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GAME THEORY AND OPTIMIZATION

Some characterizations of cone preserving Z-transformations

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Abstract Given a proper cone K in a finite dimensional real Hilbert space H, we present some results characterizing **Z**-transformations that keep K invariant. We show for example, that when K is irreducible, nonnegative multiples of the identity transformation are the only such transformations. And when K is reducible, they become 'nonnegative diagonal' transformations. We apply these results to symmetric cones in Euclidean Jordan algebras, and, in particular, obtain conditions on the Lyapunov transformation L_A and the Stein transformation S_A that keep the semidefinite cone invariant.

Keywords Proper cone · **Z** and Lyapunov-like transformations · Cone invariance · Irreducible cone

Mathematics Subject Classification 47L07 · 17C20 · 90C33

This paper is dedicated, with deep appreciation and admiration, to Professor T. Parthasarathy on the occasion of his 75th birthday. The authors wish him a long, healthy, and productive life.

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1 Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ denote a finite dimensional real Hilbert space and K be a proper cone in H, that is, K is a pointed closed convex cone with nonempty interior. We say that a linear transformation $L : H \to H$ is a **Z**-*transformation* on K and write $L \in \mathbf{Z}(K)$ if

$$x \in K,\ y \in K^*,\ \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0, \tag{1}$$

where $K^* := \{z \in H : \langle z, x\rangle \geq 0\ \forall\ x \in K\}$ is the dual of K. If both L and $-L$ belong to $\mathbf{Z}(K)$, we say that L is a *Lyapunov-like* transformation on K and write $L \in \mathbf{LL}(K)$. Such transformations appear, e.g., in linear algebra (in the form of **Z**-matrices) (Berman and Plemmons 1994), optimization (in cone linear complementarity problems) (Cottle et al. 1992; Gowda and Tao 2009), continuous and discrete dynamical systems (Farina and Rinaldi 2000), economics (Arrow 1977), Lie algebras (Gowda et al. 2012), etc. A **Z**-matrix (which is a square real matrix with nonpositive off-diagonal entries) is a **Z**-transformation on the nonnegative orthant R^n_+. For any $A \in R^{n\times n}$, the Lyapunov transformation L_A defined by

$$L_A(X) := AX + XA^T \quad (X \in \mathscr{S}^n)$$

and the Stein transformation S_A defined by

$$S_A(X) := X - AXA^T \quad (X \in \mathscr{S}^n)$$

are, respectively, well-known examples of Lyapunov-like and **Z**-transformations on the semidefinite cone $\mathscr{S}^n_+$ in the Hilbert space $\mathscr{S}^n$ of all $n \times n$ real symmetric matrices (Gowda and Tao 2009).

Our main objective in this paper is to characterize **Z**-transformations that keep the corresponding cone invariant. To motivate, we consider two examples. If L is a **Z**-matrix such that $L(R^n_+) \subseteq R^n_+$, then it is a nonnegative diagonal matrix. On the other hand, if we consider a Lyapunov transformation L_A on $\mathscr{S}^n$ satisfying $L_A(\mathscr{S}^n_+) \subseteq \mathscr{S}^n_+$, then Corollary 3 in Gowda and Tao (2014) shows that L_A is a nonnegative multiple of the identity transformation. Thus, over R^n_+ (which is a product of irreducible cones) a cone preserving **Z**-transformation is a nonnegative diagonal matrix, while over $\mathscr{S}^n_+$ (which is an irreducible cone), a cone preserving Lyapunov-like transformation is a nonnegative multiple of the identity transformation. Do such results hold for (general) **Z**-transformations that keep the corresponding cone invariant? In particular, is there a characterization of S_A when $S_A(\mathscr{S}^n_+) \subseteq \mathscr{S}^n_+$? Our motivation also comes from two recent papers of Song (2015, 2016), where the global unique solvability property for semidefinite and symmetric cone linear complementarity problems is discussed for Stein and **Z**-transformations under the assumption of cone invariance.

Motivated by the above questions and relevance, in this paper, we present the following results characterizing the (elements of the) set $\mathbf{Z}(K) \cap \Pi(\mathrm{K})$, where

$$\Pi(\mathrm{K}) := \{L \in \mathscr{B}(H) : L(K) \subseteq K\},$$

with $\mathscr{B}(H)$ denoting the set of all linear transformations on H.

