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COMMUTATION PRINCIPLES IN EUCLIDEAN JORDAN ALGEBRAS AND NORMAL DECOMPOSITION SYSTEMS*

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Abstract. The commutation principle of Ramírez, Seeger, and Sossa [SIAM J. Optim., 23 (2013), pp. 687–694] proved in the setting of Euclidean Jordan algebras says that when the sum of a Fréchet differentiable function Θ and a spectral function F is minimized (maximized) over a spectral set Ω, any local minimizer (respectively, maximizer) a operator commutes with the Fréchet derivative $\Theta'(a)$. In this paper, we extend this result to sets and functions which are (just) invariant under algebra automorphisms. We also consider a similar principle in the setting of normal decomposition systems.

Key words. Euclidean Jordan algebra, (weakly) spectral sets/functions, automorphisms, commutation principle, normal decomposition system, variational inequality problem, cone complementarity problem

AMS subject classifications. 17C20, 17C30, 52A41, 90C25

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1. Introduction. Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n [3] and $\lambda : \mathcal{V} \to \mathcal{R}^n$ denote the eigenvalue map (which takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). A set Ω in $\mathcal{V}$ is said to be a *spectral set* [1] if it is of the form $\Omega = \lambda^{-1}(Q)$, where Q is a permutation invariant set in $\mathcal{R}^n$. A function $F : \mathcal{V} \to \mathcal{R}$ is said to be a *spectral function* [1] if it is of the form $F = f \circ \lambda$, where $f : \mathcal{R}^n \to \mathcal{R}$ is a permutation invariant function.

Extending an earlier result of Iusem and Seeger [7] for real symmetric matrices, Ramírez, Seeger, and Sossa [13] prove the following commutation principle.

THEOREM 1.1. *Let $\mathcal{V}$ be a Euclidean Jordan algebra, Ω be a spectral set in $\mathcal{V}$, and $F : \mathcal{V} \to \mathcal{R}$ be a spectral function. Let $\Theta : \mathcal{V} \to R$ be Fréchet differentiable. If a is a local minimizer/maximizer of the map*

$$x \in \Omega \mapsto \Theta(x) + F(x), \tag{1}$$

then a and $\Theta'(a)$ operator commute in $\mathcal{V}$.

A number of important and interesting applications are mentioned in [13]. The proof of the above result in [13] is somewhat intricate, deep, and long. In our paper we extend the above result by assuming only the automorphism invariance of Ω and F, and at the same time provide (perhaps) a simpler and shorter proof. To elaborate, recall that an (algebra) automorphism on $\mathcal{V}$ is an invertible linear transformation on $\mathcal{V}$ that preserves the Jordan product. It is known (see [8, Theorem 2]) that spectral sets and functions are invariant under automorphisms, but the converse may fail unless the algebra is either $\mathcal{R}^n$ or simple. By defining *weakly spectral sets/functions* as those having this automorphism invariance property, we extend the above result of Ramírez, Seeger, and Sossa as follows.

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THEOREM 1.2. *Let $\mathcal{V}$ be a Euclidean Jordan algebra and suppose that Ω in $\mathcal{V}$ and $F : \mathcal{V} \to \mathcal{R}$ are weakly spectral. Let $\Theta : \mathcal{V} \to R$ be Fréchet differentiable. If a is a local minimizer/maximizer of the map*

$$x \in \Omega \mapsto \Theta(x) + F(x), \tag{2}$$

then a and $\Theta'(a)$ operator commute in $\mathcal{V}$.

By noting that a Euclidean Jordan algebra is an inner product space and the corresponding automorphism group is a subgroup of the orthogonal group (at least when the algebra is equipped with the canonical inner product), we state a similar result in the setting of a normal decomposition system. Such a system was introduced by Lewis [10] to unify various results in convex analysis on matrices. A normal decomposition system is a triple $(X, \mathcal{G}, \gamma)$, where X is a real inner product space, $\mathcal{G}$ is a (closed) subgroup of the orthogonal group of X, and $\gamma : X \to X$ is a mapping that has properties similar to those of the map $x \mapsto \lambda(x)$; see section 4. Our commutation principle on such a system is as follows.

THEOREM 1.3. *Let $(X, \mathcal{G}, \gamma)$ be a normal decomposition system. Let Ω be a convex $\mathcal{G}$-invariant set in X, $F : X \to \mathcal{R}$ be a convex $\mathcal{G}$-invariant function, and $\Theta : X \to \mathcal{R}$ be Fréchet differentiable. Suppose that a is a local minimizer of the map*

$$x \in \Omega \mapsto \Theta(x) + F(x). \tag{3}$$

Then a and $-\Theta'(a)$ commute in $(X,\, \mathcal{G},\, \gamma)$.

The organization of our paper is as follows. We cover some preliminary material in section 2. In section 3, we define weakly spectral sets/functions and present a proof of Theorem 1.2. In section 4, we describe normal decomposition systems and present a proof of Theorem 1.3. In the appendix, we state a structure theorem for the automorphism group of a Euclidean Jordan algebra and show that weakly spectral sets and spectral sets coincide only in an essentially simple algebra.

2. Preliminaries.

2.1. Euclidean Jordan algebras. *Throughout this paper, $\mathcal{V}$ denotes a Euclidean Jordan algebra* [3]. For $x, y \in \mathcal{V}$, we denote their inner product by $\langle x, y\rangle$ and Jordan product by $x \circ y$. We let e denote the unit element in $\mathcal{V}$ and $\mathcal{V}_+ := \{x \circ x : x \in \mathcal{V}\}$ denote the corresponding *symmetric cone*. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two Euclidean Jordan algebras, then $\mathcal{V}_1 \times \mathcal{V}_2$ becomes a Euclidean Jordan algebra under the Jordan and inner products, defined, respectively, by $(x_1, x_2) \circ (y_1, y_2) = \big(x_1 \circ y_1, x_2 \circ y_2\big)$ and $\langle (x_1, x_2), (y_1, y_2)\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle$. A similar definition is made for a product of several Euclidean Jordan algebras. Recall that a Euclidean Jordan algebra $\mathcal{V}$ is *simple* if it is not a direct product of nonzero Euclidean Jordan algebras (or equivalently, if it does not contain any nontrivial ideal). It is known (see [3]) that any nonzero Euclidean Jordan algebra is, in a unique way, a direct product of simple Euclidean Jordan algebras. Moreover, there are only five types of simple algebras, two of which are $\mathcal{S}^n$, the algebra of $n \times n$ real symmetric matrices, and $\mathcal{H}^n$, the algebra of $n \times n$ complex Hermitian matrices.

