.7. Our conditions (a) and (b) in the above theorem, generally, do not imply this coercivity property. For example, let $K = K^{\infty} = \mathcal{R}^2_+$ and $f(x) = Ax$, where A is the 2×2 matrix given by

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}.$$

This is an **N**-matrix of first category (which means that all principle minors of N are negative and A has some nonnegative entries). Based on the results of Kojima and Saigal [14], it can be verified that conditions (a) and (b) hold with $\mathrm{ind}(G_{K^{\infty}}, 0) = -1$. Clearly, this f is not coercive.

Remarks 3. Suppose, in the above theorem, we let $C = K = K^{\infty} = H$. As $\Pi_K(x) = \Pi_{K^{\infty}}(x) = x$ for all x, conditions (a) and (b) reduce to: (i) $f^{\infty}(x) = 0 \Rightarrow x = 0$ and (ii) $\mathrm{ind}(f^{\infty}, 0) \neq 0$. The conclusion(s) says that the equation $f(x) + q = 0$ has a solution for all q. In other words, f is surjective when (i) and (ii) hold. A result of this type is, perhaps, known in the degree theory literature.

We now provide a simple example to illustrate Theorem 4.1.

Example 1. In $\mathcal{R}^2$, consider the closed convex set $K := \{(t, s) : t, s > 0 \text{ and } ts \geq 1\}$. Then $K^{\infty} = \mathcal{R}^2_+$ (the nonnegative orthant in $\mathcal{R}^2$). Let A be a 2×2 real matrix which is an $\mathbf{R}_0$-matrix with $\deg(A) \neq 0$. (This simply means that for $\theta(x) := x - \Pi_{\mathcal{R}^2_+}(x - Ax) = \min\{x, Ax\}$, $\theta(x) = 0 \Leftrightarrow x = 0$ and $\deg(A) := \mathrm{ind}(\theta, 0) \neq 0$. For example, A could be the identity matrix, or an **R**-matrix [2], or the matrix given in the Remark 2 above.) On $C := \mathcal{R}^2_+$, let $g(x) = \sqrt{x}$ or $g(x) = \sin x$ (which are defined componentwise), and $f(x) := Ax + g(x)$. Then, f is weakly homogeneous and satisfies the conditions of the above theorem. We conclude that for all q, $\mathrm{VI}(f, K, q)$ has a nonempty compact solution set.

5 A generalization of Karamardian's theorem

A well-known result of Karamardian [13] deals with a proper cone C and a nonconstant positively homogeneous continuous map $h : C \to H$. It asserts that, if $\mathrm{CP}(h, C, 0)$ and $\mathrm{CP}(h, C, d)$ have (only) trivial (that is, zero) solutions for some $d \in \mathrm{int}(C^*)$, then $\mathrm{CP}(h, C, q)$ has nonempty compact solution set for all q. Below, we generalize it to weakly homogeneous variational inequalities.

Theorem 5.1. *Let $(K, C, f, f^{\infty}) \in \Gamma$ with K^{∞} pointed. Suppose there is a vector $d \in \mathrm{int}\left((K^{\infty})^*\right)$ such that one of the following conditions holds:*

(a) $\mathrm{SOL}(f^{\infty}, K^{\infty}, 0) = \{0\} = \mathrm{SOL}(f^{\infty}, K^{\infty}, d)$.

(b) $\mathrm{SOL}(f^{\infty}, K^{\infty}, 0) = \{0\} = \mathrm{SOL}(f, K^{\infty}, d)$.

Then, $\text{ind}\,(G_{K^\infty}, 0) = 1$. *Hence, for all* $q \in H$*,* $\text{VI}(f, K, q)$ *and* $\text{CP}(f, K^\infty, q)$ *have nonempty compact solution sets.*

Proof. In order to handle both cases (a) and (b) together, we let ϕ denote f^∞ when (a) holds or f when (b) holds. Let, correspondingly, Φ denote G or F (which are continuous extensions of f^∞ and f, respectively). In either case, for any $t \in [0, 1]$, the leading term of the weakly homogeneous map $(1-t)f^\infty(x) + t[\phi(x) + d]$ is $f^\infty(x)$. Now consider the homotopy

$$\mathcal{H}(x,t) := x - \Pi_{K^\infty}\Big(x - \Big\{(1-t)G(x) + t[\Phi(x) + d]\Big\}\Big).$$