In Theorem 1, analogous to the 'exponential' characterization of Schneider and Vidyasagar (1970) that

$$L \in \mathbf{Z}(K) \iff e^{-tL}(K) \subseteq K \text{ for all } t \geq 0,$$

we present a 'linear' characterization:

$$L \in \mathbf{Z}(K) \iff \text{for all large } t,\ \ (L + tI) \text{ is invertible and } (L + tI)^{-1}(K) \subseteq K.$$

In Theorem 2, we show that $\mathbf{Z}(K) \cap \Pi(\mathrm{K}) = \mathbf{LL}(K) \cap \Pi(\mathrm{K})$ and

$$L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K}) \Longleftrightarrow L + tI \in \mathrm{Aut}(K) \text{ for all } t > 0,$$

where $\mathrm{Aut}(K)$ denotes the *automorphism group* of K (consisting of all invertible linear transformations preserving K). Theorem 3 deals with a special case of Theorem 2:

$$\text{When } K \text{ is irreducible,} \quad \mathbf{Z}(K) \cap \Pi(\mathrm{K}) = R_+ I.$$

Illustrating Theorem 3, among other results, we show that $L_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$ if and only if $A = \gamma I$ for some $\gamma \geq 0$ and $S_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$ if and only if $A = \gamma I$ for some γ with $|\gamma| \leq 1$.

Extending Theorem 3, in Theorem 4, we show that every $\mathbf{Z}$-transformation that leaves a (general) proper cone invariant is a 'nonnegative diagonal' transformation. As an application of Theorem 4, we give short proofs of two results of Song (2015, 2016).

2 Preliminaries

Let C denote a closed convex cone. We say that C is pointed if C does not contain any line passing through the origin, or equivalently, $C \cap -C = \{0\}$. We write $\mathrm{span}(C) = C - C$ for the span of C. A nonzero vector u in C is an *extreme direction* of C if $u = x + y$; $x, y \in C$ imply that x and y are nonnegative multiples of u. In this case, we write $u \in \mathrm{Ext}(C)$.

A closed convex cone C in H is *reducible* if there exist nonzero closed convex cones C_1 and C_2 such that $C = C_1 + C_2$ and $\mathrm{span}(C_1) \cap \mathrm{span}(C_2) = \{0\}$; In this situation, we say that C is a direct sum of C_1 and C_2. If C is not reducible, it is said to be *irreducible.* We have the following result regarding a reducible proper cone.

Proposition 1 *Let K be a reducible proper cone. Then there exists a unique set of nonzero closed convex (pointed) irreducible cones $K_1, K_2, \ldots, K_l$ such that*

$$K = K_1 + K_2 + \cdots + K_l \quad \textit{and} \quad \mathrm{span}(K_i) \cap \left(\sum_{j \neq i} \mathrm{span}(K_j) \right) = \{0\} \textit{ for all } i.$$

Moreover,

(i) H is the direct sum of spaces $\mathrm{span}(K_i)$, $i = 1, 2, \ldots, l$, and
(ii) $\mathrm{Ext}(K) = \bigcup_{i=1}^{l} \mathrm{Ext}(K_i)$.

In the above result, the decomposition part is well-known, see for example, Hauser and Güler (2002), Theorem 4.3. The additional statements are easy to verify.

Throughout this paper, depending on the context, I denotes either the identity transformation or the identity matrix. For a linear transformation L on H, we denote the spectrum of L by $\sigma(L)$; $Re\,\sigma(L)$ denotes the set of the real parts of elements in $\sigma(L)$. We say that L is *positive stable* if $Re(\lambda) > 0$ for all $\lambda \in \sigma(L)$.

For any proper cone K, we let $\mathbf{Z}(K)$ and $\mathbf{LL}(K)$ denote, respectively, the set of all $\mathbf{Z}$-transformations and Lyapunov-like transformations on K. We note that

$$\mathbf{Z}(K) \text{ is a closed convex cone and } \mathbf{LL}(K) = \mathbf{Z}(K) \cap -\mathbf{Z}(K). \tag{2}$$

For ease of reference, we recall the following results.