We say that $\mathcal{V}$ is *essentially simple* if it is either $\mathcal{R}^n$ or simple.

An element $c \in \mathcal{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say a finite set $\{e_1,\, e_2,\, \dots,\, e_n\}$ of primitive idempotents in $\mathcal{V}$ is a *Jordan frame* if $e_i \circ e_j = 0$ if $i \neq j$ and $\sum_{i=1}^{n} e_i = e$. It turns out that the number of elements in any Jordan frame is the same; this common number is called the *rank* of $\mathcal{V}$.

PROPOSITION 2.1 (spectral decomposition theorem [3]). *Suppose $\mathcal{V}$ is a Euclidean Jordan algebra of rank n. Then, for every $x \in \mathcal{V}$, there exist uniquely determined real numbers $\lambda_1(x), \ldots, \lambda_n(x)$ (called the* eigenvalues *of x) and a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ such that*

$$x = \lambda_1(x)e_1 + \cdots + \lambda_n(x)e_n.$$

Conversely, given any Jordan frame $\{e_1, \ldots, e_n\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, the sum $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ defines an element of $\mathcal{V}$ whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

We define the *eigenvalue map* $\lambda : \mathcal{V} \to \mathcal{R}^n$ by

$$\lambda(x) = \Big(\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)\Big),$$

where $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$. This is well-defined and continuous [1].

We define the *trace* of an element $x \in \mathcal{V}$ by $\mathrm{tr}(x) := \lambda_1(x) + \cdots + \lambda_n(x)$. Correspondingly, the *canonical (or trace) inner product* on $\mathcal{V}$ is defined by

$$\langle x, y\rangle_{tr} := \mathrm{tr}(x \circ y).$$

This defines an inner product on $\mathcal{V}$ that is compatible with the given Jordan structure. With respect to this inner product, the norm of any primitive element is one.

Throughout this paper, for a linear transformation $A : \mathcal{V} \to \mathcal{V}$ and $x \in \mathcal{V}$, we use, depending on the context, both the function notation $A(x)$ as well as the operator notation Ax.

Given $a \in \mathcal{V}$, we define the corresponding transformation $L_a : \mathcal{V} \to \mathcal{V}$ by $L_a(x) = a \circ x$. We say that two elements a and b *operator commute* in $\mathcal{V}$ if $L_a L_b = L_b L_a$. We remark that a and b operator commute if and only if there exist a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and real numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ such that

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \quad \text{and} \quad b = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n;$$

see [3, Lemma X.2.2]. (Note that the a_is and b_is need not be in decreasing order.) In particular, in $\mathcal{S}^n$ or $\mathcal{H}^n$, operator commutativity reduces to the ordinary (matrix) commutativity.

A linear transformation between two Euclidean Jordan algebras is a (Jordan algebra) *homomorphism* if it preserves Jordan products. If it is also one-to-one and onto, then it is an *isomorphism*. If the algebras are the same, we call such an isomorphism an *automorphism*. Thus, a linear transformation $A : \mathcal{V} \to \mathcal{V}$ is an *automorphism* of $\mathcal{V}$ if it is invertible and

$$A(x \circ y) = Ax \circ Ay \quad \text{for all } x,\, y \in \mathcal{V}.$$

The set of all automorphisms of $\mathcal{V}$ is denoted by $\mathrm{Aut}(\mathcal{V})$. When $\mathcal{V}$ is a product, say, $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, for $\phi_i \in \mathrm{Aut}(\mathcal{V}_i)$, it is easy to see that ϕ defined by

$$\phi(x) := \Big(\phi_1(x_1), \phi_2(x_2)\Big), \quad x = (x_1, x_2) \in \mathcal{V}_1 \times \mathcal{V}_2$$

belongs to $\mathrm{Aut}(\mathcal{V})$. Thus,

$$\mathrm{Aut}(\mathcal{V}_1) \times \mathrm{Aut}(\mathcal{V}_2) \subseteq \mathrm{Aut}(\mathcal{V}_1 \times \mathcal{V}_2).$$

A similar statement can be made when $\mathcal{V}$ is a product of several factors.

When $\mathcal{V}$ carries the canonical inner product, every automorphism is inner product preserving and so $\mathrm{Aut}(\mathcal{V})$ is a closed subgroup of the orthogonal group of $\mathcal{V}$. A linear transformation $D : \mathcal{V} \to \mathcal{V}$ is a *derivation* if

$$D(x \circ y) = D(x) \circ y + x \circ D(y) \quad \text{for all } x,\, y \in \mathcal{V}.$$

We recall the following result from [8] (essentially, [3, Theorem IV.2.5]).

PROPOSITION 2.2. *Let $\mathcal{V}$ be essentially simple. If $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ are any two Jordan frames in $\mathcal{V}$, then there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $\phi(e_i) = e'_i$ for all $i = 1, \ldots, n$.*

3. Weakly spectral sets and functions.

DEFINITION 3.1. *We say that a set E in $\mathcal{V}$ is* weakly spectral *if*

$$A(E) \subseteq E \quad \text{for all } A \in \mathrm{Aut}(\mathcal{V}).$$

A function $F : \mathcal{V} \to \mathcal{R}$ is said to be weakly spectral *if*

$$F(Ax) = F(x) \text{ for all } x \in \mathcal{V},\, A \in \mathrm{Aut}(\mathcal{V}).$$

Remark 3.2. Suppose E is a spectral set, that is, $E = \lambda^{-1}(Q)$ for some permutation invariant set Q in $\mathcal{R}^n$. Then,

$$x \in E,\, y \in \mathcal{V},\, \lambda(y) = \lambda(x) \Rightarrow y \in E. \tag{4}$$

Now, let $x \in E$ and $A \in \mathrm{Aut}(\mathcal{V})$. As A maps Jordan frames to Jordan frames, $\lambda(Ax) = \lambda(x)$. From (4), $Ax \in E$. This proves that E is weakly spectral. Hence, *every spectral set is weakly spectral.* Now suppose $F : \mathcal{V} \to \mathcal{R}$ is a spectral function so that for some permutation invariant function $f : \mathcal{R}^n \to \mathcal{R}$, $F = f \circ \lambda$. It follows that $F(Ax) = f(\lambda(Ax)) = f(\lambda(x)) = F(x)$ for any $A \in \mathrm{Aut}(\mathcal{V})$. Thus, F is weakly spectral. This proves that *every spectral function is weakly spectral.*

Remark 3.3. It has been observed in [8, Theorem 2] that *in any essentially simple algebra, every weakly spectral set is spectral.* The following example shows that weakly spectral sets/functions can be different from spectral sets/functions in general algebras.