Since $\text{SOL}(f^\infty, K^\infty, 0) = \{0\}$, by a normalization argument (as in the Proof of Theorem 4.1), we see that the zero set

$$\{x : \mathcal{H}(x,t) = 0 \text{ for some}\, t \in [0,1]\}$$

is contained in a bounded open set Ω. Since $\mathcal{H}(\cdot, 0) = G_{K^\infty}$ and $\mathcal{H}(\cdot, 1) = \Phi_{(K^\infty,d)}$ are homotopic, by the homotopy invariance property of degree,

$$\text{ind}\,(G_{K^\infty}, 0) = \deg\,(G_{K^\infty}, \Omega, 0) = \deg\,\big(\Phi_{(K^\infty,d)}, \Omega, 0\big) = \text{ind}(\Phi_{(K^\infty,d)}, 0),$$

where the last equality holds due to $\text{SOL}(\phi, K^\infty, d) = \{0\}$ (which comes from (a) when $\phi = f^\infty$ and (b) when $\phi = f$). Now, when x is close to zero, $x - \Phi(x) - d$ is close to $0 - \phi(0) - d = -d \in -\text{int}((K^\infty)^*)$ (note that $\phi(0) = 0$ as $f^\infty(0) = 0 = f(0)$). Hence, by (4), for all x close to zero, $\Pi_{K^\infty}(x - \Phi(x) - d) = 0$. This means that $\Phi_{(K^\infty,d)}(x) = x$ near zero; hence $\text{ind}(\Phi_{(K^\infty,d)}, 0) = 1$. This implies that $\text{ind}\,(G_{K^\infty}, 0) = 1$. From Theorem 4.1, we get the stated conclusion. ☐

Remarks 4. The above result, under condition (a), even strengthens Karamardian's theorem: While Karamardian's theorem says that $\text{CP}(f^\infty, K^\infty, q)$ has a nonempty compact solution set for all q, our result gives a stronger conclusion that $\text{CP}(f, K^\infty, q)$ has a nonempty compact solution set for all q.

6 Copositivity results

In this section, we consider copositive maps. Given a set E in H, we say that a map $\phi : E \to H$ is *copositive (respectively, strictly copositive)* on E, if $\langle \phi(x), x\rangle \geq 0$ (respectively, > 0) for all $0 \neq x \in E$.

Theorem 6.1. *Let* $(K, C, f, f^\infty) \in \Gamma$*. Suppose one of the following conditions holds:*

- (a) $\text{SOL}(f^\infty, K^\infty, 0) = \{0\}$ *and* f^∞ *is copositive on* K^∞*.*
- (b) f^∞ *is strictly copositive on* K^∞*.*

Then, $\text{ind}\,(G_{K^\infty}, 0) = 1$. *Hence, for all* $q \in H$*,* $\text{VI}(f, K, q)$ *and* $\text{CP}(f, K^\infty, q)$ *have nonempty compact solution sets.*

Proof. As $f^\infty(0) = 0$, $(b) \Rightarrow (a)$. Hence, it is enough to prove the result under (a). Consider the homotopy

$$\mathcal{H}(x,t) := x - \Pi_{K^\infty}\Big(x - \Big\{(1-t)G(x) + tx\Big\}\Big).$$

Suppose that $\mathcal{H}(x,t)=0$ for some x and t. Then,

$$x \in K^\infty,\ (1-t)f^\infty(x)+tx \in (K^\infty)^*,\ \text{and}\ \langle (1-t)f^\infty(x)+tx, x\rangle = 0.$$

In particular, $(1-t)\langle f^\infty(x),x\rangle + t||x||^2 = 0$. Since f^∞ is copositive on K^∞, we have $\langle f^\infty(x),x\rangle \geq 0$ and so $t||x||^2=0$. If $t=0$, then $x\in K^\infty$, $f^\infty(x)\in (K^\infty)^*$, and $\langle f^\infty(x),x\rangle = 0$ and so $x \in \mathrm{SOL}(f^\infty,K^\infty,0)=\{0\}$. If $t\neq 0$, then $t||x||^2=0$ implies $x=0$. Hence, $\mathcal{H}(x,t)=0 \Rightarrow x=0$. Let Ω be any bounded open set containing 0. Since $\mathcal{H}(\cdot,0)=G_{K^\infty}$ and $\mathcal{H}(\cdot,1)$ (=identity map) are homotopic on Ω, by the homotopy invariance property of degree,

$$\mathrm{ind}\,(G_{K^\infty},0)=1$$

(as the degree of the identity map at zero is one). Now the stated conclusion follows from Theorem 4.1.
□

All the previous results require that $\mathrm{SOL}(f^\infty,K^\infty,0)=\{0\}$. In the next result we drop this assumption, but require that f be copositive on K. A result of this type for copositive multivalued maps on $\mathcal{R}^n_+$ appears in [9].