Proposition 2 *(Gowda and Tao (2009), Theorems 6 and 7) Let $L \in \mathbf{Z}(K)$. The following statements are equivalent:*

(a) *L is positive stable.*
(b) *L is invertible and $L^{-1}(K) \subseteq K$.*

Proposition 3 *(Schneider and Vidyasagar (1970), Theorem 6) Let $L \in \mathbf{Z}(K)$. Then, $\tau :=$ min $Re\,\sigma(L)$ is an eigenvalue of L with an eigenvector in K.*

Given a closed convex cone C and a linear transformation L, we say that L is a *cone preserving* transformation or that the *cone is invariant under* L if $L(C) \subseteq C$. If L is invertible and $L(C) = C$, we say that L is an automorphism of C and write $L \in \text{Aut}(C)$.

The space $\mathcal{S}^n$ (of all $n \times n$ real symmetric matrices) carries the inner product $\langle X, Y\rangle := trace(XY)$, where the trace of a matrix is the sum of its diagonal entries. In $\mathcal{S}^n$, the semidefinite cone $\mathcal{S}^n_+$ of all positive semidefinite matrices is an irreducible cone. We refer to Faraut and Koranyi (1984) for all things related to Euclidean Jordan algebras.

3 Some characterization results

3.1 A linear characterization of Z-transformations

A well-known result of Schneider and Vidyasagar (1970) gives an exponential characterization of **Z**-transformations. In what follows, we provide a linear characterization.

Theorem 1 *The following statements are equivalent:*

(a) $L \in \mathbf{Z}(K)$.
(b) $e^{-tL}(K) \subseteq K$ *for all* $t \geq 0$.
(c) *There exists* $t^* \geq 0$ *such that* $L + tI$ *is invertible and* $(L + tI)^{-1}(K) \subseteq K$ *for all* $t > t^*$.

Proof The equivalence of (a) and (b) is well-known, see Theorem 3 in Schneider and Vidyasagar (1970).

(a) $\Rightarrow$ (c): Assume that $L \in \mathbf{Z}(K)$. Then, by Prop 3, $\tau :=$ min $Re\,\sigma(L)$ is an eigenvalue of L.

Hence, for $t > t^* := |\tau|$, $L + tI$ (which is a **Z**-transformation) is positive stable and by Prop. 2, $L + tI$ is invertible with $(L + tI)^{-1}(K) \subseteq K$; this gives (c).

(c) $\Rightarrow$ (a): Suppose (c) holds (without loss of generality) for some $t^* > 0$. Then, $t > t^* \Rightarrow (L + tI)^{-1}(K) \subseteq K$, or equivalently (as K is a cone), $(I + sL)^{-1}(K) \subseteq K$ for all $s \in (0, \frac{1}{t^*})$. Now, let $||L||$ denote the operator norm of L. Then, for $0 < s < \min\{\frac{1}{t^*}, \frac{1}{||L||}\}$, we have

$$(I + sL)^{-1} = I - sL + s^2 A(s),$$

where $A(s) := L^2 - sL^3 + \cdots$. Now, to show that $L \in \mathbf{Z}(K)$, let $x \in K$ and $y \in K^*$ with $\langle x, y\rangle = 0$. Then, as $(I + sL)^{-1}(K) \subseteq K$,

$$0 \leq \langle (I + sL)^{-1}x, y\rangle = \langle (I - sL + s^2 A(s))x, y\rangle = -s\langle L(x), y\rangle + s^2\langle A(s)x, y\rangle$$

for all small positive s. This gives $\langle L(x), y\rangle \leq s\langle A(s)x, y\rangle$ for all small positive s. Letting $s \to 0$, we get $\langle L(x), y\rangle \leq 0$. This proves that $L \in \mathbf{Z}(K)$. □

3.2 A linear characterization of cone preserving Z-transformations

The following result is key to our characterizations of cone preserving **Z**-transformations to be presented in the subsequent sections. This result shows that the cone preserving

Z-transformations are Lyapunov-like and provides a linear characterization of such transformations.