In the product algebra $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$, let $\Omega = \mathcal{R}_+ \times \mathcal{S}^2$, and

$$x = \left(1,\, \left[\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right]\right),\; y = \left(-1,\, \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]\right).$$

Since $x \in \Omega$, $y \notin \Omega$, and $\lambda(x) = (2, 1, -1)^{\mathsf{T}} = \lambda(y)$, we see that Ω cannot be of the form $\lambda^{-1}(Q)$ for any (permutation invariant) set Q in $\mathcal{R}^3$. Thus, Ω is not a spectral set in $\mathcal{V}$. Now, identity transformation is the only automorphism of $\mathcal{R}$ and any automorphism of $\mathcal{S}^2$ is of the form $X \mapsto UXU^{\mathsf{T}}$ for some orthogonal matrix U. As $\mathcal{R}$ and $\mathcal{S}^2$ are nonisomorphic Euclidean Jordan algebras, we see (from Proposition 1 in [4] or Corollary 5.6 in the appendix) that automorphisms of $\mathcal{V}$ are of the form $(t, X) \mapsto (t, UXU^{\mathsf{T}})$ for some orthogonal matrix U. It follows that Ω is weakly spectral. The characteristic function of Ω is an example of a weakly spectral function that is not spectral.

Remark 3.4. As a consequence of Corollary 5.6 in the appendix, one can show the following: Suppose $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_m$, where $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ are nonisomorphic simple algebras. Let E_i be a spectral set in $\mathcal{V}_i$, $i = 1, 2, \ldots, m$. Then, $E_1 \times E_2 \times \cdots \times E_m$

is weakly spectral in $\mathcal{V}$. Not every weakly spectral set in $\mathcal{V}$ arises this way: Referring to the example given in Remark 3.3,

$$\{(t, X) \in \Omega : t + \mathrm{tr}(X) = 0\}$$

is weakly spectral but not a product of two spectral sets.

We also note, as a consequence of Theorem 5.1, that a product of (weakly) spectral sets need not be (weakly) spectral. The set $\{0\} \times \mathcal{S}^2$ in $\mathcal{S}^2 \times \mathcal{S}^2$ is one such example.

Remark 3.5. It will be shown in Theorem 5.7 that in any algebra that is not essentially simple, the class of weakly spectral sets is strictly larger than the class of spectral sets. This shows that Theorem 1.2 is applicable to a wider class of sets/functions than Theorem 1.1.

Proof of Theorem 1.2. We first prove the result by assuming that a is a local minimizer of the map (2). Then we have

$$\Theta(a) + F(a) \le \Theta(x) + F(x) \qquad \text{for all} \quad x \in N_a \cap \Omega,$$

where N_a denotes some open ball around a. Let u and v be arbitrary (but fixed) elements of $\mathcal{V}$. Let $D := L_u L_v - L_v L_u$. Then, Proposition II.4.1. in [3] shows that D is a derivation on $\mathcal{V}$; hence, as observed in [3, p. 36], e^{tD} is an automorphism of $\mathcal{V}$ for all $t \in R$. Therefore, by the continuity of $t \mapsto e^{tD} a$ and the automorphism invariance of Ω, $x := e^{tD} a \in N_a \cap \Omega$ for all t close to zero in $\mathcal{R}$. Then,

$$\Theta(a) + F(a) \le \Theta(e^{tD} a) + F(e^{tD} a) \text{ for all } t \text{ close to } 0.$$

As $F(e^{tD} a) = F(a)$ by the automorphism invariance of F, we see that

$$\Theta(a) \le \Theta(e^{tD} a) \qquad \text{for all} \quad t \text{ near } 0.$$

This implies that the derivative of $t \mapsto \Theta(e^{tD} a)$ at $t = 0$ is zero. Thus, we have $\langle \Theta'(a), Da \rangle = 0$. Putting $b := \Theta'(a)$ and recalling $D = L_u L_v - L_v L_u$, we get

$$\Big\langle b, (L_u L_v - L_v L_u) a \Big\rangle = 0.$$

Since L_u and L_v are self-adjoint, the above expression can be rewritten as

$$\langle v \circ a,\, u \circ b \rangle - \langle u \circ a,\, v \circ b \rangle = 0.$$

This, upon rearrangement, leads to $\langle (L_b L_a - L_a L_b) u,\, v \rangle = 0$. As this equation holds for all u and v, we see that $L_b L_a = L_a L_b$, proving the operator commutativity of a and $b\,(= \Theta'(a))$ in $\mathcal{V}$.

Now suppose that a is a local maximizer of the map (2). Then a is a local minimizer of the map $x \in \Omega \mapsto -\Theta(x) - F(x)$. From the above, we conclude that a operator commutes with $-\Theta'(a)$. It follows that a operator commutes with $\Theta'(a)$. □

Remark 3.6. As noted by a referee, the above proof only requires the invariance of Ω and F under automorphisms of the form e^{tD}, where $t \in R$, and D is a derivation. Since these belong to $\mathcal{K}$, the connected component of identity in $\mathrm{Aut}(\mathcal{V})$, we may weaken the hypothesis of the theorem by only requiring the $\mathcal{K}$-invariance of Ω and F. A result of this type for $\mathcal{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{SO}_n$ (the special orthogonal group) has already been observed in [16, Theorem 9].

Remark 3.7. As can be seen, the above proof consists of two parts. In the first part, using the automorphisms of the form e^{tD}, we prove that $\langle \Theta'(a), Da\rangle = 0$ for all derivations D. In the second part, knowing the precise form of the derivations, we further relate a and $\Theta'(a)$, thus proving the commutativity. With the observation that the set of all derivations is just the Lie algebra of $\mathrm{Aut}(\mathcal{V})$, we can extend the first part in the following way: *Let H be a finite dimensional real Hilbert space and $\mathcal{G}$ be a Lie subgroup of the group of all invertible linear transformations on H. Suppose $\Omega \subseteq H$ and $F : H \to R$ are $\mathcal{G}$-invariant, and $\Theta : H \to R$ is differentiable. If a is a minimizer (or a maximizer) of the map $x \in \Omega \mapsto \Theta(x) + F(x)$, then*

$$\langle \Theta'(a), L(a)\rangle = 0 \text{ for all } L \in Lie(\mathcal{G}), \tag{5}$$

where $Lie(\mathcal{G})$ denotes the Lie algebra of $\mathcal{G}$.