Theorem 6.2. *Let $(K,C,f,f^\infty)\in\Gamma$ such that $0\in K$ and $\mathrm{int}\,(K^*)\neq\emptyset$. Assume further that f is copositive on K. If $\mathcal{S}:=\mathrm{SOL}(f^\infty,K^\infty,0)$ and $q\in\mathrm{int}\,(\mathcal{S}^*)$, then $\mathrm{VI}(f,K,q)$ has a nonempty compact solution set.*

Proof. We fix a $d\in\mathrm{int}\,(K^*)$ and consider the homotopy

$$\mathcal{H}(x,t):=x-\Pi_{\mathcal{K}(t)}\Big(x-[F(x)+(1-t)d+tq]\Big),$$

where $\mathcal{K}(t):=tK+K^\infty$. Then,

$$\mathcal{H}(x,0)=x-\Pi_{K^\infty}\Big(x-[F(x)+d]\Big)\quad\text{and}\quad\mathcal{H}(x,1)=x-\Pi_K\Big(x-[F(x)+q]\Big).$$

Note that $\mathcal{K}(t)\subseteq K$ for all $t\in[0,1]$ because $0\in K$ and K is convex. We claim that the zero sets of $\mathcal{H}(x,t)$ are uniformly bounded. If not, we can find sequences $t_k\in[0,1]$ and $0\neq x_k\in\mathcal{K}(t_k)$ such that $||x_k||\to\infty$ and $\mathcal{H}(x_k,t_k)=0$ for all k. Since $F=f$ on K, we have, for all k, $x_k\in\mathcal{K}(t_k)$, and

$$\langle f(x_k)+(1-t_k)\,d+t_k\,q, x-x_k\rangle \geq 0\ \forall\, x\in\mathcal{K}(t_k). \tag{9}$$

Without loss of generality, we may assume that $\lim\limits_{k\to\infty}\frac{x_k}{\|x_k\|}=\bar{x}$. A normalization argument (as in the proof Theorem 4.1) shows that $\bar{x}\in\mathrm{SOL}(f^\infty,K^\infty,0)$. Thus, $\bar{x}\in\mathcal{S}$ and $||\bar{x}||=1$. By putting $x=0$ in (9), we obtain

$$\langle f(x_k),x_k\rangle+(1-t_k)\langle d,x_k\rangle+t_k\langle q,x_k\rangle\leq 0\ \forall\, k.$$

Since $\langle f(x_k),x_k\rangle\geq 0$ and $\langle d,x_k\rangle>0$ (recall $d\in\mathrm{int}(K^*)$ and $0\neq x_k\in K$), we must have $t_k>0$ and $\langle q,x_k\rangle\leq 0$. This yields $\langle q,\bar{x}\rangle\leq 0$. Since $q\in\mathrm{int}\,(\mathcal{S}^*)$, we reach a contradiction, see (1). Hence, the zero sets of $\mathcal{H}(x,t)$ are uniformly bounded. In particular, $\mathrm{SOL}(f,K,q)$ (which is the zero set of $\mathcal{H}(x,1)$) is bounded. Let Ω be a bounded open set in H that contains all these zero sets. By the homotopy invariance property,

$$\deg\Big(\mathcal{H}(\cdot,1),\Omega,0\Big)=\deg\Big(\mathcal{H}(\cdot,0),\Omega,0\Big).$$

Note that $K^\infty \subseteq K$ because $0 \in K$. Then, $d \in \text{int}\,(K^*) \subseteq \text{int}\,\big((K^\infty)^*\big)$. Hence, as f is copositive and $f(0) = 0$ (by our blanket assumption), we have $\text{SOL}(f, K^\infty, d) = \{0\}$. Now, if x is close to zero, $x - [F(x) + d]$ is close to $-d$. Since $d \in \text{int}\,\big((K^\infty)^*\big)$, by (4), for all x close to zero, $\mathcal{H}(x, 0) = x - \Pi_{K^\infty}\Big(x - [F(x) + d]\Big) = x$. This means that $\deg\,(\mathcal{H}(\cdot, 0), \Omega, 0) = 1$. Hence, $\deg\,(\mathcal{H}(\cdot, 1), \Omega, 0) = 1$ and so, $\mathcal{H}(\cdot, 1)$ has a zero in Ω, that is, $\text{SOL}(f, K, q) \neq \emptyset$. As we have already shown that this set is bounded, we conclude that $\text{SOL}(f, K, q)$ is nonempty and compact. □

Corollary 6.3. *Let $(K, C, f, f^\infty) \in \Gamma$ such that $0 \in K$ and C is pointed. Assume further that f is copositive on C. If $\mathcal{S} := \text{SOL}(f^\infty, K^\infty, 0)$ and $q \in \text{int}\,(\mathcal{S}^*)$, then $\text{VI}(f, K, q)$ and $\text{CP}(f, C, q)$ have nonempty compact solution sets.*

