

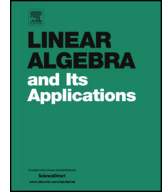


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## Spectral sets and functions on Euclidean Jordan algebras



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### ABSTRACT

Spectral sets (functions) in Euclidean Jordan algebras are generalizations of permutation invariant sets (respectively, functions) in  $\mathcal{R}^n$ . In this article, we study properties of such sets and functions and show how they are related to algebra automorphisms and majorization. We show that spectral sets/functions are indeed invariant under automorphisms, but the converse may not hold unless the algebra is  $\mathcal{R}^n$  or simple. We study Schur-convex spectral functions and provide some applications. We also discuss the transfer principle and a related metaformula.

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## 1. Introduction

This paper deals with some interconnections between spectral sets/functions, algebra automorphisms, and majorization in the setting of Euclidean Jordan algebras. Given a Euclidean Jordan algebra  $\mathcal{V}$  of rank  $n$ , a set  $E$  in  $\mathcal{V}$  is said to be a *spectral set* [1] if it is of the form

$$E = \lambda^{-1}(Q),$$

where  $Q$  is a permutation invariant set in  $\mathcal{R}^n$  and  $\lambda : \mathcal{V} \rightarrow \mathcal{R}^n$  is the eigenvalue map (that takes  $x$  to  $\lambda(x)$ , the vector of eigenvalues of  $x$  with entries written in the decreasing order). A function  $F : \mathcal{V} \rightarrow \mathcal{R}$  is said to be a *spectral function* [1] if it is of the form

$$F = f \circ \lambda,$$

where  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  is a permutation invariant function. How these are related to algebra automorphisms of  $\mathcal{V}$  (which are invertible linear transformations on  $\mathcal{V}$  that preserve the Jordan product) and majorization is the main focus of the paper.

The above concepts are generalizations of similar concepts that have been extensively studied in the setting of  $\mathcal{R}^n$  (where the concepts reduce to permutation invariant sets and functions) and in  $\mathcal{S}^n$  ( $\mathcal{H}^n$ ), the space of all  $n \times n$  real (respectively, complex) Hermitian matrices, see for example, [3–5,10–14,20], and the references therein. In the case of  $\mathcal{S}^n$  ( $\mathcal{H}^n$ ), spectral sets/functions are precisely those that are invariant under linear transformations of the form  $X \rightarrow UXU^*$ , where  $U$  is an orthogonal (respectively, unitary) matrix.

There are a few works that deal with spectral sets and functions on general Euclidean Jordan algebras. Baes [1] discusses some properties of  $Q$  that get transferred to  $E$  (such as closedness, openness, boundedness/compactness, and convexity) and properties of  $f$  that get transferred to  $F$  (such as convexity and differentiability). Sun and Sun [22] deal with the transferability of the semismoothness properties of  $f$  to  $F$ . Ramirez, Seeger, and Sossa [18] and Sossa [21] deal with a commutation principle and a number of applications.

In this paper, we present some new results on spectral sets and functions. In addition to giving some elementary characterization results, we show that spectral sets/functions are indeed invariant under automorphisms of  $\mathcal{V}$ , but the converse may not hold unless the algebra is  $\mathcal{R}^n$  or simple (e.g.,  $\mathcal{S}^n$  or  $\mathcal{H}^n$ ). We will also relate the concepts of spectral sets and functions to that of majorization. Given two elements  $x, y$  in  $\mathcal{V}$ , we say that  $x$  is *majorized by*  $y$  and write  $x \prec y$  if  $\lambda(x)$  is majorized by  $\lambda(y)$  in  $\mathcal{R}^n$  (see Section 2.1 for the definition); we say that  $x$  and  $y$  are *spectrally equivalent* and write  $x \sim y$  if  $\lambda(x) = \lambda(y)$  (or equivalently,  $x \prec y$  and  $y \prec x$ ). We show that spectral sets are characterized by the condition

$$x \sim y, y \in E \quad \Rightarrow \quad x \in E$$

and spectral functions are characterized by

$$x \sim y \quad \Rightarrow \quad F(x) = F(y).$$

Under the assumption of convexity, by replacing  $x \sim y$  by  $x \prec y$  and  $F(x) = F(y)$  by  $F(x) \leq F(y)$ , we derive similar characterizations. In particular, we observe that convex spectral functions are Schur-convex, that is,

$$x \prec y \quad \Rightarrow \quad F(x) \leq F(y),$$

and, as a consequence, show that

$$F(\Psi(x)) \leq F(x) \quad (x \in \mathcal{V})$$

whenever  $F$  is a convex spectral function and  $\Psi$  is a doubly stochastic transformation on  $\mathcal{V}$ . We also prove a converse of this latter statement.

We conclude the paper with a discussion on the *Transfer Principle* which asserts that (numerous) properties of  $Q$  (of  $f$ ) get transferred to  $\lambda^{-1}(Q)$  (respectively,  $f \circ \lambda$ ). We also prove a metaformula

$$\# \diamond = \diamond \#,$$

where  $\#$  is a set operation (such as the closure, interior, boundary, convex hull, etc.) and  $\diamond$  is the set operation defined by

$$Q^\diamond = \lambda^{-1}(Q) \text{ for } Q \subseteq \mathcal{R}^n \quad \text{and} \quad E^\diamond = \Sigma_n(\lambda(E)) \text{ for } E \subseteq \mathcal{V},$$

with  $\Sigma_n$  denoting the set of all  $n \times n$  permutation matrices.

The organization of the paper is as follows. In Section 2, we cover basic notation, definitions, and preliminary results. Sections 3 and 4 deal with some characterization and automorphism invariance properties of (convex) spectral sets and functions. Section 5 deals with Schur-convex spectral functions. Finally, in Section 6 we deal with the transfer principle and a metaformula.

## 2. Preliminaries

### 2.1. Some notation

An  $n \times n$  *permutation matrix* is a matrix obtained by permuting the rows of an  $n \times n$  identity matrix. Every row and column of such a matrix therefore contains a single 1 with 0s elsewhere. The set of all  $n \times n$  permutation matrices is denoted by  $\Sigma_n$ .

A set  $Q$  in  $\mathcal{R}^n$  is said to be *permutation invariant* if  $\sigma(x) \in Q$  whenever  $x \in Q$  and  $\sigma \in \Sigma_n$ . (We assume that the empty set vacuously satisfies this condition, hence

permutation invariant.) A function  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  is permutation invariant if  $f(\sigma(x)) = f(x)$  for all  $\sigma \in \Sigma_n$ . The word ‘symmetric’ is also used in the literature to describe permutation invariance of a set/function.

Recall that for two vectors  $u$  and  $v$  in  $\mathcal{R}^n$  with their decreasing rearrangements  $u^\downarrow$  and  $v^\downarrow$ , we say that  $u$  is *majorized* by  $v$  and write  $u \prec v$  if

$$\sum_{i=1}^k u_i^\downarrow \leq \sum_{i=1}^k v_i^\downarrow \text{ for } 1 \leq k \leq n-1 \quad \text{and} \quad \sum_{i=1}^n u_i^\downarrow = \sum_{i=1}^n v_i^\downarrow.$$

We have, by setting  $k=1$  and  $k=n-1$ ,

$$u \prec v \Rightarrow \max u_i \leq \max v_i \quad \text{and} \quad \min u_i \geq \min v_i. \quad (1)$$

## 2.2. Euclidean Jordan algebras

Throughout this paper,  $\mathcal{V}$  denotes a Euclidean Jordan algebra [7]. For  $x, y \in \mathcal{V}$ , we denote their inner product by  $\langle x, y \rangle$  and Jordan product by  $x \circ y$ . We let  $e$  denote the unit element in  $\mathcal{V}$  and  $\mathcal{V}_+ := \{x \circ x : x \in \mathcal{V}\}$  denote the corresponding *symmetric cone*. If  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are two Euclidean Jordan algebras, then,  $\mathcal{V}_1 \times \mathcal{V}_2$  becomes a Euclidean Jordan algebra under the Jordan and inner products, defined, respectively by  $(x_1, x_2) \circ (y_1, y_2) = (x_1 \circ y_1, x_2 \circ y_2)$  and  $\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$ . A similar definition is made for a product of several Euclidean Jordan algebras. Recall that a Euclidean Jordan algebra  $\mathcal{V}$  is *simple* if it is not a direct product of nonzero Euclidean Jordan algebras (or equivalently, if it does not contain any non-trivial ideal). It is known, see [7], that any nonzero Euclidean Jordan algebra is, in a unique way, a direct product of simple Euclidean Jordan algebras. Moreover, there are only five types of simple algebras, one of which is:  $\mathcal{S}^n$ , the algebra of  $n \times n$  real symmetric matrices, with inner and Jordan products given by

$$\langle X, Y \rangle = \text{tr}(XY) \quad \text{and} \quad X \circ Y = \frac{1}{2}(XY + YX).$$

The algebra  $\mathcal{R}^n$  is simply the product of  $n$  copies of  $\mathcal{S}^1$ . We say that an algebra is *essentially simple* if it is either  $\mathcal{R}^n$  or simple.

An element  $c \in \mathcal{V}$  is an *idempotent* if  $c^2 = c$ ; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say a finite set  $\{e_1, e_2, \dots, e_n\}$  of primitive idempotents in  $\mathcal{V}$  is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ when } i \neq j \quad \text{and} \quad \sum_{i=1}^n e_i = e.$$

Note that  $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$  whenever  $i \neq j$ .

It turns out that all Jordan frames in  $\mathcal{V}$  will have the same number of elements. This common number is the *rank* of  $\mathcal{V}$ .

**Proposition 1** (Spectral decomposition theorem [7]). Suppose  $\mathcal{V}$  is a Euclidean Jordan algebra of rank  $n$ . Then, for every  $x \in \mathcal{V}$ , there exist uniquely determined real numbers  $\lambda_1(x), \dots, \lambda_n(x)$  (called the eigenvalues of  $x$ ) and a Jordan frame  $\{e_1, \dots, e_n\}$  such that

$$x = \lambda_1(x)e_1 + \dots + \lambda_n(x)e_n.$$

Conversely, given any Jordan frame  $\{e_1, \dots, e_n\}$  and real numbers  $\lambda_1, \lambda_2, \dots, \lambda_n$ , the sum  $\lambda_1 e_1 + \dots + \lambda_n e_n$  defines an element of  $\mathcal{V}$  whose eigenvalues are  $\lambda_1, \lambda_2, \dots, \lambda_n$ .