This is seen by noting that $L \in Lie(\mathcal{G}) \Leftrightarrow e^{tL} \in \mathcal{G}$ for all $t \in R$ and imitating the first part of the above proof. Knowing the precise nature of $Lie(\mathcal{G})$, one can then get a refined relation between $\Theta'(a)$ and a. We remark that (5) is a type of "orbital relation" discussed in a very broad setting in [15] for extremal problems under invariance assumptions; see also [16].

An immediate special case of Theorem 1.2 is obtained by taking $F = 0$: If Ω is weakly spectral and Θ is Fréchet differentiable, then any local minimizer a of $\min_{x\in\Omega} \Theta(x)$ operator commutes with $\Theta'(a)$. This can further be specialized by assuming that Θ is linear, that is, of the form $\Theta(x) = \langle b, x\rangle$.

A number of applications mentioned in [13] have analogues for weakly spectral sets and functions. We mention one application that is especially important.

THEOREM 3.8. *Suppose $\Omega \subseteq \mathcal{V}$ and $F : \mathcal{V} \to \mathcal{R}$ are weakly spectral. Let $G : \mathcal{V} \to \mathcal{V}$ be arbitrary. Consider the* variational inequality problem *VI(G, Ω, F): Find $x^* \in \Omega$ such that*

$$\langle G(x^*), x - x^*\rangle + F(x) - F(x^*) \geq 0 \quad \text{for all } x \in \Omega.$$

If a solves VI(G, Ω, F), then a operator commutes with $G(a)$.

Proof. The proof is similar to the one given in [13, Proposition 1.9]. If a solves $\mathrm{VI}(G, \Omega, F)$, then

$$\langle G(a), x - a\rangle + F(x) - F(a) \geq 0 \text{ for all } x \in \Omega.$$

This implies

$$\langle G(a), x\rangle + F(x) \geq \langle G(a), a\rangle + F(a) \text{ for all } x \in \Omega.$$

So, a minimizes $\langle G(a), x\rangle + F(x)$ over Ω. By Theorem 1.2 applied to $\Theta(x) := \langle G(a), x\rangle$, we see that a operator commutes with $G(a)$. ∎

As an illustration of the above result, let K be a closed convex cone in $\mathcal{V}$ and $G : \mathcal{V} \to \mathcal{V}$ be arbitrary. Consider the *cone complementarity problem* $\mathrm{CP}(G, K)$ of finding an $x^* \in K$ such that

$$x^* \in K,\ G(x^*) \in K^*, \text{ and } \langle x^*, G(x^*)\rangle = 0,$$

where K^* is the dual of K defined by $K^* = \{y \in \mathcal{V} : \langle y, x\rangle \geq 0 \text{ for all } x \in K\}$.

COROLLARY 3.9. *Suppose K is weakly spectral. If a solves the cone complementarity problem $CP(G, K)$, then a operator commutes with $G(a)$.*

Remark 3.10. The above corollary yields the following: *Suppose K is a closed convex cone in $\mathcal{V}$ that is weakly spectral. If $a \in K$ and $b \in K^*$ satisfy $\langle a, b\rangle = 0$, then a and b operator commute.* Such a result for $K = \mathcal{V}_+$ (the symmetric cone of $\mathcal{V}$) is well-known; see Proposition 6 in [5].

4. Normal decomposition systems. Before giving a proof of Theorem 1.3, we briefly recall the definition of a normal decomposition system and mention relevant properties.

Definition 4.1. *Let X be a real inner product space, $\mathcal{G}$ be a closed subgroup of the orthogonal group of X, and $\gamma : X \to X$ be a mapping satisfying the following properties:*

(a) *γ is $\mathcal{G}$-invariant, that is, $\gamma(Ax) = \gamma(x)$ for all $x \in X$, $A \in \mathcal{G}$,*
(b) *for each $x \in X$, there exists $A \in \mathcal{G}$ such that $x = A\gamma(x)$, and*
(c) *for all $x, w \in X$, we have $\langle x, w\rangle \leq \langle \gamma(x), \gamma(w)\rangle$.*

Then, $(X, \mathcal{G}, \gamma)$ is called a normal decomposition system [10]*. In such a system, a set $\Omega \subseteq X$ is said to be $\mathcal{G}$-*invariant *if $A(\Omega) \subseteq \Omega$ for all $A \in \mathcal{G}$; a function $F : X \to \mathcal{R}$ is said to be $\mathcal{G}$-*invariant *if $F(Ax) = F(x)$ for all $x \in X$ and $A \in \mathcal{G}$.*

In [10], various results on normal decomposition systems are given. In particular, the following is proved.

Proposition 4.2 (see [10, Proposition 2.3]). *In a normal decomposition system, for any two elements x and w, we have*

$$\max_{A\in\mathcal{G}} \langle Ax, w\rangle = \langle \gamma(x), \gamma(w)\rangle .$$

Also, $\langle x, w\rangle = \langle \gamma(x), \gamma(w)\rangle$ holds for two elements x and w if and only if there exists an $A \in \mathcal{G}$ such that $x = A\gamma(x)$ and $w = A\gamma(w)$.

Motivated by the above proposition, we say that x and w *commute in* $(X, \mathcal{G}, \gamma)$ if there exists an $A \in \mathcal{G}$ such that $x = A\gamma(x)$ and $w = A\gamma(w)$.

Now consider an essentially simple Euclidean Jordan algebra $\mathcal{V}$. We assume that $\mathcal{V}$ carries the canonical inner product and let $\mathcal{G} = \mathrm{Aut}(\mathcal{V})$. Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ be a fixed Jordan frame in $\mathcal{V}$. Define for any $x \in \mathcal{V}$,

$$\gamma(x) := \sum_{i=1}^{n} \lambda_i(x)\bar{e}_i, \tag{6}$$

where $\lambda_i(x)$ are components of $\lambda(x)$. As eigenvalues are preserved under automorphisms, we see that γ satisfies condition (a) in the definition of normal decomposition system. Since $\mathcal{V}$ is essentially simple, any Jordan frame can be mapped onto any other by an element of $\mathcal{G}$ (by Proposition 2.2). Thus, given any $x \in \mathcal{V}$ with its spectral decomposition $x = \sum_1^n \lambda_i(x) f_i$, we can find $A \in \mathcal{G}$ such that $A(\overline{e_i}) = f_i$ for all i. Then,

$$x = A\left(\sum_{i=1}^{n} \lambda_i(x)\bar{e}_i\right) = A\gamma(x).$$

This verifies condition (b) in the definition of normal decomposition system. Finally, for all $x, w \in X$, we have the so-called Theobald–von Neumann inequality $\langle x, w\rangle \leq \langle \gamma(x), \gamma(w)\rangle$; see, for example, [12], [1], or [6]. Putting all these together, we have the following result.