Proof. As C is pointed, C^* has nonempty interior. Since $K \subseteq C \Rightarrow C^* \subseteq K^*$, we see that K^* has nonempty interior. Also, f is copositive on K. If $q \in \text{int}\,(\mathcal{S}^*)$, the conditions of the above theorem are satisfied. Hence, $\text{VI}(f, K, q)$ has nonempty compact solution set. Now, applying the above theorem with C in place of K, we see that $\text{VI}(f, C, q)$, that is, $\text{CP}(f, C, q)$ has nonempty compact solution set. □

7 A uniqueness result

Theorem 7.1. *Suppose C is a pointed closed convex cone and $f : C \to H$ is weakly homogeneous of positive degree with leading term f^∞. Suppose that $\text{SOL}(f^\infty, C, 0) = \{0\}$. Then the following are equivalent:*

- *(a) $\text{CP}(f, C, q)$ has a unique solution for every $q \in H$.*
- *(b) $\text{CP}(f, C, q)$ has at most one solution for every $q \in H$.*

Proof. Clearly, $(a) \Rightarrow (b)$. Suppose (b) holds. By our blanket assumption, $f(0) = 0$. Then for any $d \in \text{int}(C^*)$, $\{0\} \subseteq \text{SOL}(f, C, d)$, and by (b), $\text{SOL}(f, C, d) = \{0\}$. Since (by assumption) $\text{SOL}(f^\infty, C, 0) = \{0\}$, by Theorem 5.1 (applied to $C = K = K^\infty$), for every q, $\text{CP}(f, C, q)$ has a solution; this solution is unique by (b). Thus (a) holds. □

Remark 5. It is not clear if the above result holds without the assumption that $\text{SOL}(f^\infty, C, 0) = \{0\}$.

8 Solvability of nonlinear equations over cones

As an application of our results, we now describe a method of proving the solvability of certain equations over cones. To motivate, first we consider linear equations in matrix variables. Let $\mathcal{S}^n$ and $\mathcal{H}^n$ denote, respectively, the spaces of all $n \times n$ real symmetric matrices and $n \times n$ complex Hermitian matrices. With the inner product given by $\langle X, Y \rangle := \text{tr}(XY)$, both are real Hilbert spaces. $\mathcal{S}^n_+$ and $\mathcal{H}^n_+$ denote, respectively, the closed convex cones of positive semidefinite matrices in $\mathcal{S}^n$ and $\mathcal{H}^n$. (Either of these cones will be called a semidefinite cone.) We write $X \succeq 0$ for positive semidefinite matrices and $X \succ 0$ for positive definite matrices. Both $\mathcal{S}^n_+$ and $\mathcal{H}^n_+$ are self-dual cones and the following implication holds [10]:

$$X \succeq 0,\ Y \succeq 0 \text{ and } \langle X, Y \rangle = 0 \Rightarrow XY = 0. \tag{10}$$

(1) Given a matrix $A \in \mathcal{R}^{n\times n}$, the Lyapunov transformation $L_A : \mathcal{S}^n \to \mathcal{S}^n$ is defined by $L_A(X) := AX + XA^T$. This appears in continuous dynamical systems. It is well known that for any/some positive definite Q, the equation $L_A(X) = Q$ has a positive definite solution X if and only if A is positive stable, that is, all the eigenvalues of A have positive real parts. One interesting feature of L_A is that it satisfies the following implication ([11], Example 2):

$$X \succeq 0,\ Y \succeq 0 \text{ and } \langle X, Y\rangle = 0 \Rightarrow \langle L_A(X), Y\rangle = 0.$$

(2) Given a matrix $A \in \mathcal{R}^{n\times n}$, the Stein transformation $S_A : \mathcal{S}^n \to \mathcal{S}^n$ is defined by $S_A(X) := X - AXA^T$. This appears in discrete dynamical systems. Similar to the Lyapunov transformation, the following statement holds: For any/some positive definite Q, the equation $S_A(X) = Q$ has a positive definite solution X if and only if A is Schur stable, that is, all the eigenvalues of A lie in the open unit disc of the xy-plane. We also have the following implication ([11], Example 3) :

$$X \succeq 0,\ Y \succeq 0 \text{ and } \langle X, Y\rangle = 0 \Rightarrow \langle S_A(X), Y\rangle \leq 0.$$

(3) Suppose C is a proper cone in H, $L : H \to H$ is linear and satisfies the **Z**-property:

$$x \in C, y \in C^*,\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

It has been shown in [11], Theorems 6 and 7, that for every/some $q \in \text{int}(C)$, the equation $L(x) = q$ has a solution in $\text{int}(C)$ if and only if L is positive stable.

In the result below, we consider an extension of the **Z**-property for nonlinear maps.