We note that

$$\mathcal{V}_+ = \{x \in \mathcal{V} : \lambda_i(x) \geq 0 \quad \forall i = 1, 2, \dots, n\}.$$

We define the *trace* of an element  $x \in \mathcal{V}$  by:

$$\text{tr}(x) := \sum_{i=1}^n \lambda_i(x).$$

From now on,  $\lambda(x)$  denotes the vector of eigenvalues of  $x$  with components written in the decreasing order. Note that the mapping  $x \mapsto \lambda(x)$  is well-defined and continuous, see [1].

Recall by definition,

$$x \sim y \text{ in } \mathcal{V} \Leftrightarrow \lambda(x) = \lambda(y) \text{ in } \mathcal{R}^n.$$

As this is an equivalence relation on  $\mathcal{V}$ , we define the equivalence class of an  $x \in \mathcal{V}$  by:

$$[x] := \{y \in \mathcal{V} : y \sim x\}.$$

Let  $\phi$  and  $\Psi$  be linear transformations on  $\mathcal{V}$ . We say that

- $\phi$  is an *algebra automorphism* (or ‘automorphism’ for short) if it is invertible and

$$\phi(x \circ y) = \phi(x) \circ \phi(y) \quad \forall x, y \in \mathcal{V}.$$

The set of all automorphisms of  $\mathcal{V}$  is denoted by  $\text{Aut}(\mathcal{V})$ . The automorphisms of  $\mathcal{R}^n$  are precisely permutation matrices. Also, automorphisms of  $\mathcal{S}^n(\mathcal{H}^n)$  are of the form  $X \rightarrow UXU^*$ , where  $U$  is an orthogonal (respectively, unitary) matrix. See [8] for an explicit description of automorphisms of  $\mathcal{L}^n$ .

We note that automorphisms map Jordan frames to Jordan frames. Thus, eigenvalues of an element remain the same under the action of an automorphism. In particular,

$$\phi(x) \sim x \quad \forall x \in \mathcal{V}, \phi \in \text{Aut}(\mathcal{V}).$$

We say that a set  $\Omega$  in  $\mathcal{V}$  is *invariant under automorphisms* (or *automorphism invariant*) if

$$\phi(\Omega) \subseteq \Omega \quad \forall \phi \in \text{Aut}(\mathcal{V});$$

similarly, a function  $F : \mathcal{V} \rightarrow \mathcal{R}$  is invariant under automorphisms (or automorphism invariant) if

$$F \circ \phi = F \quad \forall \phi \in \text{Aut}(\mathcal{V}).$$

If  $\phi$  is an automorphism of  $\mathcal{V}$ , then

$$\phi(\mathcal{V}_+) = \mathcal{V}_+.$$

Linear transformations satisfying the latter property are called *cone automorphisms*; the set of all such transformations will be denoted by  $\text{Aut}(\mathcal{V}_+)$ .

- $\Psi$  is *doubly stochastic* [9] if it is
  - (i) positive, i.e.,  $\Psi(\mathcal{V}_+) \subseteq \mathcal{V}_+$ ,
  - (ii) unital, i.e.,  $\Psi(e) = e$ , and
  - (iii) trace preserving, i.e.,  $\text{tr}(\Psi(x)) = \text{tr}(x)$  for all  $x \in \mathcal{V}$ .

When  $\mathcal{V} = \mathcal{R}^n$ , such a  $\Psi$  is a *doubly stochastic matrix* (which is an  $n \times n$  nonnegative matrix with each row/column sum one).

We write  $\text{DS}(\mathcal{V})$  for the set of all doubly stochastic transformations on  $\mathcal{V}$ .

The following result will be used in many of our theorems, particularly, in the converse statements. Here and elsewhere, we implicitly assume that a Jordan frame, in addition to being a set, is also an ordered listing of its objects. Recall that  $\mathcal{V}$  is essentially simple if it is either  $\mathcal{R}^n$  or simple.

**Proposition 2.** *Let  $\mathcal{V}$  be essentially simple. If  $\{e_1, \dots, e_n\}$  and  $\{e'_1, \dots, e'_n\}$  are any two Jordan frames in  $\mathcal{V}$ , then there exists  $\phi \in \text{Aut}(\mathcal{V})$  such that  $\phi(e_i) = e'_i$  for all  $i = 1, \dots, n$ . In particular, if  $x \sim y$  in  $\mathcal{V}$ , then there exists  $\phi \in \text{Aut}(\mathcal{V})$  such that  $x = \phi(y)$ .*

**Proof.** We observe that in  $\mathcal{R}^n$ , up to permutations, there is only one Jordan frame, namely, the standard coordinate system. In this case, we can take  $\phi$  to be a permutation matrix. If  $\mathcal{V}$  is simple, the existence of an automorphism mapping one Jordan frame to another comes from Theorem IV.2.5, [7]. The last statement comes from an application of the spectral decomposition theorem (Proposition 1).  $\square$

It is interesting to note that the converse in the above result holds: If every Jordan frame can be mapped onto another by an automorphism, then the algebra is essentially simple. While we do not require this result in the main body of the paper, we provide a proof in the Appendix.

We also recall the following result from [9].

**Proposition 3.** *Let  $\mathcal{V}_1$  and  $\mathcal{V}_2$  be two non-isomorphic simple Euclidean Jordan algebras. If  $\phi \in \text{Aut}(\mathcal{V}_1 \times \mathcal{V}_2)$ , then  $\phi$  is of the form  $(\phi_1, \phi_2)$  for some  $\phi_i \in \text{Aut}(\mathcal{V}_i)$ ,  $i = 1, 2$ , that is,*

$$\phi(x) = (\phi_1(x_1), \phi_2(x_2)) \quad \forall x = (x_1, x_2) \in \mathcal{V}_1 \times \mathcal{V}_2.$$

### 2.3. Majorization in $\mathcal{R}^n$ and Euclidean Jordan algebras

We start by recalling two classical results in matrix theory. The first one is due to Hardy, Littlewood, and Pólya and the second one is due to Birkhoff.

**Proposition 4** ([2], Theorem II.1.10). *Let  $x, y \in \mathcal{R}^n$ . A necessary and sufficient condition for  $x \prec y$  is that there exists a doubly stochastic matrix  $A$  such that  $x = A(y)$ .*

**Proposition 5** ([2], Theorem II.2.3). *The set of all  $n \times n$  doubly stochastic matrices is a compact convex set whose extreme points are permutation matrices. In particular, every doubly stochastic matrix is a convex combination of permutation matrices.*

The next proposition, which is essential for Section 4, is somewhat classical and well known. It easily follows from the above two propositions.

#### Proposition 6.

(i) *If  $Q$  is convex and permutation invariant in  $\mathcal{R}^n$ , then*

$$u \prec v, v \in Q \quad \Rightarrow \quad u \in Q.$$

(ii) *If  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  is convex and permutation invariant, then  $f$  is Schur-convex, that is,*

$$u \prec v \quad \Rightarrow \quad f(u) \leq f(v).$$

The result below describes a connection between majorization, automorphisms, and doubly stochastic transformations in the setting of Euclidean Jordan algebras. Recall, by definition,

$$x \prec y \text{ in } \mathcal{V} \quad \Leftrightarrow \quad \lambda(x) \prec \lambda(y) \text{ in } \mathcal{R}^n.$$

**Proposition 7** (Gowda [9]). *For  $x, y \in \mathcal{V}$ , consider the following statements:*

- (a)  $x = \Phi(y)$ , where  $\Phi$  is a convex combination of automorphisms of  $\mathcal{V}$ .
- (b)  $x = \Psi(y)$ , where  $\Psi$  is doubly stochastic on  $\mathcal{V}$ .
- (c)  $x \prec y$ .

Then,  $(a) \Rightarrow (b) \Rightarrow (c)$ . Furthermore, when  $\mathcal{V}$  is essentially simple, reverse implications hold.

When  $\mathcal{V}$  is a simple Euclidean Jordan algebra, it is known [17] that  $\lambda(x + y) \prec \lambda(x) + \lambda(y)$  for all  $x, y \in \mathcal{V}$ . Here is a generalization of this result.

**Proposition 8.** *Let  $\mathcal{V}$  be a Euclidean Jordan algebra of rank  $n$ . Then for  $x, y \in \mathcal{V}$ , there exist doubly stochastic matrices  $A$  and  $B$  such that*

$$\lambda(x + y) = A\lambda(x) + B\lambda(y). \quad (2)$$

When  $\mathcal{V}$  is simple, we can take  $A = B$ .

**Proof.** As the result for a simple algebra is known, we assume that  $\mathcal{V}$  is non-simple, that is,  $\mathcal{V}$  is a product of simple algebras. For simplicity, we assume  $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ , where  $\mathcal{V}_1, \mathcal{V}_2$  are simple algebras of rank  $n_1, n_2$ , respectively. Now, let  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$ , where  $x_i, y_i \in \mathcal{V}_i$  for  $i = 1, 2$ . Define  $u_i = \lambda(x_i) \in \mathcal{R}^{n_i}$  for  $i = 1, 2$  and  $u = (u_1, u_2) \in \mathcal{R}^{n_1+n_2} = \mathcal{R}^n$ . As eigenvalues of  $x$  come from the eigenvalues of  $x_1$  and  $x_2$ , we have  $\lambda(x) = \sigma_1(u)$  for some  $\sigma_1 \in \Sigma_n$ . Similarly, there exist  $\sigma_2, \sigma_3 \in \Sigma_n$  such that

$$\begin{array}{ll} \lambda(y) = \sigma_2(v) & \text{where } v = (v_1, v_2), v_i = \lambda(y_i) \\ \lambda(x + y) = \sigma_3(w) & w = (w_1, w_2), w_i = \lambda(x_i + y_i) \end{array} \quad \text{for } i = 1, 2.$$

Since  $\mathcal{V}_1, \mathcal{V}_2$  are simple,

$$w_i = \lambda(x_i + y_i) \prec \lambda(x_i) + \lambda(y_i) = u_i + v_i \quad \text{for } i = 1, 2.$$

Thus, there are doubly stochastic matrices  $C_i$  on  $\mathcal{R}^{n_i}$  such that  $w_i = C_i(u_i + v_i)$  for  $i = 1, 2$ . So,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}.$$

Letting  $C$  denote the block diagonal matrix that appears above, one can easily verify that  $C$  is doubly stochastic on  $\mathcal{R}^n$ . Moreover, we have  $w = C(u + v) = C(u) + C(v)$ . Now, define matrices  $A$  and  $B$  by

$$A = \sigma_3 C \sigma_1^{-1} \quad \text{and} \quad B = \sigma_3 C \sigma_2^{-1}.$$

As products of doubly stochastic matrices are doubly stochastic,  $A$  and  $B$  are doubly stochastic. Finally, we have  $A\lambda(x) + B\lambda(y) = \lambda(x + y)$ .  $\square$



### 3. Spectral sets

Throughout this paper,  $\mathcal{V}$  is assumed to be of rank  $n$ .