Theorem 2 *The following statements are equivalent:*

(i) $L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K})$.
(ii) $L \in \mathbf{LL}(K) \cap \Pi(\mathrm{K})$.
(iii) $e^{-tL}(K) \subseteq K$ *for all* $t \geq 0$ *and* $L(K) \subseteq K$.
(iv) $e^{tL}(K) \subseteq K$ *for all* $t \in R$ *and* $L(K) \subseteq K$.
(v) $L + tI \in \mathrm{Aut}(K)$ *for all* $t > 0$.

Proof (i) ⇒ (ii): Suppose $L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K})$. We claim that L is Lyapunov-like on K. Let $x \in K$, $y \in K^*$, and $\langle x, y\rangle = 0$. By the **Z**-property, $\langle L(x), y\rangle \leq 0$. But, as $L(K) \subseteq K$, we have $L(x) \in K$ and therefore, $\langle L(x), y\rangle \geq 0$. Hence $\langle L(x), y\rangle = 0$. Thus, $L \in \mathbf{LL}(K) \cap \Pi(\mathrm{K})$.

(ii) ⇒ (i): This is obvious as $\mathbf{LL}(K) \cap \Pi(\mathrm{K}) \subseteq \mathbf{Z}(K) \cap \Pi(\mathrm{K})$.

The equivalence of (a) and (b) in the previous result gives that of (i) and (iii). As $L \in \mathbf{LL}(K)$ if and only if $L, -L \in \mathbf{Z}(K)$, we also get the equivalence of (ii) and (iv).

(i) ⇒ (v): Now suppose $L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K})$. Since $L \in \mathbf{Z}(K)$, $\tau := \min\, Re\, \sigma(L)$ is an eigenvalue of L and there exists a nonzero $u \in K$ such that $L(u) = \tau\, u$, see Prop 3. Since $L(K) \subseteq K$, we have $\tau \geq 0$. Thus, for $t > \tau \geq 0$, $L + tI$ (which is a **Z**-transformation) is positive stable and so $(L + tI)^{-1}(K) \subseteq K$. Also, $(L + tI)(K) \subseteq K$ for all $t > 0$. Hence, $(L + tI)(K) = K$, that is,

$$L + tI \in \mathrm{Aut}(K) \text{ for all } t > 0.$$

(v) ⇒ (i): Suppose $L + tI \in \mathrm{Aut}(K)$ for all $t > 0$. Then, $(L + tI)^{-1}(K) \subseteq K$ for all $t > 0$. By the previous theorem, $L \in \mathbf{Z}(K)$. Also, from $(L + tI)(K) \subseteq K$ for all $t > 0$, we get $L(K) \subseteq K$. Hence we have $L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K})$. □

3.3 A characterization of cone preserving Z-transformations over an irreducible cone

In this subsection, we characterize cone preserving **Z**-transformations over an irreducible cone and provide several examples.

Theorem 3 *Suppose* K *is an irreducible proper cone in* H. *Then,*

$$\mathbf{Z}(K) \cap \Pi(\mathrm{K}) = R_+\, I.$$

In particular, the above statement holds if K *is a symmetric cone in a simple Euclidean Jordan algebra.*

Proof From Theorem 2, $\mathbf{Z}(K) \cap \Pi(\mathrm{K}) = \mathbf{LL}(K) \cap \Pi(\mathrm{K})$. When K is irreducible, Corollary 3 in Gowda and Tao (2014) shows that $\mathbf{LL}(K) \cap \Pi(\mathrm{K}) = R_+\, I$. This gives the stated result. Since a symmetric cone in a simple Euclidean Jordan algebra is irreducible (Faraut and Koranyi 1984), the additional statement follows. □

Remark 1 Suppose K is an irreducible proper cone in H. An easy consequence of the above theorem is that if L and $I - L$ are in $\Pi(\mathrm{K})$, then L is a nonnegative multiple of I. This is equivalent to saying that I is an extreme direction of the convex cone $\Pi(\mathrm{K})$. (This in turn, is equivalent to saying that (any) L in $\Pi(\mathrm{K})$ that satisfies $L(K) = K$ is an extreme direction of $\Pi(\mathrm{K})$.) That these statements are actually equivalent to the irreducibility of K is shown in Theorem 3.3 of Loewy and Schneider (1975). Thus, the equality $\mathbf{Z}(K) \cap \Pi(\mathrm{K}) = R_+\, I$ holds for a proper cone if and only if it is irreducible. (This also can be seen from our Theorem 4 given in the next Section.) We note that the results of Loewy and Schneider (1975) are proved in a purely algebraic way (without relying on the inner product structure of H) and are valid for convex cones more general than proper cones.