Proposition 8.1. *Suppose C is a closed convex cone and $f : C \to H$ satisfies:*

$$x \in C,\ y \in C^*, \text{ and } \langle x, y\rangle = 0 \Rightarrow \langle f(x), y\rangle \leq 0. \tag{11}$$

Then, the following statements hold:

- $q \in C,\ x^* \in \text{SOL}(f, C, -q) \Rightarrow f(x^*) = q.$
- $x^* \in C,\ f(x^*) \in \text{int}(C) \Rightarrow x^* \in \text{int}(C).$

Proof. Assume $q \in C$ and $x^* \in \text{SOL}(f, C, -q)$. Letting $y^* := f(x^*) - q$, we see that $x^* \in C$, $y^* \in C^*$ and $\langle x^*, y^*\rangle = 0$. By the condition imposed on f, we have $\langle f(x^*), y^*\rangle \leq 0$. From this we get $\langle q + y^*, y^*\rangle \leq 0$. As $q \in C$ (which implies $\langle q, y^*\rangle \geq 0$), this leads to $y^* = 0$ and to $f(x^*) = q$.
To see the second item, suppose $x^* \in C$ and $q := f(x^*) \in \text{int}(C)$. Suppose, if possible, $x^* \notin \text{int}(C)$. Then, x^* lies on the boundary of C. By the well-known separation theorem (also called supporting hyperplane theorem, see [19, Theorem 11.6]), there exists a nonzero $y^* \in C^*$ such that $\langle x^*, y^*\rangle = 0$. By (11),

$$\langle q, y^*\rangle = \langle f(x^*), y^*\rangle \leq 0.$$

On the other hand, since $q \in \text{int}(C)$ and $0 \neq y^* \in C^*$, we must have $\langle q, y^*\rangle > 0$, yielding a contradiction. Hence, $x^* \in \text{int}(C)$. ☐

Remarks 6. The set of all maps satisfying (11) is closed under nonnegative linear combinations. It also contains maps of the form $-\phi$, where $\phi(C) \subseteq C$.

Corollary 8.2. *Suppose conditions of Theorem 4.1 (or Theorem 5.1 or Theorem 6.1) hold with $K = K^\infty = C$ and f satisfies (11). Then for all q in C (in $\mathrm{int}(C)$), the equation $f(x) = q$ has a solution in C (respectively, in $\mathrm{int}(C)$).*

Proof. $\mathrm{CP}(f, C, q)$ has a solution for all q; in particular, for any $q \in C$, $\mathrm{CP}(f, C, -q)$ has a solution. This solution, by the above proposition, solves the equation $f(x) = q$ in C. If $q \in \mathrm{int}(C)$, then such an x belongs to $\mathrm{int}(C)$. □

We now illustrate the above corollary by some examples.

Example 2. On $H = \mathcal{H}^n$, with $K = C = \mathcal{H}^n_+$, let

$$f(X) := X - \sum_1^N A_i X^{\delta_i} A_i^*,$$

where A_i is an $n \times n$ complex matrix and $0 < \delta_i < 1$ for all $i = 1, 2, \ldots, N$. Then, for every $Q \in \mathcal{H}^n_+$, the equation $f(X) = Q$ has a solution in $\mathcal{H}^n_+$. Moreover, if Q is positive definite, then so is X. (A similar result holds for $H = \mathcal{S}^n$.) To see these statements, we first observe that f is a sum of homogeneous maps, hence weakly homogeneous. Also, as $0 < \delta_i < 1$, the leading term f^∞ is the identity map and $g := f - f^\infty$ satisfies $-g(\mathcal{H}^n_+) \subseteq \mathcal{H}^n_+$. We see that f satisfies condition (11) in Proposition 8.1. Because f^∞ is the identity map, it is strictly copositive on C, hence conditions of Theorem 6.1 hold. From the above corollary, we conclude that for every Q in $\mathcal{H}^n_+$, there is a solution X of $f(X) = Q$ in $\mathcal{H}^n_+$. If Q is positive definite, then $X = \sum_1^N A_i X^{\delta_i} A_i^* + Q$ is also positive definite.

Note: In [16], Lim considers the case $0 < |\delta_i| < 1$. Using the Hilbert metric on $\mathrm{int}(\mathcal{H}^n_+)$ and the contraction principle, he proves the existence and uniqueness of solutions. See [3] for another fixed point proof.

Example 3. On $H = \mathcal{S}^n$, with $K = C = \mathcal{S}^n_+$, let

$$f(X) = X + \alpha \sin X,$$

where $\alpha \in \mathcal{R}$. Since $\sin t$ is bounded over $\mathcal{R}$, we see that $\sin X$ is bounded, hence $\sin X = o(||X||)$. With $f^\infty(X) = X$, f is weakly homogeneous on $\mathcal{S}^n_+$. By an application of Theorem 6.1, we conclude that $\mathrm{CP}(f, \mathcal{S}^n_+, Q)$ has a solution for all Q. Now, to verify the property (11) in Proposition 8.1. Let $X \succeq 0$, $Y \succeq 0$ and $\langle X, Y\rangle = 0$. From (10), $XY = 0$. From the series expansion of $\sin X$, we verify that $(\sin X)Y = 0$. Thus, $\langle \sin X, Y\rangle = 0$, that is, $\langle f(X), Y\rangle = 0$. Since we have verified all conditions in the above corollary, it follows that for every $Q \in \mathcal{S}^n_+$, $f(X) = Q$ has a semidefinite solution. We note that this semidefinite solution is positive definite if Q is positive definite.