**Definition 1.** A set  $E$  in  $\mathcal{V}$  is a *spectral set* if there exists a permutation invariant set  $Q$  in  $\mathcal{R}^n$  such that  $E = \lambda^{-1}(Q)$ .

It is easy to see that arbitrary union and/or intersection of spectral sets is spectral. So is the complement of a spectral set. Clearly, the symmetric cone  $\mathcal{V}_+$  is a spectral set in  $\mathcal{V}$  as it corresponds to  $\mathcal{R}_+^n$ .

In this section, we describe some properties of spectral sets. We characterize spectral sets via spectral equivalence. First, we introduce a ‘diamond’ operation on subsets of  $\mathcal{R}^n$  and  $\mathcal{V}$  and state a simple proposition that will facilitate our study. This operation is motivated by the question of recovering  $Q$  from  $E$ . We note that in the case of  $\mathcal{V} = \mathcal{S}^n$ , if  $E$  is a spectral set, then the corresponding  $Q$  is given by  $Q = \{u \in \mathcal{R}^n : \text{Diag}(u) \in E\}$ , where  $\text{Diag}(u)$  is a diagonal matrix with  $u$  as the diagonal [11].

**Definition 2.** For a set  $Q$  in  $\mathcal{R}^n$ , we define the set  $Q^\diamond$  in  $\mathcal{V}$  by

$$Q^\diamond := \lambda^{-1}(Q) = \{x \in \mathcal{V} : \lambda(x) \in Q\}. \quad (3)$$

For a set  $E$  in  $\mathcal{V}$ , we let  $E^\diamond$  in  $\mathcal{R}^n$  be

$$E^\diamond := \Sigma_n(\lambda(E)) = \{u \in \mathcal{R}^n : u^\downarrow = \lambda(x) \text{ for some } x \in E\}. \quad (4)$$

For simplicity, we write  $Q^{\diamond\diamond}$  in place of  $(Q^\diamond)^\diamond$ , etc.

**Proposition 9.** *The following statements hold:*

- (i) *For any  $Q$  that is permutation invariant in  $\mathcal{R}^n$ ,  $Q^\diamond$  is a spectral set in  $\mathcal{V}$  and  $Q^{\diamond\diamond} = Q$ .*
- (ii) *For any set  $E$  in  $\mathcal{V}$ ,  $E^\diamond$  is permutation invariant in  $\mathcal{R}^n$ .*
- (iii)  *$E^{\diamond\diamond} = \{x \in \mathcal{V} : x \sim y \text{ for some } y \in E\} = \bigcup_{y \in E} [y]$ , where  $[x]$  is the equivalence class of  $x$  induced by  $\sim$ .*

**Proof.** For any set  $Q$  in  $\mathcal{R}^n$ , we define the *core* of  $Q$  by

$$Q^\downarrow := \{u^\downarrow : u \in Q\}.$$

We immediately note the following when  $Q$  is permutation invariant:

$$Q^\downarrow \subseteq Q, \quad Q = \Sigma_n(Q^\downarrow), \quad \text{and} \quad \lambda(\lambda^{-1}(Q)) = Q^\downarrow.$$

(i): Let  $Q$  be permutation invariant. By definition,  $E := Q^\diamond$  is a spectral set. We have

$$(Q^\diamond)^\diamond = E^\diamond = \Sigma_n(\lambda(E)) = \Sigma_n(\lambda(\lambda^{-1}(Q))) = \Sigma_n(Q^\downarrow) = Q.$$

(ii): Since  $\Sigma_n$  is a group,  $\Sigma_n(E^\diamond) = \Sigma_n(\Sigma_n(\lambda(E))) = \Sigma_n(\lambda(E)) = E^\diamond$ . Thus,  $E^\diamond$  is permutation invariant.

(iii): This follows from  $(E^\diamond)^\diamond = \{x \in \mathcal{V} : \lambda(x) \in E^\diamond\} = \{x \in \mathcal{V} : \lambda(x) = \lambda(y) \text{ for some } y \in E\}$ .  $\square$

**Example 1.** This example shows that in Item (i) above, one needs the permutation invariance property of  $Q$  to get the equality  $Q^{\diamond\diamond} = Q$ . Let  $\mathcal{V} = \mathcal{S}^2$  and  $Q_1 = \{u \in \mathcal{R}_+^2 : u_1 \geq u_2\}$ . Then, it is easy to see that  $Q_1^\diamond = \mathcal{S}_+^2$  and so  $Q_1^{\diamond\diamond} = \mathcal{R}_+^2$ . Thus, we have  $Q_1 \subsetneq Q_1^{\diamond\diamond}$ . On the other hand, consider  $Q_2 = \{u \in \mathcal{R}_+^2 : u_1 \leq u_2\}$ . Then,  $Q_2^\diamond = \{aI : a \geq 0\}$ ; therefore  $Q_2^{\diamond\diamond} = \{u \in \mathcal{R}_+^2 : u_1 = u_2\}$ . This means  $Q_2^{\diamond\diamond} \subsetneq Q_2$ .

We now characterize spectral sets.

**Theorem 1.** *The following are equivalent for any set  $E$  in  $\mathcal{V}$ .*

- (a)  $E$  is spectral.
- (b)  $x \sim y, y \in E \Rightarrow x \in E$ .
- (c)  $E^{\diamond\diamond} = E$ .

**Proof.** (a)  $\Rightarrow$  (b): Let  $E = \lambda^{-1}(Q)$ , where  $Q$  is permutation invariant. If  $x \sim y$  with  $y \in E$ , then  $\lambda(x) = \lambda(y) \in Q$ . Hence,  $x \in \lambda^{-1}(Q) = E$ .

(b)  $\Leftrightarrow$  (c): This comes from Item (iii) in Proposition 9.

(c)  $\Rightarrow$  (a): When  $E^{\diamond\diamond} = E$ , we let  $Q := E^\diamond$ . By Item (ii) in Proposition 9,  $Q$  is permutation invariant; we see that  $E = Q^\diamond = \lambda^{-1}(Q)$  is spectral.  $\square$

**Remarks.** The above two results show that  $Q \mapsto Q^\diamond$  sets up is a one-to-one correspondence between permutation invariant sets in  $\mathcal{R}^n$  and spectral sets in  $\mathcal{V}$ .

We now describe spectral sets via automorphisms. This result explains why in  $\mathcal{S}^n$  or  $\mathcal{H}^n$ , spectral sets are completely characterized by automorphism invariance.

**Theorem 2.** *Every spectral set in  $\mathcal{V}$  is invariant under automorphisms. Converse holds when  $\mathcal{V}$  is essentially simple.*

**Proof.** Suppose  $E$  is a spectral set,  $x \in E$ , and  $\phi \in \text{Aut}(\mathcal{V})$ . As eigenvalues remain the same under the action of automorphisms, we see that  $\phi(x) \sim x$ . By Item (b) in Theorem 1,  $\phi(x) \in E$ . This proves that  $E$  is invariant under automorphisms.

To see the converse, assume that  $E$  is invariant under automorphisms and  $\mathcal{V}$  is essentially simple. We verify Item (b) in Theorem 1 to show that  $E$  is spectral. To this end, let  $x \sim y, y \in E$ . Then,  $\lambda(x) = \lambda(y)$ . By Proposition 2, there exists  $\phi \in \text{Aut}(\mathcal{V})$  such that

$x = \phi(y)$ . As  $E$  is invariant under automorphisms, we must have  $x \in E$ . This concludes the proof.  $\square$

**Example 2.** In the theorem above, the converse may not hold for general algebras. To see this, let  $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$  and  $E = \mathcal{R} \times \mathcal{S}_+^2$ . From Proposition 3, every automorphism  $\phi$  on  $\mathcal{V}$  is of the form  $\phi = (\phi_1, \phi_2)$ , where  $\phi_1 \in \text{Aut}(\mathcal{R})$ ,  $\phi_2 \in \text{Aut}(\mathcal{S}^2)$ . It is easy to check that  $E$  is invariant under automorphisms. Now consider two elements,

$$x = \left(0, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \notin E \quad \text{and} \quad y = \left(-1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \in E.$$

As  $\lambda(x) = \lambda(y) = (1, 0, -1)^\top$ , we have  $x \sim y$ . Thus,  $E$  violates condition (b) in Theorem 1. Hence,  $E$  is not a spectral set. It is easy to verify that  $x \in E^{\diamond\diamond}$ . Thus,  $E \neq E^{\diamond\diamond}$  even though  $E$  is invariant under automorphisms.

We now characterize spectral sets that are convex.

**Theorem 3.** For a set  $E$  in  $\mathcal{V}$ , consider the following:

- (a)  $E = \lambda^{-1}(Q)$ , where  $Q$  is permutation invariant and convex in  $\mathcal{R}^n$ .
- (b)  $E$  is convex, and  $x \prec y, y \in E \Rightarrow x \in E$ .
- (c)  $E$  is convex, and  $x \sim y, y \in E \Rightarrow x \in E$ .
- (d)  $E$  is convex and invariant under automorphisms.

Then (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c) and (c)  $\Rightarrow$  (d). Furthermore, all statements are equivalent when  $\mathcal{V}$  is essentially simple.

**Proof.** (a)  $\Rightarrow$  (b): Let  $E = \lambda^{-1}(Q)$ , where  $Q$  is permutation invariant and convex in  $\mathcal{R}^n$ . The convexity of  $E$  has already been proved in Theorem 27, [1]. For completeness, we provide a (slightly different) proof. Let  $x, y \in E$  and  $t \in [0, 1]$ . By Proposition 8,  $\lambda(tx + (1-t)y) = tu + (1-t)v$ , where  $u = A(\lambda(x))$ ,  $v = B(\lambda(y))$  for some doubly stochastic matrices  $A$  and  $B$ . By Proposition 6, we see that  $u, v \in Q$ . As  $Q$  is convex,  $\lambda(tx + (1-t)y) \in Q$ . Thus,  $tx + (1-t)y \in E$ . This proves the convexity of  $E$ .

Now, suppose  $x \prec y$  and  $y \in E$ . Then,  $\lambda(x) \prec \lambda(y)$  and  $\lambda(y) \in Q$ . By Proposition 6,  $\lambda(x) \in Q$  and hence  $x \in E$ .

(b)  $\Rightarrow$  (c): This is obvious as  $x \sim y$  implies  $x \prec y$ .