We now illustrate the above theorem by some examples. Recall that $\mathcal{S}^n_+$ is an irreducible cone in $\mathcal{S}^n$.

Example 1 Let $H = \mathcal{S}^n$ with $K = \mathcal{S}^n_+$. Consider, for any $A \in R^{n\times n}$, the Lyapunov transformation L_A. We claim that

$$L_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+ \Longleftrightarrow A = \gamma I \text{ with } \gamma \geq 0.$$

Clearly, $L_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$ holds when $A = \gamma I$ where $\gamma \geq 0$. Now suppose $L_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$. Since L_A is a Lyapunov-like transformation, it is a **Z**-transformation. By the above theorem, $L_A = \alpha I$ for some $\alpha \geq 0$. Then, $AX + XA^T = \alpha X$ for all $X \in \mathcal{S}^n$. By taking X to be an arbitrary diagonal matrix, one can easily show that A must be a diagonal matrix with a constant diagonal. Writing $A = \gamma I$, we have $2\gamma X = L_A(X) \in \mathcal{S}^n_+$ for all $X \in \mathcal{S}^n_+$; hence $\gamma \geq 0$.

Example 2 Let $H = \mathcal{S}^n$ with $K = \mathcal{S}^n_+$. Consider, for any $A \in R^{n\times n}$, the so-called (two-sided) *multiplication transformation* M_A defined by $M_A(X) = AXA^T$. One important (easily verifiable) property of M_A is that $M_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$. Now, we claim that

$$M_A \in \mathbf{Z}(\mathcal{S}^n_+) \Longleftrightarrow A = \gamma I \text{ where } \gamma \in R.$$

One implication is obvious: When $A = \gamma I$, M_A is a multiple of the identity transformation, hence a **Z**-transformation. Now suppose that $M_A \in \mathbf{Z}(\mathcal{S}^n_+)$. Since $M_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$, by Theorem 3, $M_A = \alpha I$ for some $\alpha \geq 0$. This means that $AXA^T = \alpha X$ for all $X \in \mathcal{S}^n$. If $\alpha = 0$, we put $X = I$ (the identity matrix) to get $AA^T = 0$ and $A = 0$. If $\alpha > 0$, let $B = \frac{1}{\sqrt{\alpha}} A$ so that $BXB^T = X$ for all $X \in \mathcal{S}^n$. Since this holds when $X = I$, B is invertible, so $BX = XC$ for all $X \in \mathcal{S}^n$ where $C = (B^T)^{-1}$. Putting $X = I$, we see that $B = C$. So, B commutes with every $X \in \mathcal{S}^n$. From this it is elementary to see that B is a multiple of the identity matrix. Thus, A is a multiple of the identity matrix.

Example 3 Let $H = \mathcal{S}^n$ with $K = \mathcal{S}^n_+$. Consider, for any $A \in R^{n\times n}$, the Stein transformation S_A. We claim that

$$S_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+ \Longleftrightarrow A = \gamma I \text{ where } |\gamma| \leq 1.$$

As S_A is a **Z**-transformation, by Theorem 3, $S_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+ \Rightarrow S_A = \alpha I$ for some $\alpha \geq 0$. We show that $A = \gamma I$ with $|\gamma| \leq 1$. Since $S_A(X) = X - AXA^T = \alpha X$ for all $X \in \mathcal{S}^n$, $AXA^T = \beta X$ for all $X \in \mathcal{S}^n$, where $\beta = 1 - \alpha$. We argue as in Example 2 to see that $A = \gamma I$ for some $\gamma \in R$. Then, $S_A = (1 - \gamma^2)\, I$. From the assumption $S_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$, we get $|\gamma| \leq 1$. The reverse implication in our claim is obvious.