PROPOSITION 4.3. *Every essentially simple Euclidean Jordan algebra $\mathcal{V}$ is a normal decomposition system with $X=\mathcal{V}$, $\mathcal{G} = \mathrm{Aut}(\mathcal{V})$, and $\gamma : \mathcal{V} \to \mathcal{V}$ defined as in* (6).

In this setting, two elements $x, y \in X$ commute if and only if there exists a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ such that

$$x = \sum_1^n \lambda_i(x) f_i \quad \text{and} \quad y = \sum_1^n \lambda_i(y) f_i. \tag{7}$$

We note that this is stronger than the operator commutativity of x and y. For example, in $\mathcal{V} = \mathcal{R}^2$, $x = (1,0)^\mathsf{T}$ and $y = (0,1)^\mathsf{T}$ operator commute but do not commute in the above sense.

Remark 4.4. Lim, Kim, and Faybusovich [12, Corollary 4] show that when $\mathcal{V}$ is a simple Euclidean Jordan algebra, $(\mathcal{V}, \mathcal{K}, \gamma)$ is a normal decomposition system, where $\mathcal{K}$ is the connected component of identity in $\mathrm{Aut}(\mathcal{V})$ and γ is defined as in (6).

In [10], Lewis provides numerous examples of normal decomposition systems. In particular, the algebras $\mathcal{S}^n$ and $\mathcal{H}^n$ (see section 2) are normal decomposition systems where $\mathcal{G}$ is the corresponding automorphism group, and $\gamma(X)$ is the diagonal matrix with $\lambda(X)$ as the diagonal. Another example is the space $M_{m,n}$ of all real $m \times n$ matrices with inner product $\langle X, Y \rangle := \mathrm{tr}(X^T Y)$, with $\mathcal{G}$ consisting of transformations of the form $X \mapsto UXV^T$, where U and V are orthogonal matrices, and $\gamma(X)$ is an $m \times n$ matrix with diagonal consisting of singular values of X written in the decreasing order and zeros elsewhere. These examples suggest the use of the phrase "simultaneous ordered diagonalization" to describe "commutation in $(X, \mathcal{G}, \gamma)$."

Proof of Theorem 1.3. Since a is a local minimizer of problem (3), we have

$$\Theta(a) + F(a) \leq \Theta(x) + F(x) \qquad \text{for all} \quad x \in N_a \cap \Omega,$$

where N_a denotes (some) neighborhood of a. Let A be an arbitrary element of $\mathcal{G}$. As Ω is convex and $\mathcal{G}$-invariant, we have, for all positive t near zero, $(1-t)a + tAa \in N_a \cap \Omega$. Thus,

$$\Theta(a) + F(a) \leq \Theta\Big((1-t)a + tAa\Big) + F\Big((1-t)a + tAa\Big)$$

for all positive t near 0. As F is convex and $\mathcal{G}$-invariant,

$$F\Big((1-t)a + tAa\Big) \leq (1-t)F(a) + tF(Aa) = (1-t)F(a) + tF(a) = F(a);$$

hence,

$$\Theta(a) \leq \Theta\Big((1-t)a + tAa\Big)$$

for all positive t near 0. This implies that $\langle \Theta'(a),\, Aa - a \rangle \geq 0$, that is, $\langle \Theta'(a),\, Aa \rangle \geq \langle \Theta'(a),\, a \rangle$. Now let $b := -\Theta'(a)$ so that $\langle b,\, Aa \rangle \leq \langle b,\, a \rangle$. Then, as $A \in \mathcal{G}$ is arbitrary, we have

$$\max_{A \in \mathcal{G}} \langle b,\, Aa \rangle \leq \langle b,\, a \rangle .$$

Using Proposition 4.2, we see that $\langle \gamma(b),\, \gamma(a) \rangle \leq \langle b,\, a \rangle$. Since the reverse inequality always holds in a normal decomposition system, the above inequality turns into an equality. By Proposition 4.2, a and b commute in $(X,\, \mathcal{G},\, \gamma)$. ∎

Remark 4.5. One might ask if the commutativity of a and $-\Theta'(a)$ in the above theorem could be replaced by that of a and $\Theta'(a)$. To answer this, we consider $X = \mathcal{R}^2$ with the usual inner product, let $\mathcal{G}$ be the set of all 2×2 signed permutation matrices (having exactly one nonzero entry, either 1 or -1, in each row/column), and $\gamma(x) = |x|^{\downarrow}$ (which is the vector of absolute values of entries of x written in decreasing order). Let $a = (-1, 1)^{\mathsf{T}}$, $b = (3, -2)^{\mathsf{T}}$, $c = -b$, $\Omega :=$ convex-hull$\{Aa : A \in \mathcal{G}\}$, $\Theta(x) = \langle b, x \rangle$, and $F = 0$. Then, it is easy to see that a minimizes Θ over Ω and commutes with $-b$ (which is $-\Theta'(a)$) but does not commute with b in $(X, \mathcal{G}, \gamma)$. We further note that a maximizes the map $x \mapsto \Psi(x) := \langle c, x \rangle$ over Ω but does not commute with $-\Psi'(a)$ in $(X, \mathcal{G}, \gamma)$.

Remark 4.6. We note that the conclusion in Theorem 1.2 is the same whether we consider a minimization or a maximization problem. However, in Theorem 1.3, the conclusion is specific to minimization problems. If in Theorem 1.3 we retain the convexity of Ω and F but consider a maximization problem, we may not get the commutativity of a and $-\Theta'(a)$ in $(X, \mathcal{G}, \gamma)$; see the example in the above remark. Also, if we assume convexity of Ω and concavity of F and consider a maximization problem, we would get the commutativity of a and $\Theta'(a)$ in $(X, \mathcal{G}, \gamma)$.

We now state analogues of Theorem 3.8 and Corollary 3.9 in normal decomposition systems.

THEOREM 4.7. *Suppose $\Omega \subseteq X$ and $F : X \to \mathcal{R}$ are convex and $\mathcal{G}$-invariant. Let $G : X \to X$ be arbitrary. Consider the variational inequality problem VI(G, Ω, F) on X: Find $x^* \in \Omega$ such that*

$$\langle G(x^*), x - x^* \rangle + F(x) - F(x^*) \geq 0 \quad \text{for all } x \in \Omega.$$

If a solves VI(G, Ω, F), then a commutes with $-G(a)$ in $(X, \mathcal{G}, \gamma)$.