(c)  $\Rightarrow$  (a): When (c) holds, by Theorem 1,  $E$  is spectral; let  $E = \lambda^{-1}(Q)$ , where  $Q$  is permutation invariant. To show that  $Q$  is convex, let  $u, v \in Q$  and  $t \in [0, 1]$ . Then there exist  $x, y \in E$  such that  $x := \sum_1^n u_i e_i$  and  $y := \sum_1^n v_i e'_i$  for some Jordan frames  $\{e_1, e_2, \dots, e_n\}$  and  $\{e'_1, e'_2, \dots, e'_n\}$ . Now, define  $\bar{x} = \sum_1^n u_i e'_i$ . Then  $\lambda(\bar{x}) = \lambda(x)$ , hence  $\bar{x} \in E$  by (c). As  $\bar{x}, y \in E$  and  $E$  is convex, we get

$$\sum_{i=1}^n (tu_i + (1-t)v_i)e'_i = t \sum_{i=1}^n u_i e'_i + (1-t) \sum_{i=1}^n v_i e'_i = t\bar{x} + (1-t)y \in E.$$

This proves that  $tu + (1-t)v \in Q$ . Thus,  $Q$  is convex.

(c)  $\Rightarrow$  (d): Assume (c), let  $\phi \in \text{Aut}(\mathcal{V})$  and  $x \in E$ . Then  $\lambda(x) = \lambda(\phi(x))$  and hence  $\phi(x) \sim x$ . From (c), we must have  $\phi(x) \in E$ , proving (d).

Assuming that  $\mathcal{V}$  is essentially simple, we prove (d)  $\Rightarrow$  (c). Suppose  $x, y \in \mathcal{V}$  such that  $x \sim y$  and  $y \in E$ . As  $\mathcal{V}$  is essentially simple, by Proposition 2, there exists  $\phi \in \text{Aut}(\mathcal{V})$  such that  $x = \phi(y)$ . By (d),  $x \in E$ .  $\square$

**Example 3.** We can use Example 2 to show that in a general  $\mathcal{V}$ , (d) may not imply (a), (b), or (c) in the above theorem.

#### 4. Spectral functions

**Definition 3.** A function  $F : \mathcal{V} \rightarrow \mathcal{R}$  is a *spectral function* if there exists a permutation invariant function  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  such that  $F = f \circ \lambda$ .

By way of an example, we observe that  $F(x) = \lambda_{\max}(x)$  (the maximum of eigenvalues of  $x$ ) is a spectral function on  $\mathcal{V}$  as it corresponds to  $f(u) = \max u_i$  on  $\mathcal{R}^n$ .

**Proposition 10.** A set  $E$  in  $\mathcal{V}$  is a spectral set if and only if its characteristic function  $\chi_E$  (which takes the value one on  $E$  and zero outside of  $E$ ) is a spectral function.

**Proof.** Suppose  $E$  is a spectral set, say  $E = \lambda^{-1}(Q)$ , where  $Q$  is a permutation invariant set in  $\mathcal{R}^n$ . Then  $\chi_E = \chi_Q \circ \lambda$ . As  $\chi_Q$  is a permutation invariant,  $\chi_E$  is a spectral function.

Conversely, if  $\chi_E = f \circ \lambda$  is a spectral function, then let  $Q$  be the set where  $f$  takes the value one. If  $u \in Q$ , then  $f(u) = 1$ . As  $f$  is permutation invariant, we have  $f(Pu) = 1$  for all  $P \in \Sigma_n$ . This gives  $Pu \in Q$ ; hence  $Q$  is a permutation invariant set. We now show  $E = \lambda^{-1}(Q)$ . Suppose  $x \in E$ . Then  $1 = \chi_E(x) = f(\lambda(x))$ , which implies  $\lambda(x) \in Q$  by our construction. Thus,  $x \in \lambda^{-1}(Q)$ . For the reverse implication, let  $x \in \lambda^{-1}(Q)$ . As  $\lambda(x) \in Q$ , we get  $\chi_E(x) = f(\lambda(x)) = 1$ , and so  $x \in E$ .  $\square$

In this section, we describe some properties of spectral functions. We characterize spectral sets via spectral equivalence. As with sets, motivated by the question of recovering  $f$  from  $F$ , we introduce a ‘diamond’ operation on functions on  $\mathcal{R}^n$  and  $\mathcal{V}$ . We note that in the case of  $\mathcal{V} = \mathcal{S}^n$ , if  $F$  is a spectral function on  $\mathcal{S}^n$ , then the corresponding  $f$  is given by  $f(u) = F(\text{Diag}(u))$ .

**Definition 4.** Given a function  $f : \mathcal{R}^n \rightarrow \mathcal{R}$ , we define  $f^\diamond : \mathcal{V} \rightarrow \mathcal{R}$  by

$$f^\diamond(x) := f(\lambda(x)) \quad \text{for } x \in \mathcal{V}. \quad (5)$$

Fix a Jordan frame  $\{\bar{e}_1, \dots, \bar{e}_n\}$  in  $\mathcal{V}$ . Then, given a function  $F : \mathcal{V} \rightarrow \mathcal{R}$ , we define  $F^\diamond : \mathcal{R}^n \rightarrow \mathcal{R}$  by

$$F^\diamond(u) := F\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right) \quad \text{where } u = (u_1, \dots, u_n)^\top \in \mathcal{R}^n. \quad (6)$$

For simplicity, we write  $f^{\diamond\diamond}$  in place of  $(f^\diamond)^\diamond$ , etc. In the result below,  $F^\diamond$  and  $(f^\diamond)^\diamond$  are defined as in (6) with respect to a (fixed) Jordan frame  $\{\bar{e}_1, \dots, \bar{e}_n\}$ .

**Proposition 11.** *The following statements hold:*

- (i) *For any permutation invariant function  $f : \mathcal{R}^n \rightarrow \mathcal{R}$ ,  $f^\diamond$  is a spectral function and  $f^{\diamond\diamond} = f$ .*
- (ii) *For any function  $F : \mathcal{V} \rightarrow \mathcal{R}$ ,  $F^\diamond$  is permutation invariant.*

**Proof.** (i): When  $f$  is permutation invariant, by definition,  $f^\diamond$  is a spectral function. Now, define  $(f^\diamond)^\diamond$  as in (6). Then, for any  $u \in \mathcal{R}^n$ , we have

$$f^{\diamond\diamond}(u) = f^\diamond\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right) = f\left(\lambda\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right)\right) = f(u^\downarrow) = f(u).$$

Since  $u$  is arbitrary,  $f^{\diamond\diamond} = f$ .

(ii): Take  $u \in \mathcal{R}^n$  and  $\sigma \in \Sigma_n$ . As  $u^\downarrow = \sigma(u)^\downarrow$ , we get

$$F^\diamond(\sigma(u)) = F\left(\sum_{i=1}^n \sigma(u)_i^\downarrow \bar{e}_i\right) = F\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right) = F^\diamond(u).$$

Hence,  $F^\diamond$  is permutation invariant.  $\square$

**Example 4.** For the equality  $f^{\diamond\diamond} = f$  in Item (i) above, it is essential to have  $f$  invariant under permutations. To see this, take  $\mathcal{V} = \mathcal{S}^2$  and define  $f : \mathcal{R}^2 \rightarrow \mathcal{R}$  by  $f(u_1, u_2) = u_1$ . Then, for any  $x \in \mathcal{S}^2$ ,  $f^\diamond(x) = f(\lambda(x)) = \lambda_1(x)$ . Hence, for any  $u \in \mathcal{R}^2$ ,

$$f^{\diamond\diamond}(u) = f^\diamond\left(\sum_{i=1}^2 u_i^\downarrow \bar{e}_i\right) = f(u^\downarrow) = \max\{u_1, u_2\}.$$

This proves that  $f^{\diamond\diamond} \neq f$ .

We now characterize spectral functions.

**Theorem 4.** *The following are equivalent for a function  $F : \mathcal{V} \rightarrow \mathcal{R}$ :*

- (a)  $F$  is a spectral function.
- (b) For each  $\alpha \in \mathcal{R}$ , the level set  $F^{-1}(\{\alpha\})$  is a spectral set in  $\mathcal{V}$ .
- (c)  $F$  is constant on each equivalence class of  $\sim$ .
- (d)  $F^{\diamond\diamond} = F$ .

**Proof.** (a)  $\Rightarrow$  (b): Suppose  $F$  is a spectral function, say  $F = f \circ \lambda$ , where  $f$  is a permutation invariant function. For any  $\alpha \in \mathcal{R}$ ,  $F^{-1}(\{\alpha\}) = \lambda^{-1}(f^{-1}(\{\alpha\}))$ . As  $f$  is a permutation invariant function,  $Q := f^{-1}(\{\alpha\})$  is a permutation invariant set in  $\mathcal{R}^n$ . Thus,  $F^{-1}(\{\alpha\}) = \lambda^{-1}(Q)$  is a spectral set.

(b)  $\Rightarrow$  (c): Suppose  $F$  satisfies condition (b) and  $x \sim y$ . Let  $\alpha := F(x)$ . Then,  $F^{-1}(\{\alpha\})$  is a spectral set and  $x \in F^{-1}(\{\alpha\})$ ; hence, by [Theorem 1](#),  $y \in F^{-1}(\{\alpha\})$ . This implies that  $F(y) = \alpha$ ; hence,  $F(x) = F(y)$  whenever  $x \sim y$ .

(c)  $\Rightarrow$  (d): Let  $x \in \mathcal{V}$  with its spectral decomposition  $x = \sum_{i=1}^n \lambda_i(x) e_i$ . Define  $y = \sum_{i=1}^n \lambda_i(x) \bar{e}_i$ . Then,  $x \sim y$  and by (c),  $F(x) = F(y)$ . This implies,

$$F^{\diamond\diamond}(x) = F^{\diamond}(\lambda(x)) = F\left(\sum_{i=1}^n \lambda_i(x) \bar{e}_i\right) = F(y) = F(x).$$

Hence,  $F^{\diamond\diamond} = F$ .

(d)  $\Rightarrow$  (a): Let  $F^{\diamond\diamond} = F$ . Then,  $f := F^{\diamond}$  is permutation invariant by Item (ii) in [Proposition 11](#). Hence,  $F = f^{\diamond}$  is spectral.  $\square$

**Theorem 5.** Every spectral function is invariant under automorphisms. Converse holds when  $\mathcal{V}$  is essentially simple.

**Proof.** Suppose  $F = f \circ \lambda$  for some permutation invariant function  $f$ . Note that  $\lambda(x) = \lambda(\phi(x))$  for every  $x \in \mathcal{V}$  and  $\phi \in \text{Aut}(\mathcal{V})$ . Thus,  $F(\phi(x)) = f(\lambda(\phi(x))) = f(\lambda(x)) = F(x)$ .

