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Spectral cones in Euclidean Jordan algebras

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ABSTRACT

A spectral cone in a Euclidean Jordan algebra $\mathcal{V}$ of rank n is of the form $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$ and $\lambda : \mathcal{V} \to \mathcal{R}^n$ is the eigenvalue map (which takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). In this paper, we describe some properties of spectral cones. We show, for example, that spectral cones are invariant under automorphisms of $\mathcal{V}$, that the dual of a spectral cone is a spectral cone when $\mathcal{V}$ is simple or carries the canonical inner product, and characterize the pointedness/solidness of a spectral cone. We also show that for any spectral cone K in $\mathcal{V}$, $\dim(K) \in \{0, 1, m-1, m\}$, where $\dim(K)$ denotes the dimension of K and m is the dimension of $\mathcal{V}$.

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1. Introduction

Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n and $\lambda : \mathcal{V} \to \mathcal{R}^n$ denote the eigenvalue map (which takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). A set E in $\mathcal{V}$ is said to be a *spectral set* [1] if there exists a permutation

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invariant set Q in $\mathcal{R}^n$ such that $E = \lambda^{-1}(Q)$. A function $F : \mathcal{V} \to \mathcal{R}$ is said to be a *spectral function* [1] if there is a permutation invariant function $f : \mathcal{R}^n \to \mathcal{R}$ such that $F = f \circ \lambda$. Motivated by the works of Baes [1], Sun and Sun [11], Ramirez, Seeger, and Sossa [10], in [7], we presented some new results on spectral sets and functions, specifically addressing characterization, invariance under automorphisms, Schur-convexity, etc. In the present paper, we study *spectral cones*, which, by definition, are spectral sets of the form

$$K = \lambda^{-1}(Q),$$

where Q is a *permutation invariant convex cone* in $\mathcal{R}^n$. The symmetric cone in a Euclidean Jordan algebra is an important example of a spectral cone as it comes from $Q = \mathcal{R}^n_+$ (the nonnegative orthant in $\mathcal{R}^n$). In this paper, we describe some properties of spectral cones, specifically looking for properties that are similar to those of symmetric cones. We show, for example, that spectral cones are invariant under automorphisms of $\mathcal{V}$, that the dual of a spectral cone is a spectral cone when $\mathcal{V}$ is simple or carries the canonical inner product, and characterize the pointedness/solidness of a spectral cone. We also show that for any spectral cone K in $\mathcal{V}$, $\dim(K) \in \{0, 1, m-1, m\}$, where $\dim(K)$ denotes the dimension of K and m is the dimension of $\mathcal{V}$.

The paper is organized as follows. In Section 2, we cover some preliminary material. Section 3 deals with the invariance of spectral sets under automorphisms, establishes the 'linearity' of λ^{-1} over permutation invariant sets in $\mathcal{R}^n$, and describes a duality result. In Section 4, we provide examples of polyhedral and non-polyhedral self-dual permutation invariant cones in $\mathcal{R}^n$. Section 5 deals with a characterization of spectral cones and studies the closure, interior, dual, and orthogonal complement of a spectral cone. The dimensionality of a spectral cone is described in Section 6. Finally, in Section 7, we characterize pointedness/solidness of spectral cones.

2. Preliminaries

For a set S in a real inner product space $(H, \langle \cdot, \cdot \rangle)$, the closure, interior, and boundary are denoted, respectively, by $\overline{S}$, S°, and ∂S. The dual and the orthogonal complement of S are defined by

$$S^* = \{x \in H : \langle x, y \rangle \geq 0 \ \forall \ y \in S\} \quad \text{and} \quad S^\perp = S^* \cap -S^*.$$

We say that S is a *cone* if $0 < \alpha \in \mathcal{R}$, $x \in S \Rightarrow \alpha x \in S$ and *convex* if $x, y \in S$, $0 \leq \alpha, \beta \in \mathcal{R}$, $\alpha + \beta = 1 \Rightarrow \alpha x + \beta y \in S$. Also, a convex cone S is *pointed* if $S \cap -S \subseteq \{0\}$ and *solid* if $S^\circ \neq \emptyset$. Given a set S, we write $\mathrm{conv}(S)$ for its *convex hull* and $\mathrm{cone}(S)$ for its *convex conic hull*. For sets S_1 and S_2 and real numbers α_1 and α_2, we let $\alpha_1 S_1 + \alpha_2 S_2 := \{\alpha_1 x + \alpha_2 y : x \in S_1, y \in S_2\}$.