Example 4. On $H = \mathcal{S}^n$, with $C = K = \mathcal{S}^n_+$, we consider the Riccati equation $XAX + BX + XB^T = Q$, where A is positive definite, Q is positive semidefinite, and $B \in \mathcal{R}^{n\times n}$. With homogeneous maps $f^\infty(X) = XAX$, $g(X) = BX + XB^T$, we see that $f = f^\infty + g$ is weakly homogeneous. If $X \succeq 0$, $Y \succeq 0$ and $\langle X, Y\rangle = 0$, we have $XY = 0$ and hence, $\langle f(X), Y\rangle = 0$. (Note: $g(X) = L_B(X)$ is a Lyapunov transformation, see Item (1) at the beginning of this Section.) We verify conditions of Theorem 6.1. Suppose $X \in \mathrm{SOL}(f^\infty, \mathcal{S}^n_+, 0)$ so that $X \succeq 0$, $XAX \succeq 0$ and $\langle X, XAX\rangle = 0$. An application of (10) gives $X\,XAX = 0$ and consequently,

$X^2AX^2 = 0$. Since A is positive definite, we conclude that $X = 0$. The verification of copositivity of f^∞ is easy: Since A is positive definite, $XAX \succeq 0$ for any X. Thus, for all $X \in \mathcal{S}^n_+$, $\langle XAX, X\rangle \geq 0$. (We have actually proved that f^∞ is strictly copositive on $\mathcal{S}^n_+$.)
Note: If $B = 0$ and Q is positive definite, the equation $XAX = Q$ has a positive definite solution X which is called the geometric mean of A^{-1} and Q [15].

Our next section is devoted to symmetric word equations.

9 Solvability of symmetric word equations in Euclidean Jordan algebras

Given letters X and $A_1, \ldots, A_m$, a *symmetric word* $W(X, A_1, \ldots, A_m)$ is a juxtaposition of letters A_i that alternate with positive powers of X in a symmetric way, that is, it is of the form

$$X^{r_m}A_m \cdots X^{r_2}A_2X^{r_1}A_1X^{r_1}A_2X^{r_2} \cdots A_mX^{r_m},$$

where the exponents $r_1, \ldots, r_m$ are positive real numbers.
A *symmetric word equation* in $\mathcal{S}^n$ (or in $\mathcal{H}^n$) is of the form $W(X, A_1, \ldots, A_m) = Q$, where $A_1, \ldots, A_m$ are positive definite matrices and Q is positive semidefinite. Any positive semidefinite matrix X for which this equation holds is called a solution. A symmetric word equation (corresponding to m and exponents $r_1, \ldots, r_m$) is called *solvable* if there exists a solution for every collection of positive definite matrices $A_1, \ldots, A_m$ and positive semidefinite matrix Q. The solvability of symmetric word equation arises in many situations. In [12], C. Hillar and C. R. Johnson prove that every symmetric word equation is solvable via a fixed point argument. Armstrong and Hillar [1] give an alternate proof based on degree theory and the following lemma:

Lemma 9.1. *([1], Lemma 6.1) Suppose $A_1, \ldots, A_m$ are positive definite matrices. Then, for any positive semidefinite X (either in $\mathcal{S}^n$ or $\mathcal{H}^n$, depending on the context),*

$$Ker(X) = Ker\Big(X^{l_1}A_1X^{l_2}A_2 \cdots X^{l_{k-1}}A_{k-1}X^{l_k}\Big),$$

where 'Ker' denotes the kernel and $l_1, l_2, \ldots, l_k$ are positive real numbers.

Based on this lemma, one can give a complementarity proof of the result of Hillar and Johnson by observing that $f(X) := X^{r_m}A_m \cdots X^{r_2}A_2X^{r_1}A_1X^{r_1}A_2X^{r_2} \cdots A_mX^{r_m}$ is strictly copositive and satisfies (11) on the semidefinite cone. We will use this idea to extend the result of Hillar and Johnson to Euclidean Jordan algebras.
Let $(V, \langle\cdot,\cdot\rangle, \circ)$ be a Euclidean Jordan algebra of rank r and unit element e [6]. Let $V_+ := \{x \circ x : x \in V\}$ denote the symmetric cone of V with V_{++} denoting its interior. We note that

(i) V_+ is a self-dual cone; in particular, $\langle x, y\rangle \geq 0$ for all $x, y \in V_+$.