For the converse, let  $F$  be invariant under automorphism and  $\mathcal{V}$  be essentially simple. If  $x \sim y$ , then by [Proposition 2](#), there exists  $\phi \in \text{Aut}(\mathcal{V})$  such that  $x = \phi(y)$ . This implies that  $F(x) = F(y)$ . We now use Item (c) in [Theorem 4](#) to see that  $F$  is spectral.  $\square$

**Example 5.** The converse in [Theorem 5](#) may not hold in general. To see this, take  $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$  and let  $F : \mathcal{V} \rightarrow \mathcal{R}$  be defined by

$$F((a, A)) = \text{tr}(A) \quad \forall a \in \mathcal{R}, A \in \mathcal{S}^2.$$

Since we know the (explicit) description of automorphisms of  $\mathcal{V}$  (via [Proposition 3](#)), we easily see that  $F$  is automorphism invariant. However, for  $x$  and  $y$  of [Example 2](#),  $\lambda(x) = \lambda(y)$  and so  $x \sim y$ . One can easily check that  $F(x) = 0 \neq 1 = F(y)$ . As  $F$  violates condition (c) in [Theorem 4](#),  $F$  is not a spectral function.

A celebrated result of Davis [\[6\]](#) says that a unitarily invariant function on  $\mathcal{H}^n$  is convex if and only if its restriction to diagonal matrices is convex. This result has numerous

applications in various fields. A generalization of this result for Euclidean Jordan algebras has already been observed by Baes [1]. In what follows, we consider equivalent formulations of this generalization and give some applications.

**Theorem 6.** *Let  $E$  be a convex spectral set in  $\mathcal{V}$  so that  $E = \lambda^{-1}(Q)$  with  $Q$  convex and permutation invariant in  $\mathcal{R}^n$ . Let  $F : E \rightarrow \mathcal{R}$ . Consider the following statements:*

- (a)  $F = f \circ \lambda$ , where  $f : Q \rightarrow \mathcal{R}$  is convex and permutation invariant.
- (b)  $F$  is convex, and  $x \prec y$  implies  $F(x) \leq F(y)$ .
- (c)  $F$  is convex, and  $x \sim y$  implies  $F(x) = F(y)$ .
- (d)  $F$  is convex and  $F \circ \phi = F$  for all  $\phi \in \text{Aut}(\mathcal{V})$ .

Then (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c) and (c)  $\Rightarrow$  (d). Moreover, all these statements are equivalent when  $\mathcal{V}$  is essentially simple.

**Proof.** (a)  $\Rightarrow$  (b): The convexity of  $F$  has already been proved in Theorem 41, [1]. Capturing its essence, we present our proof based on Proposition 8. For  $t \in [0, 1]$  and  $x, y \in E$ , we write  $\lambda(tx + (1-t)y) = tA\lambda(x) + (1-t)B\lambda(y)$ , where  $A$  and  $B$  are doubly stochastic matrices. As these matrices are convex combinations of permutation matrices (see Proposition 6), by permutation invariance and convexity of  $f$  on  $Q$ , we see that  $f(A\lambda(x)) \leq f(\lambda(x))$  and  $f(B\lambda(y)) \leq f(\lambda(y))$ . It follows that

$$\begin{aligned} F(tx + (1-t)y) &= f(tA\lambda(x) + (1-t)B\lambda(y)) \\ &\leq tf(A\lambda(x)) + (1-t)f(B\lambda(y)) \\ &\leq tf(\lambda(x)) + (1-t)f(\lambda(y)) \\ &= tF(x) + (1-t)F(y). \end{aligned}$$

Now for the second part of (b), suppose  $x, y \in E$  with  $x \prec y$ . Writing  $u = \lambda(x)$  and  $v = \lambda(y)$ , we get  $u \prec v$  in  $Q$  and hence (from Propositions 4, 5)  $u = \sum_j \alpha_j \sigma_j(v)$  for some  $\alpha_j > 0$  with  $\sum_j \alpha_j = 1$  and permutation matrices  $\sigma_j \in \Sigma_n$ . As  $f$  is convex and permutation invariant,

$$f(u) = f\left(\sum \alpha_j \sigma_j(v)\right) \leq \sum \alpha_j f(\sigma_j(v)) = \sum \alpha_j f(v) = f(v).$$

Since  $F = f \circ \lambda$ , we get  $F(x) = f(\lambda(x)) = f(u) \leq f(v) = f(\lambda(y)) = F(y)$ .

(b)  $\Rightarrow$  (c): This is obvious as  $x \sim y$  implies  $x \prec y$ .

(c)  $\Rightarrow$  (a): Given  $F$  satisfying (c), we define  $f : Q \rightarrow \mathcal{R}$  as follows. For any  $u \in Q$ , there exists  $x \in E$  such that  $\lambda(x) = u^\downarrow$ . We let  $f(u) := F(x)$ . This is well defined: if there is a  $y \in E$  with  $\lambda(y) = u^\downarrow$ , then  $x \sim y$  and so by (c),  $F(x) = F(y)$ . Now, for any permutation  $\sigma$ ,  $u^\downarrow = \sigma(u)^\downarrow$ ; hence,  $f(\sigma(u)) = F(x) = f(u)$ , proving the permutation invariance of  $f$ .

We also see  $F(x) = f(u) = f(u^\perp) = f(\lambda(x))$ . Note that these (relations) hold if we start with an  $x \in E$  and let  $u = \lambda(x)$ . Thus we have  $F = f \circ \lambda$ .

We now show that  $f$  is convex. Let  $u, v \in Q$  and  $t \in [0, 1]$ . As  $w := tu + (1 - t)v \in Q$ , there exists  $z \in E$  such that  $\lambda(z) = w^\perp$ . Let  $\sum_1^n w_i e_i$  be the spectral decomposition of  $z$ . Define  $x = \sum_1^n u_i e_i$  and  $y = \sum_1^n v_i e_i$ . Note that  $\lambda(x) = u^\perp \in Q$  and  $\lambda(y) = v^\perp \in Q$ ; hence,  $x, y \in E$  with  $z = tx + (1 - t)y$ . Then, as  $f$  is permutation invariant and  $F$  is convex, we get

$$\begin{aligned} f(tu + (1 - t)v) &= f(w) = f(w^\perp) = F(z) \\ &= F(tx + (1 - t)y) \\ &\leq tF(x) + (1 - t)F(y) \\ &= tf(u) + (1 - t)f(v). \end{aligned}$$

Hence,  $f$  is convex.

(c)  $\Rightarrow$  (d): Let  $\phi$  be an automorphism. For  $x \in \mathcal{V}$ ,  $\lambda(\phi(x)) = \lambda(x)$  and so  $\phi(x) \sim x$ . Then, by (c), we must have  $F(\phi(x)) = F(x)$ .

Now, we prove (d)  $\Rightarrow$  (c) when  $\mathcal{V}$  is essentially simple. Suppose  $x, y \in E$  such that  $x \sim y$ . By Proposition 2, there exists an automorphism  $\phi$  such that  $x = \phi(y)$ . Then, from (d),  $F(x) = F(\phi(y)) = F(y)$ . Thus, condition (c) holds. This completes the proof.  $\square$

**Remarks.** In the case of a general  $\mathcal{V}$ , (d) may not imply other statements. This can be seen by taking  $\mathcal{V}$  and  $F$  as in Example 5. Clearly, this  $F$  is automorphism invariant and convex (indeed, linear). However, as seen in Example 5,  $F$  is not a spectral function.

We now provide two applications of the above theorem.

**Example 6.** For  $p \in [1, \infty]$ , let  $F(x) = \|x\|_{sp, p} := \|\lambda(x)\|_p$ , where  $\|u\|_p$  denotes the  $p$ th-norm of a vector  $u$  in  $\mathcal{R}^n$ . Then,  $F$  is a convex spectral function which is also positive homogeneous. Since  $F(x) = 0 \implies x = 0$ , we see that

$$\|\cdot\|_{sp, p} \text{ is a norm on } \mathcal{V}.$$

This fact has already been observed in [23], where it is proved via Thompson's triangle inequality, majorization techniques, and case-by-case analysis.

**Example 7** (A Golden–Thompson type inequality). The Golden–Thompson inequality ([2], p. 261) says that for  $A, B \in \mathcal{H}^n$ ,

$$\operatorname{tr} \left( \exp(A + B) \right) \leq \operatorname{tr} \left( \exp(A) \exp(B) \right).$$



It is not known if an analogous result holds in Euclidean Jordan algebras. The Golden–Thompson inequality easily implies the (weaker) inequality

$$\operatorname{tr} \left( \exp(A + B) \right) \leq \operatorname{tr} \left( \exp(A) \right) \operatorname{tr} \left( \exp(B) \right).$$

Rivin [19], based on a result of Davis [6], gives a simple proof of this. A modification of this proof, given below, leads to the following generalization:

*For any two elements  $x, y$  in a Euclidean Jordan algebra  $\mathcal{V}$ ,*

$$\operatorname{tr} \left( \exp(x + y) \right) \leq \operatorname{tr} \left( \exp(x) \right) \operatorname{tr} \left( \exp(y) \right), \quad (7)$$

where  $\exp(x)$  is defined by  $\exp(x) := \sum_1^n \exp(\lambda_i(x)) e_i$  when  $x$  has the spectral decomposition  $x = \sum_1^n \lambda_i(x) e_i$ .

We prove (7) as follows. For  $u \in \mathcal{R}^n$  with components  $u_1, \dots, u_n$ , let  $f(u) = \ln(\sum_1^n \exp(u_i))$ , which is known to be convex [19]. Since  $f$  is also permutation invariant,  $F := f \circ \lambda$  is convex by the above theorem. It follows that

$$F \left( \frac{x + y}{2} \right) \leq \frac{F(x) + F(y)}{2},$$

for all  $x, y \in \mathcal{V}$ . As  $F(x) = \ln \left( \operatorname{tr} \left( \exp(x) \right) \right)$ , this leads to

$$\left[ \operatorname{tr} \left( \exp \left( \frac{x + y}{2} \right) \right) \right]^2 \leq \operatorname{tr} \left( \exp(x) \right) \operatorname{tr} \left( \exp(y) \right).$$

This, with the observation  $\operatorname{tr} \left( \exp(2z) \right) \leq \left[ \operatorname{tr} \left( \exp(z) \right) \right]^2$  for any  $z \in \mathcal{V}$ , yields (7).