2.1. Permutation matrices and majorization in $\mathcal{R}^n$

Vectors in $\mathcal{R}^n$ are considered as column vectors and $\mathcal{R}^n$ carries the usual inner product. An $n \times n$ *permutation matrix* is a matrix obtained by permuting the rows of an $n \times n$ identity matrix. The set of all $n \times n$ permutation matrices is denoted by Σ_n. A set Q in $\mathcal{R}^n$ is said to be *permutation invariant* if $\sigma(Q) = Q$ for all $\sigma \in \Sigma_n$.

For any $u = (u_1, u_2, \ldots, u_n)^{\mathsf{T}}$ in $\mathcal{R}^n$ consider $\Sigma_n(u) = \{\sigma(u) : \sigma \in \Sigma_n\}$, the set of all possible permutations of u. If we look (only) at the first components of vectors in this collection, we see u_i (for $i = 1, 2, \ldots, n$) appearing exactly $(n-1)!$-times. Hence, adding all these first components, we get the sum $(n-1)!\ \mathrm{tr}(u)$, where $\mathrm{tr}(u) = u_1 + u_2 + \cdots + u_n$. The same sum is obtained when other components are considered. Thus,

$$\sum_{\sigma \in \Sigma_n} \sigma(u) = (n-1)!\ \mathrm{tr}(u)\, \mathbf{1}, \tag{1}$$

where $\mathbf{1}$ denotes the vector in $\mathcal{R}^n$ with all entries 1.

Given $u \in \mathcal{R}^n$, we write $u^{\downarrow}$ for its decreasing rearrangement: that is the vector obtained by rearranging the coordinates of u in the decreasing order. Likewise, we write $u^{\uparrow}$ for the increasing rearrangement of u. For two vectors u and v in $\mathcal{R}^n$ with their decreasing rearrangements $u^{\downarrow}$ and $v^{\downarrow}$, we say that u is *majorized* by v [9] and write $u \prec v$ if

$$\sum_{i=1}^{k} u_i^{\downarrow} \le \sum_{i=1}^{k} v_i^{\downarrow} \text{ for } 1 \le k \le n-1 \qquad \text{and} \qquad \sum_{i=1}^{n} u_i^{\downarrow} = \sum_{i=1}^{n} v_i^{\downarrow}.$$

A theorem of Hardy, Littlewood, and Pólya ([2], Theorem II.1.10) says that $u \prec v$ if and only if $u = Av$, for some doubly stochastic matrix A (which is a nonnegative matrix with row and column sums one). Also, a theorem of Birkhoff ([2], Theorem II.2.3) says that every doubly stochastic matrix is a convex combination of permutation matrices.

The following elementary proposition will be useful.

Proposition 2.1. *Suppose $u\,(\neq 0)$ and v be vectors in $\mathcal{R}^n$ with decreasing entries and $\sum_1^n u_i = \sum_1^n v_i = 0$. Then,*

(i) $\sum_1^k u_i > 0$ for all k with $1 \le k \le n-1$, and
(ii) $v \prec \alpha\, u$ for some positive number α.

Proof. Suppose $\sum_1^k u_i \le 0$ for some k with $1 \le k \le n-1$. As $\sum_1^n u_i = 0$, we have $\sum_{k+1}^n u_i \ge 0$. Since the entries of u are decreasing, we must have $u_{k+1} \ge 0$. This implies that $u_1 \ge u_2 \ge \cdots \ge u_k \ge 0$. But then, $\sum_1^k u_i \le 0$ implies that $u_1 = u_2 = \cdots = u_k = 0$. From this we get $0 \ge u_{k+1} \ge u_{k+2} \ge \cdots \ge u_n$. As these inequalities imply $0 \ge \sum_{k+1}^n u_i$, we see that $0 = \sum_{k+1}^n u_i$ from which we get $0 = u_{k+1} = u_{k+2} = \cdots = u_n$. Thus, $u = 0$, leading to a contradiction. Hence we have (i).