(ii) When $a \in V_{++}$ and $0 \neq x \in V_+$, $\langle a, x\rangle > 0$.

Given any $a \in V$, we define the corresponding *quadratic representation* by

$$P_a(x) := 2\,a \circ (a \circ x) - a^2 \circ x.$$

Then, a *symmetric word* in V is defined by

$$W(x, a_1, \ldots, a_m) := P_{x^{r_m}} P_{a_m} \cdots P_{x^{r_2}} P_{a_2} P_{x^{r_1}}(a_1),$$

where m is a natural number, the variable x various over V_+, $a_1, \ldots, a_m$ are in V_{++}, and the exponents $r_1, \ldots, r_m$ are positive real numbers. As $P_x(a)$ is quadratic in x, the *homogeneity degree* of $W(x, a_1, \ldots, a_m)$ is

$$\gamma := 2(r_1 + r_2 + \cdots + r_m).$$

In the setting of $V = \mathcal{S}^n$ (or $\mathcal{H}^n$), $P_A(X) = AXA$ and the above word reduces to a symmetric word in $\mathcal{S}^n$ (or $\mathcal{H}^n$).

We collect some relevant properties of these transformations which are either well-known or can be deduced from known results.

Proposition 9.2. *The following statements hold:*

(a) *For any $x \in V$, P_x is a self-adjoint linear operator on V.*

(b) *V_+ is invariant under any quadratic representation, that is, $P_x(V_+) \subseteq V_+$ for any x. Same holds for any composition of quadratic representations. In particular, such a composition is copositive on V_+.*

(c) *If $x \in V_+$ ($x \in V_{++}$), then P_x is positive semidefinite (respectively, positive definite) on V.*

(d) *When $x, y \in V_+$, $\langle x, y\rangle = 0 \Rightarrow P_x(y) = 0$.*

(e) *$P_{x^\alpha} = (P_x)^\alpha$ and $P_x P_{x^\alpha} = P_{x^{\alpha+1}}$ for every $x \in V_+$ and every $\alpha > 0$.*

(f) *$P_x(a)$ is invertible if and only if both x and a are invertible.*

(h) *$P_u(v) = 0 \Rightarrow P_{u^\alpha}(v) = 0$, when $u, v \in V_+$ and $\alpha > 0$.*

Proposition 9.3. *Let $L(x)$ denote a finite linear combination of symmetric words in the variable x (which varies over V_+). Then, the following statements hold:*

(i) $u, v \in V_+$, $\langle u, v\rangle = 0 \Rightarrow \langle L(u), v\rangle = 0$.

(ii) $x \in V_+$, $L(x) \in V_{++} \Rightarrow x \in V_{++}$.

Proof. (i) Suppose $u, v \in V_+$, $\langle u, v\rangle = 0$, and let $P_{x^{r_m}} P_{a_m} \cdots P_{x^{r_2}} P_{a_2} P_{x^{r_1}}(a_1)$ be a term in $L(x)$ (ignoring the scalar factor). As $P_u(v) = 0$ and $P_{u^{r_m}}(v) = 0$ (from Items (d) and (h) in the above proposition), we have

$$\langle P_{u^{r_m}} P_{a_m} \cdots P_{u^{r_2}} P_{a_2} P_{u^{r_1}}\,(a_1), v\rangle = \langle P_{a_m} \cdots P_{u^{r_2}} P_{a_2} P_{u^{r_1}}\,(a_1), P_{u^{r_m}}(v)\rangle = 0.$$

It follows that $\langle L(u), v\rangle = 0$.
From Item (i), we see that L satisfies the condition (11). Item (ii) now follows from Proposition 8.1. $\square$

We now extend the result of Hillar and Johnson as follows.

Theorem 9.4. *Let $a_1, \ldots, a_m$ be in V_{++}, $r_1, \ldots, r_m$ be positive real numbers and $\gamma := 2(r_1 + r_2 + \cdots + r_m)$. Let continuous maps $g_1, g_2 : V_+ \to V$ satisfy the following conditions:*

- *g_1 is a finite linear combination of symmetric words,*
- *$g_2(0) = 0$, $g_2(V_+) \subseteq V_+$, and*
- *$(g_1 - g_2)(x) = o(||x||^\gamma)$ as $||x|| \to \infty$ in V_+.*

Then, for every $q \in V_+$, there exists $x \in V_+$ such that

$$W(x, a_1 \ldots, a_m) + g_1(x) - g_2(x) = q.$$

If $q \in V_{++}$, such an x is in V_{++}.