## 5. Schur-convex functions

**Definition 5.** A function  $F : \mathcal{V} \rightarrow \mathcal{R}$  is said to be *Schur-convex* if

$$x \prec y \Rightarrow F(x) \leq F(y).$$

**Theorem 6** (together with **Proposition 7**) immediately yields the following.

**Theorem 7.** *Every convex spectral function on  $\mathcal{V}$  is Schur-convex. In particular, for any doubly stochastic transformation  $\Psi$  on  $\mathcal{V}$ , and for any convex spectral function  $F$  on  $\mathcal{V}$ , we have*

$$F(\Psi(x)) \leq F(x) \quad \text{for all } x \in \mathcal{V}.$$

We illustrate the above result with a number of examples.

**Example 8.** For  $p \in [1, \infty]$ , as in [Example 6](#), consider  $F(x) = \|x\|_{sp, p} := \|\lambda(x)\|_p$ . Then,  $F$  is a convex spectral function and hence, for any doubly stochastic transformation  $\Psi$  on  $\mathcal{V}$ ,

$$\|\Psi(x)\|_{sp, p} \leq \|x\|_{sp, p}.$$

This extends Proposition 2 in [\[9\]](#), where it is shown that  $\|\Psi(x)\|_{sp, 2} \leq \|x\|_{sp, 2}$  for all  $x$  in a simple Euclidean Jordan algebra.

**Example 9.** Consider an idempotent  $c$  ( $\neq 0, e$ ) in  $\mathcal{V}$  and the corresponding orthogonal decomposition [\[7\]](#)

$$\mathcal{V} = \mathcal{V}(c, 1) + \mathcal{V}(c, \tfrac{1}{2}) + \mathcal{V}(c, 0),$$

where  $\mathcal{V}(c, \gamma) = \{x \in \mathcal{V} : x \circ c = \gamma x\}$ ,  $\gamma \in \{0, \frac{1}{2}, 1\}$ . For each  $x \in \mathcal{V}$ , we write

$$x = u + v + w \quad \text{where} \quad u \in \mathcal{V}(c, 1), v \in \mathcal{V}(c, \tfrac{1}{2}), w \in \mathcal{V}(c, 0).$$

For any  $a \in \mathcal{V}$ , let  $P_a$  denote the corresponding quadratic representation defined by  $P_a(x) = 2a \circ (a \circ x) - a^2 \circ x$ . Then for  $\varepsilon = 2c - e$ , one verifies that  $P_\varepsilon(x) = u - v + w$  and  $(\frac{P_\varepsilon + I}{2})(x) = u + w$ . As  $\varepsilon^2 = e$ ,  $P_\varepsilon$  is an automorphism of  $\mathcal{V}$  ([\[7\]](#), Prop. II.4.4). Thus,  $P_\varepsilon$  and  $\frac{P_\varepsilon + I}{2}$  are doubly stochastic on  $\mathcal{V}$ . Then, for any convex spectral function  $F$  on  $\mathcal{V}$ ,

$$F(u - v + w) \leq F(x) \quad \text{and} \quad F(u + w) \leq F(x).$$

We note that the process of going from  $x = w + v + w$  to  $u + w$  is a ‘pinching’ process; in the context of block matrices, this ‘pinching’ is obtained by setting the off-diagonal blocks to zero.

**Example 10.** Let  $A$  be an  $n \times n$  real symmetric positive semidefinite matrix with each diagonal entry one. In  $\mathcal{V}$ , we fix a Jordan frame  $\{e_1, e_2, \dots, e_n\}$  and consider the corresponding Peirce decomposition (Theorem IV.2.1 [\[7\]](#)) of any  $x \in \mathcal{V}$ :  $x = \sum_{i \leq j} x_{ij} = \sum_1^n x_i e_i + \sum_{i < j} x_{ij}$ . Then the transformation

$$\Psi(x) := A \bullet x = \sum_{i \leq j} a_{ij} x_{ij}$$

is doubly stochastic, see [\[9\]](#). It follows that when  $F$  is a convex spectral function on  $\mathcal{V}$ ,

$$F(A \bullet x) \leq F(x).$$

By taking  $A$  to be the identity matrix, this inequality reduces to

$$F\left(\sum_{i=1}^n x_i e_i\right) \leq F(x).$$

**Example 11.** Let  $g : \mathcal{R} \rightarrow \mathcal{R}$  be any function. Then the corresponding *Löwner mapping*  $L_g : \mathcal{V} \rightarrow \mathcal{V}$  [22] is defined as follows: For any  $x \in \mathcal{V}$  with spectral decomposition  $x = \sum \lambda_i(x) e_i$ ,

$$L_g(x) := \sum_{i=1}^n g(\lambda_i(x)) e_i.$$

Clearly,  $F(x) := \text{tr}(L_g(x)) = \sum g(\lambda_i(x)) = f \circ \lambda$ , where  $f(u_1, \dots, u_n) = \sum g(u_i)$ , is a spectral function. Thus, when  $g$  is convex,  $F$  is a convex spectral function. Hence, the function  $x \mapsto \text{tr}(L_g(x))$  is Schur-convex for any convex function  $g : \mathcal{R} \rightarrow \mathcal{R}$ .

The second part of Theorem 7 says that a doubly stochastic transformation decreases the value of a convex spectral function. Below, we present a converse to this statement:

**Theorem 8.** *The following statements hold:*

- (i) *If  $x, y \in \mathcal{V}$  with  $F(x) \leq F(y)$  for all convex spectral functions  $F$  on  $\mathcal{V}$ , then  $x \prec y$ .*
- (ii) *If  $x, y \in \mathcal{V}$  with  $F(x) = F(y)$  for all convex spectral functions  $F$  on  $\mathcal{V}$ , then  $x \sim y$ .*
- (iii) *If  $\Psi$  is a linear transformation on  $\mathcal{V}$  such that  $F(\Psi(x)) \leq F(x)$  for all  $x \in \mathcal{V}$  and for all convex spectral functions  $F$ , then  $\Psi$  is a doubly stochastic transformation on  $\mathcal{V}$ .*

This result is based on two lemmas. The first lemma (noted on page 159 in [16]) is a consequence of a result of Hardy, Littlewood, and Pólya.

**Lemma 1.** *Suppose  $u, v \in \mathcal{R}^n$  with the property that  $f(u) \leq f(v)$  for all convex permutation invariant functions  $f : \mathcal{R}^n \rightarrow \mathcal{R}$ . Then,  $u \prec v$ .*

Our second lemma is a generalization of a well-known result: An  $n \times n$  matrix  $A$  is doubly stochastic if and only if  $Ax \prec x$  for all  $x \in \mathcal{R}^n$ , see Theorem A.4 in [16].

**Lemma 2.** *A linear transformation  $\Psi : \mathcal{V} \rightarrow \mathcal{V}$  is doubly stochastic if and only if  $\Psi(x) \prec x$  for all  $x \in \mathcal{V}$ .*

**Proof.** First, suppose  $\Psi$  is doubly stochastic. Then by Proposition 7, we have  $\Psi(x) \prec x$ . To prove the converse, suppose  $\Psi(x) \prec x$  for all  $x \in \mathcal{V}$ . First, let  $x \in \mathcal{V}_+$ . Then all the eigenvalues of  $x$  are nonnegative. Now,  $\Psi(x) \prec x$  implies, from (1) in Section 2.1,  $\lambda_n(\Psi(x)) \geq \lambda_n(x) \geq 0$ . As  $\lambda_n(\Psi(x))$  is the smallest of the eigenvalues of  $\Psi(x)$ , we see that all eigenvalues of  $\Psi(x)$  are nonnegative; hence  $\Psi(x) \in \mathcal{V}_+$ . Thus,  $\Psi(\mathcal{V}_+) \subseteq \mathcal{V}_+$ .

Next, we show that  $\Psi(e) = e$ . As  $\Psi(e) \prec e$  and every eigenvalue of  $e$  is one, from (1), the smallest and largest eigenvalues of  $\Psi(e)$  coincide with 1. Thus, all eigenvalues of  $\Psi(e)$  are equal to one. By the spectral decomposition of  $e$ , we see that  $\Psi(e) = e$ .

We now show  $\Psi$  is trace-preserving: For any Jordan frame  $\{e_1, \dots, e_n\}$ , we have  $\Psi(e_i) \prec e_i$ , so  $\text{tr}(\Psi(e_i)) = \text{tr}(e_i) = 1$ . Thus for  $x \in \mathcal{V}$  with (spectral decomposition)  $x = \sum_i x_i e_i$ ,

$$\text{tr}(\Psi(x)) = \text{tr} \left( \Psi \left( \sum_{i=1}^n x_i e_i \right) \right) = \sum_{i=1}^n x_i \text{tr}(\Psi(e_i)) = \sum_{i=1}^n x_i = \text{tr}(x).$$

Thus,  $\Psi$  is doubly stochastic on  $\mathcal{V}$ .  $\square$

We now come to the proof of the theorem.

**Proof.** (i) Suppose  $x, y \in \mathcal{V}$  with  $F(x) \leq F(y)$  for all convex spectral functions  $F$  on  $\mathcal{V}$ . Then, for any  $f$  that is convex and permutation invariant on  $\mathcal{R}^n$ , we let  $F = f \circ \lambda$  and get  $f(\lambda(x)) \leq f(\lambda(y))$ . Now from Lemma 1,  $\lambda(x) \prec \lambda(y)$ . This means that  $x \prec y$ .

(ii) From the proof of Item (i),  $\lambda(x) \prec \lambda(y)$  and  $\lambda(y) \prec \lambda(x)$ . From the defining (majorization) inequalities of Section 2.1,  $\lambda(x) = \lambda(y)$ . Thus,  $x \sim y$ .

(iii) Now suppose that  $\Psi$  is linear and  $F(\Psi(x)) \leq F(x)$  for all  $x \in \mathcal{V}$  and all convex spectral functions  $F$  on  $\mathcal{V}$ . By (i),  $\Psi(x) \prec x$  for all  $x$ . By Lemma 2,  $\Psi$  is doubly stochastic.  $\square$

It has been observed before that automorphisms preserve eigenvalues. The following result shows that in the setting of essentially simple algebras, automorphisms are the only linear transformations that preserve eigenvalues.

**Corollary 1.** Suppose  $\mathcal{V}$  is essentially simple and  $\phi : \mathcal{V} \rightarrow \mathcal{V}$  is linear. If  $\phi(x) \sim x$  for all  $x \in \mathcal{V}$ , then  $\phi \in \text{Aut}(\mathcal{V})$ .