Now, because of (i), we can find a positive α such that $\sum_1^k v_i \leq \alpha(\sum_1^k u_i)$ for all k with $1 \leq k \leq n-1$. Since $\alpha(\sum_1^n u_i) = \sum_1^n v_i = 0$, we see that $v \prec \alpha\, u$. This gives (ii). □

2.2. Euclidean Jordan algebras

Throughout this paper, $\mathcal{V}$ denotes a Euclidean Jordan algebra [3]; for short, we say that $\mathcal{V}$ is an algebra. For $x, y \in \mathcal{V}$, we denote their inner product by $\langle x, y\rangle$ and Jordan product by $x \circ y$. We let e denote the unit element in $\mathcal{V}$ and

$$\mathcal{V}_+ := \{x \circ x : x \in \mathcal{V}\}$$

denote the corresponding *symmetric cone*. Recall that a Euclidean Jordan algebra $\mathcal{V}$ is *simple* if it is not a direct sum of nonzero Euclidean Jordan algebras. It is known, see [3], that any nonzero Euclidean Jordan algebra is, in a unique way, a direct sum/product of simple Euclidean Jordan algebras. Moreover, every simple algebra is isomorphic to one of the following five algebras:

(i) the algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices,
(ii) the algebra $\mathcal{H}^n$ of $n \times n$ complex Hermitian matrices,
(iii) the algebra $\mathcal{Q}^n$ of $n \times n$ quaternion Hermitian matrices,
(iv) the algebra $\mathcal{O}^3$ of 3×3 octonion Hermitian matrices,
(v) the Jordan spin algebra $\mathcal{L}^n$ for $n \geq 3$.

The Euclidean space $\mathcal{R}^n$ is a Euclidean Jordan algebra under componentwise product of vectors and the usual inner product.

We say that an algebra $\mathcal{V}$ is *essentially simple* if it is either simple or isomorphic to $\mathcal{R}^n$.

An element $c \in \mathcal{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say a finite set $\{e_1, e_2, \ldots, e_n\}$ of primitive idempotents in $\mathcal{V}$ is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j \quad \text{and} \quad \sum_{i=1}^{n} e_i = e.$$

It turns out that the number of elements in any Jordan frame is the same; this common number is called the *rank* of $\mathcal{V}$.

Proposition 2.2 *(Spectral decomposition theorem [3]). Suppose $\mathcal{V}$ is a Euclidean Jordan algebra of rank n. Then, for every $x \in \mathcal{V}$, there exist uniquely determined real numbers $\lambda_1(x), \ldots, \lambda_n(x)$ (called the* eigenvalues *of x) and a Jordan frame $\{e_1, \ldots, e_n\}$ such that*

$$x = \lambda_1(x)e_1 + \cdots + \lambda_n(x)e_n.$$

Conversely, given any Jordan frame $\{e_1, \ldots, e_n\}$ *and real numbers* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *the sum* $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ *defines an element of* $\mathcal{V}$ *whose eigenvalues are* $\lambda_1, \lambda_2, \ldots, \lambda_n$.

The *trace* of any $x \in \mathcal{V}$ is defined by:

$$\mathrm{tr}(x) := \sum_{i=1}^{n} \lambda_i(x).$$

Note that when $\mathcal{V} = \mathcal{R}^n$, for any $u = (u_1, u_2, \ldots, u_n)^{\mathsf{T}}$, $\mathrm{tr}(u) = u_1 + u_2 + \cdots + u_n$.

From now on, we assume that the eigenvalues of $x \in \mathcal{V}$ are arranged so that $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$ and let

$$\lambda(x) = \Big(\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)\Big)^{\mathsf{T}}.$$

We note that the mapping $x \mapsto \lambda(x)$, called the *eigenvalue map*, is well-defined, continuous [1] and positively homogeneous.