Example 4 This example can be considered as a combination of the previous examples. Here we consider

$$L = L_A - \sum_{i=1}^{N} M_{B_i},$$

which is a **Z**-transformation on $\mathcal{S}^n_+$. We show that if $L(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$, then A and all B_i are multiples of the identity transformation. To see this, suppose $L(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$. In view of

Example 1, we may assume that the sum in L is nonvacuous. Since L is a $\mathbf{Z}$-transformation on $\mathcal{S}^n_+$, by Theorem 3, $L = \alpha I$ for some $\alpha \geq 0$. Then, $M_{B_1} = L_A - \alpha I$ or $M_{B_1} = L_A - \alpha I - \sum_{i=2}^N M_{B_i}$, depending on whether $N = 1$ or $N > 1$. In both cases, M_{B_1} is a $\mathbf{Z}$-transformation; hence from Example 2, B_1 is a multiple of the identity matrix. Similarly, we can show that every B_i is a multiple of the identity matrix. Now, since L and every M_{B_i} keep the semidefinite cone $\mathcal{S}^n_+$ invariant, we see that $L_A = L + \sum_{i=1}^N M_{B_i}$ also keeps the semidefinite cone invariant. Hence, from Example 1, A is a multiple of the identity matrix. This completes the proof.

We note a special case: In $\mathcal{S}^n$,

$$L_A = \sum_{i=1}^{N} M_{B_i} \implies A \text{ and all } B_i \text{ are multiples of } I.$$

This means that a Lyapunov transformation can be written as a nonnegative linear combination of multiplication transformations only in some trivial cases. A similar statement can be made by replacing the multiplication transformations by Stein transformations.

3.4 A 'diagonal' characterization of cone preserving Z-transformations

In this subsection, we consider the problem of characterizing cone preserving $\mathbf{Z}$-transformations over general cones. Our next result says that such a transformation is a 'nonnegative diagonal' transformation and is an extension of Theorem 3 to a direct sum of irreducible cones.

Theorem 4 *Suppose K is a reducible proper cone. Let K be the direct sum of irreducible cones K_i, $i = 1, 2 \ldots, l$ (as in Proposition 1). If $L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K})$, then on each K_i, L is a nonnegative multiple of the identity transformation. Hence, there exist nonnegative real numbers α_i, $i = 1, 2 \ldots, l$, such that*

$$L(x) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_l x_l \quad \textit{for all} \quad x = x_1 + x_2 + \cdots + x_l,$$
$$\textit{where } x_i \in \mathrm{span}(K_i), \; i = 1, 2 \ldots, l. \tag{3}$$

Note: As $K^* \subseteq \cap K_i^*$, it is easy to see that (3) defines a linear transformation L which is in $\mathbf{Z}(K) \cap \Pi(\mathrm{K})$ when all the α_is are nonnegative.

Proof Suppose $L \in \mathbf{Z}(K) \cap \Pi(\mathrm{K})$. We show that for each i, $L(K_i) \subseteq K_i$ and the restriction of L to $\mathrm{span}(K_i)$ is a $\mathbf{Z}$-transformation on cone K_i (which is a proper cone in $\mathrm{span}(K_i)$). Then, we invoke Theorem 3 to say that on K_i, $L = \alpha_i I$ for some $\alpha_i \geq 0$. The diagonal representation given in the theorem follows immediately as H is the direct sum of the spaces $\mathrm{span}(K_i)$.

Now, from Theorem 2, Item (v),

$$L + tI \in \mathrm{Aut}(K) \text{ for all } t > 0.$$

Putting $t = \frac{1}{s}$, we see that $I + sL \in \mathrm{Aut}(K)$ for all $s > 0$. Note that the last statement holds even for $s = 0$. Thus, $(I + sL)(\mathrm{Ext}(K)) = \mathrm{Ext}(K)$ for all $s \geq 0$. We now show, via a connectedness argument, that