**Proof.** Suppose that  $\phi(x) \sim x$  for all  $x \in \mathcal{V}$ . Then,  $\phi$  is invertible: If  $\phi(x) = 0$  for some  $x$ , then  $0 \sim x$  and so  $x = 0$ . Now, take any convex spectral function  $F$  on  $\mathcal{V}$ . As  $\phi(x) \sim x$ , we have  $F(x) = F(\phi(x))$ . From Theorem 8, we see that  $\phi$  and  $\phi^{-1}$  are doubly stochastic on  $\mathcal{V}$ . Recall, by definition, these are positive transformations, that is,  $\phi(\mathcal{V}_+) \subseteq \mathcal{V}_+$  and  $\phi^{-1}(\mathcal{V}_+) \subseteq \mathcal{V}_+$ . Hence,  $\phi(\mathcal{V}_+) = \mathcal{V}_+$ , that is,  $\phi$  belongs to  $\text{Aut}(\mathcal{V}_+)$ . Since  $\mathcal{V}$  is essentially simple, from Theorem 8 in [9] we have

$$\text{Aut}(\mathcal{V}_+) \cap \text{DS}(\mathcal{V}) = \text{Aut}(\mathcal{V}),$$

where  $\text{DS}(\mathcal{V})$  denotes the set of all doubly stochastic transformations on  $\mathcal{V}$ . This proves that  $\phi \in \text{Aut}(\mathcal{V})$ .  $\square$

## 6. The transfer principle and a metaformula

In the context of spectral sets  $E = \lambda^{-1}(Q)$  and spectral functions  $F = f \circ \lambda$ , the *Transfer Principle* asserts that (many) properties of  $Q$  (of  $f$ ) get transferred to  $E$  (respectively, to  $F$ ). [Theorems 3 and 6](#), where convexity gets transferred, illustrate this principle. In addition to this, Baes ([\[1\]](#), Theorem 27) shows that closedness, openness, boundedness and compactness properties of  $Q$  are carried over to  $E$ . Sun and Sun [\[22\]](#) show that semismoothness property of  $f$  gets transferred to  $F$  in the setting of a Euclidean Jordan algebras. Numerous specialized results exist in the setting of  $\mathcal{S}^n$  and  $\mathcal{H}^n$ ; see the recent article [\[5\]](#) and the references therein for a discussion on the transferability of  $C^\infty$ -manifold property of  $Q$  and various types differentiability properties of  $f$  (e.g., prox-regularity, Clarke-regularity, and smoothness). Related to this, in the context of normal decomposition systems, Lewis [\[12\]](#) has observed that the ‘metaformula’

$$\#(\lambda^{-1}(Q)) = \lambda^{-1}(\#(Q))$$

holds for sets  $Q$  that are invariant under a group of orthogonal transformations and certain set operations  $\#$ . In particular, it was shown in Theorem 5.4, [\[12\]](#), that the formula holds when  $\#$  is the closure/interior/boundary operation. Consequently, because of a result in [\[15\]](#), it is also valid in any (essentially) simple Euclidean Jordan algebra; see [\[10,11\]](#) for results of this type for spectral cones in  $\mathcal{S}^n$ . Motivated by these, we present the following ‘metaformula’ in the setting of general Euclidean Jordan algebras.

In what follows, for a set  $S$  (either in  $\mathcal{R}^n$  or in  $\mathcal{V}$ ), we consider closure, interior, boundary, and convex hull operations, which are respectively denoted by  $\overline{S}$ ,  $S^\circ$ ,  $\partial S$ , and  $\text{conv}(S)$ . Recall that for a set  $Q$  in  $\mathcal{R}^n$  and a set  $E$  in  $\mathcal{V}$ ,

$$Q^\diamond := \lambda^{-1}(Q) \quad \text{and} \quad E^\diamond := \Sigma_n(\lambda(E)).$$

(We remark that when  $\mathcal{V} = \mathcal{R}^n$  and  $E = Q$ , these two definitions coincide.)

**Theorem 9.** *Let  $\#$  denote one of the operations of closure, interior, boundary, or convex hull. Then, over permutation invariant sets in  $\mathcal{R}^n$  and spectral sets in  $\mathcal{V}$ , the operations  $\#$  and  $\diamond$  commute; symbolically,*

$$\# \diamond = \diamond \#.$$

In preparation for the proof, we state (and prove, for the record) the following ‘folkloric’ result:

**Proposition 12.** *Let  $Q$  be a permutation invariant set in  $\mathcal{R}^n$  and  $E = \lambda^{-1}(Q)$  in  $\mathcal{V}$ . Then,*

- (i)  $\overline{E} = \lambda^{-1}(\overline{Q})$ .
- (ii)  $E^\circ = \lambda^{-1}(Q^\circ)$ .

- (iii)  $\partial E = \lambda^{-1}(\partial Q)$ .
- (iv)  $\text{conv}(\lambda^{-1}(Q)) = \lambda^{-1}(\text{conv}(Q))$ .

Consequently, the closure/interior/boundary/convex hull of a spectral set is spectral.

**Proof.** (i) By continuity of  $\lambda$ ,  $\overline{E} = \overline{\lambda^{-1}(Q)} \subseteq \lambda^{-1}(\overline{Q})$ . To see the reverse inclusion, let  $x \in \lambda^{-1}(\overline{Q})$  so that  $\lambda(x) \in \overline{Q}$ . Let  $\lambda(x) = \lim q_k$ , where  $q_k \in Q$ . We consider the spectral decomposition  $x = \sum_1^n \lambda_i(x) e_i$  and define  $x_k := \sum_1^n (q_k)_i e_i$ , for  $k = 1, 2, \dots$ . Then,  $\lambda(x_k) = q_k^\downarrow \in Q$  (recall that  $Q$  is permutation invariant). Thus,  $x_k \in \lambda^{-1}(Q)$  for all  $k$ . As  $x_k \rightarrow x$ , we see that  $x \in \overline{E}$ . Hence,  $\overline{E} = \lambda^{-1}(\overline{Q})$ .

(ii) As  $Q^c$  is permutation invariant and  $\lambda^{-1}$  preserves complements, we see, by (i), that  $\overline{E^c} = \lambda^{-1}(\overline{Q^c})$ . Taking complements and using the identity  $E^\circ = (\overline{E^c})^c$ , we get  $E^\circ = \lambda^{-1}(Q^\circ)$ .

(iii) This comes from the previous items using the definition  $\partial E = \overline{E} \setminus E^\circ$  and the fact that  $\lambda^{-1}$  preserves set differences.

(iv) We first prove that  $\text{conv}(\lambda^{-1}(Q)) \subseteq \lambda^{-1}(\text{conv}(Q))$  in  $\mathcal{V}$ . It is clear that  $\text{conv}(Q)$  is convex and permutation invariant. Hence, by Theorem 3,  $\lambda^{-1}(\text{conv}(Q))$  is convex. As  $\lambda^{-1}(Q) \subseteq \lambda^{-1}(\text{conv}(Q))$ , we see that  $\text{conv}(\lambda^{-1}(Q)) \subseteq \lambda^{-1}(\text{conv}(Q))$ . To prove the reverse implication, let  $x \in \lambda^{-1}(\text{conv}(Q))$ . Then,  $\lambda(x) = \sum_{k=1}^N \alpha_k q_k$ , where  $\alpha_k$ s are positive with sum one and  $q_k \in Q$  for all  $k$ . Writing the spectral decomposition of  $x = \sum_{i=1}^n \lambda_i(x) e_i$ , we see that

$$x = \sum_{i=1}^n \left( \sum_{k=1}^N \alpha_k (q_k)_i \right) e_i = \sum_{k=1}^N \alpha_k \left( \sum_{i=1}^n (q_k)_i e_i \right).$$

Letting  $x_k := \sum_{i=1}^n (q_k)_i e_i$ , we get  $\lambda(x_k) = q_k^\downarrow \in Q$ . Hence,  $x_k \in \lambda^{-1}(Q)$  and  $x$  is now a convex combination of elements of  $\lambda^{-1}(Q)$ . Thus,  $x \in \text{conv}(\lambda^{-1}(Q))$  proving the required reverse inclusion.

Finally, the last statement follows from Items (i)–(iv) together with the observation that the closure/interior/boundary/convex hull of a permutation invariant set in  $\mathcal{R}^n$  is permutation invariant.  $\square$

**Proof of the Theorem.** Suppose  $Q$  is any permutation invariant set in  $\mathcal{R}^n$ . Then from Proposition 12 we have  $\lambda^{-1}(\#(Q)) = \#(\lambda^{-1}(Q))$ , that is,

$$[\#(Q)]^\diamond = \#(Q^\diamond).$$

This means that on permutation invariant sets in  $\mathcal{R}^n$ , the operations  $\#$  and  $\diamond$  commute.

Now suppose that  $E$  is any spectral set in  $\mathcal{V}$ . Then,  $Q := E^\diamond$  is permutation invariant and  $Q^\diamond = E^{\diamond\diamond} = E$  (by Theorem 1). Hence,  $\#(E) = \#(Q^\diamond) = [\#(Q)]^\diamond$ . As  $\#(Q)$  is permutation invariant, from Proposition 9,  $[\#(Q)]^{\diamond\diamond} = \#(Q)$ . Thus,

$$[\#(E)]^\diamond = [\#(Q)]^{\diamond\diamond} = \#(Q) = \#(E^\diamond).$$

This proves that the operations  $\#$  and  $\diamond$  commute on spectral sets in  $\mathcal{V}$ .  $\square$

It will be shown below that the equality  $\overline{E^\diamond} = \overline{E}^\diamond$  holds for any set  $E \subseteq \mathcal{V}$ . We now provide examples to show that in the above theorem, permutation invariance of  $Q$  and/or spectrality of  $E$  is needed to get the remaining results.

### Example 12.

- Let  $\mathcal{V} = \mathcal{S}^n$  and  $Q = \{u \in \mathcal{R}_+^n : u_1 < u_2 < \cdots < u_n\}$ . Clearly,  $Q$  is not permutation invariant. As  $Q^\diamond = \lambda^{-1}(Q) = \emptyset$ , we have  $\overline{Q^\diamond} = \emptyset$ . On the other hand, we have  $\overline{Q} = \{u \in \mathcal{R}_+^n : u_1 \leq u_2 \leq \cdots \leq u_n\}$  and  $\overline{Q}^\diamond = \lambda^{-1}(\overline{Q})$  is the (nonempty) set of all nonnegative multiples of the identity matrix. Thus,  $\overline{Q^\diamond} \neq \overline{Q}^\diamond$ .
- One consequence of Theorem 3 is that when  $E$  is spectral and convex,  $E^\diamond$  is convex. This may fail if  $E$  is not spectral: In  $\mathcal{V} = \mathcal{R}^2$ , let  $e_1$  and  $e_2$  denote the standard coordinate vectors. Then,  $E = \{e_1\}$  is convex, while  $E^\diamond = \{e_1, e_2\}$  is not. In particular,  $[\text{conv}(E)]^\diamond \neq \text{conv}(E^\diamond)$ .
- Let  $\mathcal{V} = \mathcal{S}^2$  and let  $E$  be the set of all nonnegative diagonal matrices. Then, one can easily verify that  $E^\diamond = \mathcal{R}_+^2$  and so  $(E^\diamond)^\circ = \mathcal{R}_{++}^2$ , where  $\mathcal{R}_{++}^2$  denotes the set of all positive vectors in  $\mathcal{R}^2$ . However, we have  $E^\circ = \emptyset$  and  $(E^\circ)^\diamond = \emptyset$ . Hence,  $(E^\diamond)^\circ \neq (E^\circ)^\diamond$ .