Proof. We let $f^\infty(x) := W(x, a_1, \ldots, a_n)$, $g := g_1 - g_2$, and $f := f^\infty + g$. Clearly, f is weakly homogeneous on V_+ of degree $\gamma > 0$ and $f(0) = 0$. Letting $H = V$, $C = K = K^\infty = V_+$ (which is a proper cone as it is self-dual), we show that f satisfies condition (b) in Theorem 6.1 and (11). Then an application of Corollary 8.2 yields the stated conclusion.

First, let $x \in V_+$. As $a_1 \in V_{++}$, from Item (b) in Proposition 9.2,

$$f^\infty(x) = \Big(P_{x^{r_m}} P_{a_m} \cdots P_{x^{r_2}} P_{a_2} P_{x^{r_1}}\Big)(a_1) \in V_+$$

hence $\langle f^\infty(x), x\rangle \geq 0$. Thus, f^∞ is copositive on V_+.

Now, let $0 \neq x \in V_+$. We show that $\langle f^\infty(x), x\rangle > 0$. Suppose, on the contrary that $\langle f^\infty(x), x\rangle = 0$. Putting $y := f^\infty(x)$, we see that

$$x \in V_+,\ y \in V_+, \text{ and } \langle x, y\rangle = 0.$$

By Item (d) in Proposition 9.2, $P_x(y) = 0$, that is,

$$P_x\Big(P_{x^{r_m}} P_{a_m} \cdots P_{x^{r_2}} P_{a_2} P_{x^{r_1}}\Big)(a_1) = 0.$$

From (e), this shows that

$$a_1 \in Ker\Big((P_x)^{r_m+1} P_{a_m} \cdots (P_x)^{r_2} P_{a_2} (P_x)^{r_1}\Big).$$

Noting that each P_{a_i} is a positive definite operator on V (by Item (c) in Proposition 9.2), and P_x is positive semidefinite (as $x \in V_+$), we may apply the operator version of Lemma 9.1 and get

$$a_1 \in Ker(P_x).$$

This means that $P_x(a_1) = 0$. Now, with e denoting the unit element of V, we have

$$0 = \langle P_x(a_1), e\rangle = \langle a_1, P_x(e)\rangle = \langle a_1, x^2\rangle.$$

As $a_1 \in V_{++}$ and $x^2 \in V_+$, we see that $x^2 = 0$, that is, $x = 0$. We conclude that f^∞ is strictly copositive on V_+. Hence, conditions of Theorem 6.1 are satisfied.

Now for the verification of (11). Let $u \in V_+$, $v \in V_+$ and $\langle u, v\rangle = 0$. Then,

$$L(x) := W(x, a_1 \ldots, a_m) + g_1(x)$$

is a finite linear combination of symmetric words. By the previous proposition, $\langle L(u), v\rangle = 0$. From this we get,

$$\langle f(u), v\rangle = \langle L(u), v\rangle - \langle g_2(u), v\rangle = -\langle g_2(u), v\rangle \leq 0,$$

as $g_2(u) \in V_+$. This proves that f satisfies (11).
Thus, we have verified all conditions in Corollary 8.2. Hence, for any $q \in V_+$, the equation $f(x) = q$ has a solution x in V_+. Now suppose $q \in V_{++}$. Then, there is an $x \in V_+$ such that $f^\infty(x) + g_1(x) - g_2(x) = q$. This implies that $L(x) = g_2(x) + q$. As $g_2(x) \in V_+$ and $q \in V_{++}$, we must have $L(x) \in V_{++}$. As $x \in V_+$, by Item (ii) in the previous proposition, $x \in V_{++}$. □

Example 5. To illustrate the above result, let $V = \mathcal{S}^n$ and consider positive definite matrices A, B, C, and a positive semidefinite matrix D. Let $f(X) := XAXBXAX + XCX - DX^{\frac{1}{2}}D$. Then, for any positive semidefinite $Q \in \mathcal{S}^n$, the equation $f(X) = Q$ has a positive semidefinite solution. If Q is positive definite, such a solution is positive definite.

Remarks 7. We note that in the above theorem/proof, the condition that g_1 is a finite linear combination of symmetric words was primarily used to get the conclusions of Proposition 9.3 for $L(x) = f^\infty(x) + g_1(x)$. In some settings, e.g., $V = \mathcal{S}^n$ and $g_1(X) := \sin X$, we could get the same conclusions and the above result could be modified.

Concluding Remarks. In this paper, we studied variational inequalities corresponding to weakly homogeneous maps. We showed that under appropriate settings, the study of variational inequality problems could be reduced to that of corresponding recession map/cone complementarity problems. We described a method of solving nonlinear equations over cones. We note that all the results of this paper are applicable to polynomial maps. Since the problem of minimizing a (real valued) polynomial function over a closed convex set is closely related to the variational inequality problem of its gradient, we anticipate that our results will be useful in the study of polynomial optimization.

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