When $\Omega = K$ is a closed convex cone and $F = 0$, we write CP(G, K) for VI(G, Ω, F).

COROLLARY 4.8. *Suppose K is closed convex cone in X that is $\mathcal{G}$-invariant and let $G : \mathcal{V} \to \mathcal{V}$ be arbitrary. If a solves the cone complementarity problem CP(G, K), then a commutes with $-G(a)$ in $(X, \mathcal{G}, \gamma)$.*

Remark 4.9. The above corollary yields the following: *Suppose K is a closed convex cone in X that is $\mathcal{G}$-invariant. If $a \in K$ and $b \in K^*$ satisfy $\langle a, b \rangle = 0$, then a and $-b$ commute in $(X, \mathcal{G}, \gamma)$.*

Remark 4.10. We specialize the above remark to essentially simple algebras. Let $\mathcal{V}$ be such an algebra and let K be a spectral cone (which is a closed convex cone that is spectral) in $\mathcal{V}$. If $a \in K$ and $b \in K^*$ satisfy $\langle a, b \rangle = 0$, then there exists a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ such that

$$a = \sum_1^n \lambda_i(a) f_i, \quad b = \sum_1^n \lambda_{n+1-i}(b) f_i, \quad \text{and} \quad \sum_1^n \lambda_i(a)\, \lambda_{n+1-i}(b) = 0.$$

This comes from (7) by noting $-\lambda_i(-b) = \lambda_{n+1-i}(b)$ and $\langle a, b \rangle = 0$.

Remark 4.11. Another consequence of Remark 4.9 is the following: Suppose $(X, \mathcal{G}, \gamma)$ is a normal decomposition system where

$$\langle \gamma(x), \gamma(y) \rangle = 0 \Rightarrow x = 0 \text{ or } y = 0.$$

(We note that $M_{m,n}$ and the system considered in Remark 4.5 are such systems.) *If K is a closed convex cone in X that is $\mathcal{G}$-invariant, then $K = \{0\}$ or X.* This can be seen as follows. Suppose K is different from $\{0\}$ and X. Let a be a nonzero element in the boundary of K. By an application of the supporting hyperplane theorem, we can find a nonzero $b \in K^*$ such that $\langle a, b\rangle = 0$. By Remark 4.9, a and $-b$ commute in $(X, \mathcal{G}, \gamma)$, hence, $a = A\gamma(a)$, $-b = A\gamma(-b)$ for some $A \in \mathcal{G}$. Then, $\langle \gamma(a), \gamma(-b)\rangle = \langle a, -b\rangle = 0$. It follows that $a = 0$ or $b = 0$, leading to a contradiction.

5. Appendix. Here, we describe a result on the automorphism group of a Euclidean Jordan algebra which is written as a product of simple algebras. While this result can be deduced from Theorem VI.18 in [9], for completeness, we present a direct (perhaps, elementary) proof. Using this result, we show that a Euclidean Jordan algebra $\mathcal{V}$ is essentially simple if and only if every weakly spectral set in $\mathcal{V}$ is spectral.

Consider a (general) Euclidean Jordan algebra $\mathcal{V}$. We assume that $\mathcal{V}$ is product of distinct nonisomorphic simple algebras $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ (with possible repetitions). Regrouping the factors, we assume that

$$\mathcal{V} = \Big(\mathcal{V}_1 \times \mathcal{V}_1 \times \cdots \times \mathcal{V}_1\Big) \times \Big(\mathcal{V}_2 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_2\Big) \times \cdots \times \Big(\mathcal{V}_m \times \mathcal{V}_m \times \cdots \times \mathcal{V}_m\Big). \tag{8}$$

By letting $\mathcal{W}_i := \mathcal{V}_i \times \mathcal{V}_i \times \cdots \times \mathcal{V}_i$, we write

$$\mathcal{V} = \mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_m. \tag{9}$$

THEOREM 5.1.

$$\text{Aut}(\mathcal{V}) = \text{Aut}(\mathcal{W}_1) \times \text{Aut}(\mathcal{W}_2) \times \cdots \times \text{Aut}(\mathcal{W}_m).$$

Moreover, any automorphism ϕ of $\text{Aut}(\mathcal{W}_i)$ *has the following form:*

$$\phi = \Big(\phi_1, \phi_2, \ldots, \phi_k\Big) \circ \sigma,$$

where k is the number of factors in $\mathcal{W}_i$, $\phi_j \in \text{Aut}(\mathcal{V}_i)$, $j = 1, 2, \ldots, k$, and σ is a $k \times k$ permutation matrix.

Note. The explicit form of the automorphism ϕ written with a permutation σ is

$$\phi(x) = \Big(\phi_1(x_{\sigma(1)}), \phi_2(x_{\sigma(2)}), \ldots, \phi_k(x_{\sigma(k)})\Big) \text{ for all } x = (x_1, x_2, \ldots, x_k) \in \mathcal{V}_i \times \mathcal{V}_i \times \cdots \times \mathcal{V}_i.$$

Before giving a proof, we present two lemmas. In what follows, we write $\dim(X)$ for the dimension of a space X.

LEMMA 5.2. *Suppose that $\mathcal{V}$ and $\mathcal{W}$ are simple Euclidean Jordan algebras and $A : \mathcal{V} \to \mathcal{W}$ is a nonzero Jordan homomorphism. Then,*

(i) $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$,

(ii) *if* $\dim(\mathcal{V}) = \dim(\mathcal{W})$, *$A$ is an isomorphism,*

(iii) *if* $\dim(\mathcal{V}) < \dim(\mathcal{W})$, *then zero is the only homomorphism from $\mathcal{W}$ to $\mathcal{V}$.*

Proof. (i) Since the kernel of a homomorphism is an ideal of $\mathcal{V}$ and $\mathcal{V}$ is simple, we see that A is either zero or one-to-one. Since our A is nonzero, its kernel is $\{0\}$; hence it is one-to-one and so $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$.

(ii) When $\dim(\mathcal{V}) = \dim(\mathcal{W})$, this A is also onto; hence it is an isomorphism.