We say that two elements $x, y \in \mathcal{V}$ are *spectrally equivalent* if they have the same eigenvalues; symbolically,

$$x \sim y \text{ in } \mathcal{V} \;\Leftrightarrow\; \lambda(x) = \lambda(y) \text{ in } \mathcal{R}^n.$$

Starting from the given inner product on $\mathcal{V}$, we can define the *canonical inner product* by

$$\langle x, y \rangle_c := tr(x \circ y).$$

This inner product is compatible with the given Jordan product; idempotents, Jordan frames, spectral decomposition and eigenvalues of an element, etc., remain the same under this new inner product. We note that *with respect to the canonical inner product, the norm of any primitive idempotent is one.* It is known, see [3], Prop. III,4,1, that in any simple algebra, the inner product is a (positive) multiple of the canonical inner product. Thus, *when* $\mathcal{V}$ *is either simple or carries the canonical inner product, all primitive idempotents have the same norm.*

Proposition 2.3 *(Peirce decomposition theorem [3]). Let* $\mathcal{V}$ *be of rank* n *and* $\{e_1, \ldots, e_n\}$ *be a Jordan frame of* $\mathcal{V}$. *Then the space* $\mathcal{V}$ *is the orthogonal direct sum of subspaces* $\mathcal{V}_{ij}$ $(i \leq j)$, *where for* $i, j \in \{1, \ldots, n\}$,

$$\mathcal{V}_{ii} := \{x \in \mathcal{V} : x \circ e_i = x\} = \mathcal{R}e_i,$$

$$\mathcal{V}_{ij} := \{x \in \mathcal{V} : x \circ e_i = \tfrac{1}{2}x = x \circ e_j\} \text{ when } i \neq j.$$

Thus, given any Jordan frame $\{e_1, \ldots, e_n\}$, we can write any element $x \in \mathcal{V}$ as

$$x = \sum_{i=1}^{n} x_i e_i + \sum_{1 \le i < j \le n} x_{ij},$$

where $x_i \in \mathcal{R}$ and $x_{ij} \in \mathcal{V}_{ij}$. This expression is called the *Peirce decomposition* of x associated with $\{e_1, \ldots, e_n\}$.

Let ϕ and Ψ be linear transformations on $\mathcal{V}$. We say that

- ϕ is an *algebra automorphism* (or 'automorphism' for short) if it is invertible and

$$\phi(x \circ y) = \phi(x) \circ \phi(y) \quad \forall\ x, y \in \mathcal{V}.$$

  The set of all automorphisms of $\mathcal{V}$ is denoted by $\mathrm{Aut}(\mathcal{V})$. We note that automorphisms map Jordan frames to Jordan frames. Thus, eigenvalues of an element remain the same under the action of an automorphism. In particular,

$$\phi(x) \sim x \quad \forall\ x \in \mathcal{V},\ \phi \in \mathrm{Aut}(\mathcal{V}).$$

  We say that a set Ω in $\mathcal{V}$ is *invariant under automorphisms* (or *automorphism invariant*) if

$$\phi(\Omega) \subseteq \Omega \quad \forall\ \phi \in \mathrm{Aut}(\mathcal{V}).$$

- Ψ is *doubly stochastic* [4] if it is positive, unital, and trace preserving, i.e., $\Psi(\mathcal{V}_+) \subseteq \mathcal{V}_+$, $\Psi(e) = e$, and $\mathrm{tr}(\Psi(x)) = \mathrm{tr}(x)$ for all $x \in \mathcal{V}$. When $\mathcal{V} = \mathcal{R}^n$, such a Ψ is a *doubly stochastic matrix* (which is an $n \times n$ nonnegative matrix with each row/column sum one).

We recall the following two results from [7].

Proposition 2.4. *([3], Theorem IV.2.5, [7], Proposition 3) Let $\mathcal{V}$ be essentially simple. If $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ are any two Jordan frames in $\mathcal{V}$, then there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $\phi(e_i) = e'_i$ for all $i = 1, \ldots, n$. In particular, if $x \sim y$ in $\mathcal{V}$, then there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $x = \phi(y)$.*

Proposition 2.5. *([7], Proposition 9) Given $x, y \in \mathcal{V}$, there exist doubly stochastic matrices A and B such that*

$$\lambda(x + y) = A\lambda(x) + B\lambda(y). \tag{2}$$

When $\mathcal{V}$ is simple, we can take $A = B$.

Our next two results are from [4].