$$(I + sL)(K_1) \subseteq K_1 \text{ for all } s > 0.$$

Take any $x \in \mathrm{Ext}(K_1)$ (note that $x \neq 0$). Then, from Item (ii) in Proposition 1, $x \in \mathrm{Ext}(K)$ and so $(I + sL)x \in \mathrm{Ext}(K)$ for all $s \geq 0$. As $\mathrm{Ext}(K)$ is the union of $\mathrm{Ext}(K_i)$, x is nonzero, and $I + sL$ is invertible, we must have $(I + sL)x \in \bigcup_{i=1}^{l} E_i$, where $E_i := K_i \backslash \{0\}$. Clearly, the set $\{(I + sL)x : s \geq 0\}$ is connected and contains x. Also, the sets $E_1, E_2, \cdots, E_l$ are separated (that is, $E_i \bigcap \overline{\bigcup_{j \neq i} E_i} = \emptyset$ for all i). Hence, we must have $(I + sL)x \in E_1$ for all $s \geq 0$. This means that

$$x \in \text{Ext}(K_1) \Rightarrow (I + sL)x \in K_1 \ \forall \ s \geq 0.$$

Now, K_1 is a proper cone in span(K_i); hence it is the convex hull of Ext(K_1) by the cone version of Minkowski's theorem, see Holmes (1975), page 36. It follows that $(I+sL)(K_1) \subseteq K_1$ for all $s > 0$. Replacing s by $\frac{1}{t}$, we see that $(L + tI)(K_1) \subseteq K_1$ for all $t > 0$. Letting $t \to 0$, we get $L(K_1) \subseteq K_1$. A similar argument with $(I + sL)^{-1}$ in place of $I + sL$ shows that $(L + tI)^{-1}(K_1) \subseteq K_1$, that is, $K_1 \subseteq (L + tI)(K_1)$ for all $t > 0$. As $L(K_1) \subseteq K_1$, we may consider the restriction of L to span(K_1); let L_1 denote this restriction. Then,

$$(L_1 + tI)(K_1) \subseteq K_1 \text{ and } K_1 \subseteq (L_1 + tI)(K_1) \text{ for all } t > 0.$$

This means that

$$(L_1 + tI) \in \text{Aut}(K_1) \text{ for all } t > 0.$$

By Theorem 2, $L_1 \in \mathbf{Z}(K_1) \cap \Pi(K_1)$. As K_1 is irreducible, from Theorem 3, $L_1 = \alpha_1 I$ for some $\alpha_1 \geq 0$. By replacing 1 by any i, we see that $L(K_i) \subseteq K_i$ and the restriction of L to span(K_i) is a nonnegative multiple of the identity transformation. This concludes the proof. □

Remark 2 Part of the above proof becomes simpler if the spans of K_i are orthogonal, see Gowda and Ravindran (2015), Corollary 1. We now apply the above result to Euclidean Jordan algebras. Let H be a Euclidean Jordan algebra with its symmetric cone K. Then, we may decompose H as an orthogonal direct sum of simple Euclidean Jordan algebras H_i, $i = 1, 2, \ldots, l$, with $K = K_1 + K_2 + \cdots + K_l$, where K_i is the symmetric cone of H_i, see Faraut and Koranyi (1984), Propositions III.4.4 and III.4.5. Since H_i is simple, each K_i is irreducible, hence we can apply the above theorem and say that any L in $\mathbf{Z}(K) \cap \Pi(\text{K})$ has the diagonal form (3). As $span(K_i) = H_i$, we note that the restriction of L to any H_i is a nonnegative multiple of the identity transformation.

3.5 Symmetric cone linear complementarity problems

In this subsection, we show how our characterization results (Theorems 3 and 4) lead to simplified proofs of some results of Song on symmetric cone and semidefinite linear complementarity problems. First, we recall some necessary concepts and results. Let H denote a Euclidean Jordan algebra with symmetric cone K. Given a linear transformation L on H and an element $q \in H$, the *symmetric cone linear complementarity problem*, SCLCP(L, q), is to find $x \in H$ such that

$$x \in K, \ y := L(x) + q \in K, \ \text{ and } \langle x, y \rangle = 0.$$

When $H = R^n$ with $K = R^n_+$, the above problem reduces to the linear complementarity problem (Cottle et al. 1992). If $H = \mathcal{S}^n$, $K = \mathcal{S}^n_+$, the problem becomes the *semidefinite linear complementarity problem* (Gowda and Song 2000). These are particular instances of (cone) complementarity problems and variational inequalities which have been extensively studied in the literature, see e.g., Facchinei and Pang (2003a, b).