(iii) Assume $\dim(\mathcal{V}) < \dim(\mathcal{W})$. If there is a nonzero Jordan homomorphism from $\mathcal{W}$ to $\mathcal{V}$, by (i), $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$. This is a contradiction. ∎

LEMMA 5.3. *Consider a product Euclidean Jordan algebra* $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \cdots \times \mathcal{Z}_m$. *Let* $A : \mathcal{Z} \to \mathcal{Z}$ *be a linear transformation written in the matrix form* $A = [\,A_{ij}]$, *where* $A_{ij} : \mathcal{Z}_j \to \mathcal{Z}_i$ *is linear. If* A *is a Jordan homomorphism, then so is* A_{ij} *for any* i, j. *Furthermore,* $A_{ik}^{\mathsf{T}} A_{il} = 0$ *for all* i *and* $k \neq l$.

Proof. For $x, y \in \mathcal{Z}$, we have $A(x \circ y) = Ax \circ Ay$. Taking $x = (0, \ldots, 0, x_j, 0, \ldots, 0)^{\mathsf{T}}$ and $y = (0, \ldots, 0, y_j, 0, \ldots, 0)^{\mathsf{T}}$, we get, for any i, j, $A_{ij}(x_j \circ y_j) = A_{ij}x_j \circ A_{ij}y_j$. This proves that A_{ij} is a homomorphism. Now suppose $k \neq l$ and let $x = (0, \ldots, 0, x_k, 0, \ldots, 0)^{\mathsf{T}}$ and $y = (0, \ldots, 0, y_l, 0, \ldots, 0)^{\mathsf{T}}$. Then $A(x \circ y) = Ax \circ Ay$ yields, $0 = A_{ik}x_k \circ A_{il}y_l$. This leads to $\langle A_{ik}x_k, A_{il}y_l\rangle = 0$ and to $\langle x_k, A_{ik}^{\mathsf{T}} A_{il}\, y_l\rangle = 0$. As x_k and y_l are arbitrary, we get $A_{ik}^{\mathsf{T}} A_{il} = 0$. ∎

Proof of Theorem 5.1. We may assume without loss of generality that all algebras involved carry canonical inner products. Since $\mathrm{Aut}(\mathcal{W}_1) \times \mathrm{Aut}(\mathcal{W}_2) \times \cdots \times \mathrm{Aut}(\mathcal{W}_m) \subseteq \mathrm{Aut}(\mathcal{V})$, it is enough to prove the reverse inclusion. As the result is obvious for $m = 1$, we assume that $m \geq 2$. Let $A \in \mathrm{Aut}(\mathcal{V})$. Since we are given $\mathcal{V}$ by (8) and (9), we think of A as a matrix of linear transformations $A = [A_{ij}]$, where each A_{ij} is a linear transformation from some $\mathcal{V}_k$ to $\mathcal{V}_l$. By partitioning this matrix, we write $A = [\mathbf{B}_{kl}]$, where (each block) $\mathbf{B}_{kl} : \mathcal{W}_l \to \mathcal{W}_k$ is a linear transformation. The main part of our proof consists in showing

$$\mathbf{B}_{1j} = 0 \text{ for all } j \geq 2. \tag{10}$$

Once we establish this, the same argument can then be used for A^{T} (which is the inverse of A as we are using the canonical inner product). This results in $\mathbf{B}_{j1} = 0$ for all $j \geq 2$. It then follows that $\mathbf{B}_{11} \in \mathrm{Aut}(\mathcal{W}_1)$ and A could be viewed as an element of $\mathrm{Aut}(\mathcal{W}_1) \times \mathrm{Aut}(\mathcal{W}_2 \times \mathcal{W}_3 \times \cdots \times \mathcal{W}_m)$. We then invoke the induction principle to see that $A \in \mathrm{Aut}(\mathcal{W}_1) \times \mathrm{Aut}(\mathcal{W}_2) \times \cdots \times \mathrm{Aut}(\mathcal{W}_m)$.

Now toward proving (10), we make the following claims.

CLAIM 5.4.

(a) *If for some* $k \neq l$ *(the off-diagonal block)* $\mathbf{B}_{kl}$ *has a nonzero entry, then* $\dim(\mathcal{V}_l) < \dim(\mathcal{V}_k)$ *and* $\mathbf{B}_{lk} = 0$.

(b) *If* A_{ij} *is a nonzero entry in (a diagonal block)* $\mathbf{B}_{kk}$, *then all other entries in the row/column of* A *containing* A_{ij} *are zero, that is,* $A_{il} = 0$ *and* $A_{li} = 0$ *for all* $l \neq j$.

To see (a), suppose A_{ij} is a nonzero entry in $\mathbf{B}_{kl}$. Then, A_{ij} from $\mathcal{V}_l$ to $\mathcal{V}_k$ is a nonzero homomorphism (by Lemma 5.3). As $\mathcal{V}_l$ and $\mathcal{V}_k$ are simple and nonisomorphic, by Lemma 5.2, $\dim(\mathcal{V}_l) < \dim(\mathcal{V}_k)$. Lemma 5.2 also shows that there cannot be a nonzero homomorphism from $\mathcal{V}_k$ to $\mathcal{V}_l$. Thus, every entry in $\mathbf{B}_{lk}$ is zero.

To see (b), suppose that A_{ij} is a nonzero entry in a diagonal block $\mathbf{B}_{kk}$. Then, by Lemma 5.3, $A_{ij} : \mathcal{V}_k \to \mathcal{V}_k$ is an isomorphism. From the same lemma, for $l \neq j$, we have $A_{ij}^{\mathsf{T}} A_{il} = 0$ and so $A_{il} = 0$. Thus, in the row containing A_{ij}, all other entries are zero. By working with the transpose of A, we see that the column containing A_{ij} is zero except for the A_{ij}th entry. This proves the claim.

CLAIM 5.5. *Suppose for some* l *with* $1 \leq l \leq m-1$, $\mathbf{B}_{12}, \mathbf{B}_{23}, \ldots, \mathbf{B}_{l\,l+1}$ *are nonzero. Then,* $l < m-1$ *and there exists* $j > l+1$ *such that* $\mathbf{B}_{l+1\,j}$ *is nonzero.*

If this were not true, then either $l = m-1$ or $l < m-1$ and $B_{l+1\,j} = 0$ for all $j > l+1$. From Claim 5.4(a), $\dim(\mathcal{V}_{l+1}) < \dim(\mathcal{V}_l) < \cdots < \dim(\mathcal{V}_1)$. From Lemma 5.2(iii), $\mathbf{B}_{l+1\,1}, \mathbf{B}_{l+1\,2}, \ldots, \mathbf{B}_{l+1\,l}$ are all zero. This means that in the matrix with entries $\mathbf{B}_{ij}$, in the $l+1$ row, all entries except possibly $\mathbf{B}_{l+1\,l+1}$, are zero. As

A is invertible, this lone entry $\mathbf{B}_{l+1\,l+1}$ cannot be zero. In fact, no row in the matrix $\mathbf{B}_{l+1\,l+1}$ can be zero. By Claim 5.4(b), each row of $\mathbf{B}_{l+1\,l+1}$ contains exactly one nonzero entry. This implies that in the square matrix $\mathbf{B}_{l+1\,l+1}$, each column will also contain exactly one nonzero entry. By Claim 5.4(b), all columns of $\mathbf{B}_{l\,l+1}$ will be zero. This contradicts our assumption that $\mathbf{B}_{l\,l+1}$ is nonzero. This proves our claim.