The preservation of certain properties is an important feature of the ‘diamond’ operation.

**Theorem 10.** *If a set  $E$  is closed/open/bounded/compact in  $\mathcal{V}$ , then so is  $E^\diamond$  in  $\mathcal{R}^n$ .*

**Proof.** We assume, without loss of generality, that  $E$  is nonempty.

- Suppose that  $E$  is closed. Let  $\{u_k\}$  be a sequence in  $E^\diamond$  such that  $u_k \rightarrow u$  for some  $u \in \mathcal{V}$ . For each  $u_k \in E^\diamond$ , we can find  $x_k \in E$  with  $\lambda(x_k) = u_k^\downarrow$ . Let  $x_k = \sum_i \lambda_i(x_k) e_i^{(k)}$  be the spectral decomposition of  $x_k$  for each  $k$ . As the set of all primitive idempotents in  $\mathcal{V}$  forms a compact set, see [7], page 78, there exists a sequence  $k_m$  such that  $e_i^{(k_m)} \rightarrow e_i$  for all  $i = 1, 2, \dots, n$ . Thus,

$$x_{k_m} = \sum_{i=1}^n \lambda_i(x_{k_m}) e_i^{(k_m)} \rightarrow \sum_{i=1}^n u_i^\downarrow e_i.$$

Since  $E$  is closed, we have  $x := \lim x_{k_m} \in E$ , which implies that  $\lambda(x) = u^\downarrow$ . Hence,  $u \in E^\diamond$  proving the closedness of  $E^\diamond$ .

- For any (primitive) idempotent  $c$  in  $\mathcal{V}$  we observe that  $\|c\|^2 = \langle c, c \rangle = \langle c \circ c, e \rangle = \langle c, e \rangle \leq \|c\| \|e\|$  and hence  $\|c\| \leq \|e\|$ . Then, for any Jordan frame  $\{e_1, e_2, \dots, e_n\}$  in  $\mathcal{V}$  and  $u \in \mathcal{R}^n$ , we have

$$\left\| \sum_{i=1}^n u_i e_i \right\| \leq \left( \sum_{i=1}^n |u_i| \right) \|e\| \leq \sqrt{n} \|u\| \|e\| \quad (8)$$

where  $\|u\|$  denotes the 2-norm of  $u$ .

Now assume that  $E$  is open in  $\mathcal{V}$ . To show that  $E^\diamond$  open, let  $u \in E^\diamond$ . Then there exists  $x \in E$  such that  $\lambda(x) = u^\perp$ . As  $E$  is open, there exists  $\epsilon > 0$  such that  $B(x, \epsilon) := \{y \in \mathcal{V} : \|y - x\| < \epsilon\} \subseteq E$ . Putting  $\delta := \frac{\epsilon}{\sqrt{n}\|e\|}$ , we show that the ball  $B(u, \delta) := \{v \in \mathcal{R}^n : \|u - v\| < \delta\}$  is contained in  $E^\diamond$ . To this end, let  $x = \sum_i u_i^\perp e_i$  be the spectral decomposition of  $x$ . Then, there exists a permutation matrix  $\sigma \in \Sigma_n$  such that  $x = \sum_i u_i e_{\sigma(i)}$ . Now, for any  $v \in B(u, \delta)$ , define  $y := \sum_i v_i e_{\sigma(i)} \in \mathcal{V}$ . Then, from (8),

$$\|x - y\| \leq \sqrt{n} \|e\| \|u - v\| < \epsilon.$$

This proves that  $y \in E$ . As  $\lambda(y) = v^\perp$ , we see that  $v \in E^\diamond$ . This shows that  $B(u, \delta) \subseteq E^\diamond$ ; hence  $E^\diamond$  is open.

- Now let  $E$  be compact. Then  $\lambda(E)$  is compact, by the continuity of  $\lambda$ . As  $\Sigma_n$  is compact, we see that  $E^\diamond = \Sigma_n(\lambda(E))$  is also compact.
- Finally, if  $E$  is bounded, then  $\overline{E}$  is compact. Hence,  $(\overline{E})^\diamond$  is compact. Clearly,  $E^\diamond$  is bounded as it is a subset of  $(\overline{E})^\diamond$ .  $\square$

**Corollary 2.** For any set  $E$  in  $\mathcal{V}$ ,  $\overline{E^\diamond} = \overline{E}^\diamond$ .

**Proof.** Without loss of generality, we assume that  $E$  is nonempty.

Since  $E^\diamond \subseteq \overline{E}^\diamond$  and  $\overline{E}^\diamond$  is closed by the above Theorem, we have  $\overline{E^\diamond} \subseteq \overline{E}^\diamond$ .

To see the reverse inclusion, let  $u \in \overline{E}^\diamond$ . Then there exists  $x \in \overline{E}$  such that  $\lambda(x) = u^\perp$ . As  $u^\perp$  is some permutation of  $u$ , there exists a permutation  $\sigma \in \Sigma_n$  such that  $\sigma^{-1}(u) = u^\perp = \lambda(x)$ . As  $x \in \overline{E}$ , there exists a sequence  $\{x_k\}$  of  $E$  converging to  $x$ . Then, by the continuity of  $\lambda$ , we get  $\lambda(x) = \lim_{k \rightarrow \infty} \lambda(x_k)$ . This implies

$$u = \sigma(\lambda(x)) = \sigma \left( \lim_{k \rightarrow \infty} \lambda(x_k) \right) = \lim_{k \rightarrow \infty} \sigma(\lambda(x_k)).$$

Letting  $u_k := \sigma(\lambda(x_k))$ , we get  $u_k^\perp = \lambda(x_k)$ ; thus  $u_k \in E^\diamond$  for all  $k$ . As  $u_k \rightarrow u$ , we have  $u \in \overline{E^\diamond}$  proving  $\overline{E}^\diamond \subseteq \overline{E^\diamond}$ . This completes the proof.  $\square$

Our final result deals with the ‘double diamond’ operation. For a set  $E$  in  $\mathcal{V}$ , we call  $E^{\diamond\diamond}$ , the *spectral hull* of  $E$  in  $\mathcal{V}$ . Here are some properties of the spectral hull.

**Proposition 13.** For any set  $E \subseteq \mathcal{V}$ , we have

- (i)  $E \subseteq E^{\diamond\diamond}$ .
- (ii) If  $E_1 \subseteq E_2$ , then  $E_1^{\diamond\diamond} \subseteq E_2^{\diamond\diamond}$ .
- (iii)  $E^{\diamond\diamond}$  is the smallest spectral set containing  $E$ .



- (iv)  $E$  is spectral if and only if  $E = E^{\diamond\diamond}$ .
- (v)  $(E_1 \cap E_2)^{\diamond\diamond} = E_1^{\diamond\diamond} \cap E_2^{\diamond\diamond}$ ,  $(E_1 \cup E_2)^{\diamond\diamond} = E_1^{\diamond\diamond} \cup E_2^{\diamond\diamond}$ .
- (vi)  $\overline{E^{\diamond\diamond}} = (\overline{E})^{\diamond\diamond}$ . In particular, if  $E$  is closed, then so is  $E^{\diamond\diamond}$ .
- (vii) If  $E$  is open, then  $E^{\diamond\diamond}$  is open.
- (viii) If  $E$  is compact, then  $E^{\diamond\diamond}$  is compact.

**Proof.** Items (i)–(v) follow from Item (iii), Proposition 9.

(vi): As  $E^{\diamond\diamond} (= Q)$  is permutation invariant, this follows from Proposition 12 and Corollary 2.

(vii): As  $E^{\diamond\diamond}$  is permutation invariant, it is clear from Theorem 10 and Proposition 12 that  $E^{\diamond\diamond}$  is open whenever  $E$  is open.

(viii): Suppose  $E$  is compact. Then, by Theorem 10,  $Q := E^{\diamond\diamond}$  is compact and permutation invariant. By Theorem 27 in [1],  $E^{\diamond\diamond} = \lambda^{-1}(Q)$  is also compact.  $\square$

## Appendix A

**Theorem 11.**  $\mathcal{V}$  is essentially simple (that is,  $\mathcal{V}$  is either  $\mathcal{R}^n$  or simple) if and only if any Jordan frame of  $\mathcal{V}$  can be mapped onto another by an automorphism of  $\mathcal{V}$ .

**Proof.** The ‘only if’ part has been observed in Proposition 2. We prove the ‘if’ part. Suppose, if possible,  $\mathcal{V}$  is a product of simple algebras with at least one of the factors, say,  $\mathcal{V}_1$ , has rank more than one. Let  $\mathcal{V}_2$  denote the product of the remaining factors. Then,  $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ . Let  $e_1, e_2, \dots, e_m$  ( $m \geq 2$ ) and  $f_1, f_2, \dots, f_k$  be Jordan frames in  $\mathcal{V}_1$  and  $\mathcal{V}_2$  respectively. Let  $c_i = (e_i, 0)^T$  and  $d_j = (0, f_j)^T$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, k$ . Then these  $m + k$  objects, in any specified order, form a Jordan frame. We show that  $\mathcal{V}$  does not have an automorphism taking  $c_1$  to  $c_1$  and  $c_2$  to  $d_1$ . Assuming the contrary, let such an automorphism be given in the block form by

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where  $A : \mathcal{V}_1 \rightarrow \mathcal{V}_1$  is linear, etc. It is easy to verify that  $A$  is Jordan homomorphism, that is, it preserves the Jordan product on  $\mathcal{V}_1$ . Since  $\mathcal{V}_1$  is simple, the kernel of  $A$  (which is an ideal of  $\mathcal{V}_1$ ) must be either  $\{0\}$  or  $\mathcal{V}_1$ . However, this cannot happen as  $L(c_1) = c_1$  implies  $A(e_1) = e_1$  and  $L(c_2) = d_1$  implies  $A(e_2) = 0$ . Thus we have the ‘if’ part.  $\square$

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