We say that L

(1) has the ***Q***-property if for every $q \in H$, SCLCP(L, q) has a solution;
(2) has the ***GUS***-property if for every $q \in H$, SCLCP(L, q) has a unique solution;
(3) has the ***P***-property if

$$[x \circ L(x) \leq 0, \quad x \text{ and } L(x) \text{ operator commute}] \Rightarrow x = 0,$$

where we say that two elements a and b *operator commute* if $a \circ (b \circ x) = b \circ (a \circ x)$ for all $x \in H$ with $x \circ y$ denoting the *Jordan product* of elements x and y in H and $z \leq 0$ in H means that $-z \in K$;
(4) is *monotone* if $\langle L(x), x\rangle \geq 0$ for all $x \in H$;
(5) is *strictly copositive* if $\langle L(x), x\rangle > 0$ for all $0 \neq x \in K$.

The above properties have been well studied for linear complementarity problems and many inter-connections have been noted in the general case (Cottle et al. 1992; Gowda et al. 2004). While the **GUS**-property reduces to the **P**-property for linear complementarity problems, so far there is no characterization result known for general transformations. For Lyapunov-like transformations, the following are known:

Proposition 4 *(Gowda and Sznajder (2007), Theorem 7.1; Gowda et al. 2012, Theorem 5) Suppose H is a Euclidean Jordan algebra with symmetric cone K and $L \in$ **LL**(K). Then,*

*(i) L has the **GUS**-property if and only if L is positive stable and monotone;*
*(ii) L has the **Q**-property if and only if it has the **P**-property.*

Whether statements (i) *and* (ii) *hold for **Z**-transformations remains an open question.* With a view towards characterizing the **GUS**-property for **Z**-transformations, in Song (2016), Corollary 3.2, Song proves the following result:

Proposition 5 *Suppose H is a Euclidean Jordan algebra with symmetric cone K and $L \in$ **Z**(K). If $L(K) \subseteq K$, then the following are equivalent:*

(a) L is strictly copositive.
*(b) L has the **GUS**-property.*
*(c) L has the **P**-property and monotone.*

We now show that this result can be recovered via Theorem 4 and Remark 1 as follows. When $L(K) \subseteq K$, L has a representation given in (3). Additionally, in that representation, the vectors x_i, $1 \leq i \leq l$, are mutually orthogonal. When L has one of the properties (a), (b), or (c), the nonnegative scalars $\alpha_1, \alpha_2, \ldots, \alpha_l$ in (3) are positive. But then all properties (a)–(c) hold, proving the required equivalence. This argument actually shows that

*For a cone invariant **Z**-transformation, the **Q** and **P** properties are equivalent.*

We conclude the paper with another result of Song (2015) proved for the Stein transformation S_A on $\mathcal{S}^n$:

When $S_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$, S_A has the **GUS** property if and only if $I \pm A$ are positive definite.

The following result recovers this and says more.

Corollary 1 *Let $H = \mathcal{S}^n$ and $S_A(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$. Then S_A has the **GUS**-property if and only if $A = \gamma I$ with $|\gamma| < 1$.*

We note that when $A = \gamma I$ with $|\gamma| < 1$, $I \pm A$ are positive definite.

Proof Suppose $A = \gamma I$ with $|\gamma| < 1$. Then $S_A(X) = X - \gamma^2 X = (1 - \gamma^2)X$. Hence, $S_A = \alpha I$ with $\alpha > 0$, and S_A has the **GUS**-property. Conversely, suppose S_A has the **GUS** property. Since $S_A(\mathscr{S}^n_+) \subseteq \mathscr{S}^n_+$, we see that $A = \gamma I$ with $|\gamma| \leq 1$. If $|\gamma| = 1$, then $S_A(X) = (1 - \gamma^2)X = 0$. Obviously S_A cannot have the **GUS** property. Hence, $|\gamma| < 1$. This completes the proof. □

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