Now suppose, if possible, (10) is false so that $\mathbf{B}_{1j} \neq 0$ for some $j \geq 2$. We may assume, by relabeling, that $j = 2$, so $\mathbf{B}_{12}$ is nonzero. By Claim 5.5 (with $l = 1$), $2 < m$ and there exists $j > 2$ such that $\mathbf{B}_{2j}$ is nonzero. Relabeling, we may assume that $j = 3$ so that $\mathbf{B}_{23}$ is nonzero. We can use Claim 5.5 again, to see that $\mathbf{B}_{34}$ is nonzero, etc. Claim 5.5 allows us to repeat this process; however, as m is finite, this cannot continue forever. Thus, we reach a contradiction. Hence, (10) holds and, as discussed before, leads to the completion of the proof of the first part of the theorem.

We now come to the second part of the theorem. For simplicity, we let $i = 1$. We need to describe the matrix A which is now $\mathbf{B}_{11}$. As A is invertible, each row of $\mathbf{B}_{11}$ is nonzero. By Claim 5.4(b), each row of $\mathbf{B}_{11}$ contains exactly one nonzero entry which, by Lemma 5.2, is an automorphism of $\mathcal{V}_1$. (This means that each column of $\mathbf{B}_{11}$ also has the same property.) Thus, $\mathbf{B}_{11}$ can be regarded as a permutation of a diagonal matrix of transformations where each diagonal entry is an automorphism of $\mathcal{V}_1$. This gives the stated assertion. ∎

The following is immediate.

COROLLARY 5.6. *Suppose $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_m$, where $\mathcal{V}_1, \dots, \mathcal{V}_m$ are nonisomorphic simple algebras. Then,*

$$\mathrm{Aut}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}_1) \times \mathrm{Aut}(\mathcal{V}_2) \times \cdots \times \mathrm{Aut}(\mathcal{V}_m).$$

As an application of the above results, we prove the following.

THEOREM 5.7. *$\mathcal{V}$ is essentially simple if and only if every weakly spectral set in $\mathcal{V}$ is spectral.*

Proof. The "only if" part has been observed in [8, Theorem 2]. We prove the "if" part. Suppose, if possible, $\mathcal{V}$ is not essentially simple; let $\mathcal{V}$ be given by (8) and (9). We consider two cases.

Case 1: $\mathcal{V} = \mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_m$, $m \geq 2$. For $i = 1, 2, \dots, m$, let $\mathrm{rank}(\mathcal{W}_i) = n_i$, $\mathcal{P}_i$ denote the set of all primitive idempotents in $\mathcal{W}_i$, and let 0 denote the zero element in any $\mathcal{W}_i$. Since automorphisms map primitive idempotents to primitive idempotents, $\mathcal{P}_1$ is invariant under automorphisms of $\mathcal{W}_1$, and so, by Theorem 5.1, $\Omega := \mathcal{P}_1 \times \{0\} \times \{0\} \dots \times \{0\}$ is weakly spectral in $\mathcal{V}$. Let $c_1 \in \mathcal{P}_1$, $c_2 \in \mathcal{P}_2$,

$$x = \Big(c_1, 0, 0, \dots, 0\Big) \quad \text{and} \quad y = \Big(0, c_2, 0, 0 \dots, 0\Big).$$

As both x and y have eigenvalues 1 (appearing once) and 0 (appearing $n_1 + n_2 + \cdots + n_m - 1$ times), we see that $\lambda(x) = \lambda(y)$. However, $x \in \Omega$ while $y \notin \Omega$. Thus, Ω cannot be of the form $\lambda^{-1}(Q)$ for any (permutation invariant) set Q.

Case 2: $\mathcal{V} = \mathcal{W}_1 = \mathcal{V}_1 \times \mathcal{V}_1 \times \cdots \times \mathcal{V}_1$, where $\mathcal{V}_1$ is a simple algebra of rank at least 2 and the number of factors in this product, say, m, is more than one. Let $n = \mathrm{rank}(\mathcal{V}_1)$. In $\mathcal{R}^n$, let s_i denote the coordinate vector containing 1 in its ith slot and zeros elsewhere; let $Q = \{s_1, s_2, \dots, s_n\}$. As Q is permutation invariant, the set $\mathcal{P}_1 := \lambda^{-1}(Q)$ (which equals the set of all primitive elements in $\mathcal{V}_1$) is a spectral set in $\mathcal{V}_1$, where $\lambda : \mathcal{V}_1 \to \mathcal{R}^n$ is the eigenvalue map. As $\mathcal{P}_1$ is invariant under automorphisms of $\mathcal{V}_1$, an application of Theorem 5.1 shows that the set $\Omega :=$

$\mathcal{P}_1 \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_1$ is weakly spectral in $\mathcal{V}$. We now claim that Ω is not a spectral set in $\mathcal{V}$. Let e denote the unit vector in $\mathcal{V}_1$ and $\{e_1, e_2 \ldots, e_n\}$ be a Jordan frame in $\mathcal{V}_1$. Then, $\lambda(e_1) = (1, 0, 0, \ldots, 0)^\mathsf{T}$ and so the vector $x := (e_1, e_1, \ldots, e_1)$ in Ω has eigenvalues 1 (repeated m times) and 0 (repeated $m(n-1)$ times). When $m \leq n$, let $y := (e_1 + e_2 + \cdots + e_m, 0, 0, \ldots, 0) \in \mathcal{V}$. We see that $y \notin \Omega$ while $\lambda(y) = \lambda(x)$. On the other hand, when $n < m$, we write $m = nk + l$ with $0 \leq l < n$ and define $y := (e, e, \ldots, e, e_1 + e_2 + \cdots + e_l, 0, 0 \ldots, 0)$, where e is repeated k times. We see that $y \notin \Omega$ while $\lambda(y) = \lambda(x)$. Thus, Ω is not a spectral set. This completes the proof. ◻

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