Proposition 2.6. *([4], Proposition 1) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two non-isomorphic simple Euclidean Jordan algebras. If $\phi \in \mathrm{Aut}(\mathcal{V}_1 \times \mathcal{V}_2)$, then ϕ is of the form (ϕ_1, ϕ_2) for some $\phi_i \in \mathrm{Aut}(\mathcal{V}_i)$, $i = 1, 2$, that is,*

$$\phi(x) = (\phi_1(x_1), \phi_2(x_2)), \quad \forall\, x = (x_1, x_2) \in \mathcal{V}_1 \times \mathcal{V}_2.$$

The result below describes a connection between majorization, automorphisms, and doubly stochastic transformations in the setting of Euclidean Jordan algebras. We define *majorization* in $\mathcal{V}$ by

$$x \prec y \text{ in } \mathcal{V} \Leftrightarrow \ \lambda(x) \prec \lambda(y) \text{ in } \mathcal{R}^n.$$

Proposition 2.7. *([4], Theorem 6) For $x, y \in \mathcal{V}$, consider the following statements:*

(a) $x = \Phi(y)$, where Φ is a convex combination of automorphisms of $\mathcal{V}$.
(b) $x = \Psi(y)$, where Ψ is doubly stochastic on $\mathcal{V}$.
(c) $x \prec y$.

Then, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Furthermore, when $\mathcal{V}$ is essentially simple, reverse implications hold.

3. Spectral sets in Euclidean Jordan algebras

In preparation for a study of spectral cones, in this section, we describe some results on spectral sets. Recall that a set E in $\mathcal{V}$ is a spectral set if there exists a permutation invariant set Q in $\mathcal{R}^n$ such that $E = \lambda^{-1}(Q)$.

Proposition 3.1. *([7], Theorem 1) A set E in $\mathcal{V}$ is a spectral set if and only if*

$$x \sim y,\ y \in E \Rightarrow x \in E.$$

One key observation is the following: Suppose E is a spectral set, $x \in E$ with its spectral decomposition $x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$. Then for any Jordan frame $\{e'_1, e'_2, \ldots, e'_n\}$, the element $\bar{x} := \lambda_1 e'_1 + \lambda_2 e'_2 + \cdots + \lambda_n e'_n \in E$ (as $x \sim \bar{x}$).

Proposition 3.2. *([7], Theorem 3) For a set E in $\mathcal{V}$, consider the following:*

(a) $E = \lambda^{-1}(Q)$, where Q is permutation invariant and convex in $\mathcal{R}^n$.
(b) E is convex, and $x \prec y$, $y \in E \Rightarrow x \in E$.
(c) E is convex, and $x \sim y$, $y \in E \Rightarrow x \in E$.
(d) E is convex and invariant under automorphisms.

Then $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ *and* $(c) \Rightarrow (d)$. *Furthermore, all statements are equivalent when* $\mathcal{V}$ *is essentially simple.*

Let E be a spectral set in $\mathcal{V}$ so that $E = \lambda^{-1}(Q)$ for some permutation invariant set Q in $\mathcal{R}^n$. Then, $\Sigma_n(\lambda(E)) \subseteq Q$. For any $q = (q_1, q_2, \ldots, q_n)^{\mathsf{T}} \in Q$ and any Jordan frame $\{e_1, e_2, \ldots, e_n\}$, $x := \sum_1^n q_i e_i \in E$ because $\lambda(x) = q^{\downarrow} \in Q$. Since $q = \sigma(q^{\downarrow})$ for some $\sigma \in \Sigma_n$, it follows that $Q \subseteq \Sigma_n(\lambda(E))$; hence

$$Q = \Sigma_n(\lambda(E)). \tag{3}$$

This means that Q is uniquely defined for each spectral set E. In this setting, by defining $E^{\diamondsuit} := Q$ and $Q^{\diamondsuit} := E$, we have (from [7], Section 6) the metaformula(s)

$$\#(E^{\diamondsuit}) = [\#(E)]^{\diamondsuit} \quad \text{and} \quad \#(Q^{\diamondsuit}) = [\#(Q)]^{\diamondsuit}, \tag{4}$$

where $\#$ is one of the operations of closure, interior, boundary, and convex hull. In particular, closure/interior/boundary/convex hull of a spectral set is a spectral set.

The following result shows that on permutation invariant (convex) sets, λ^{-1} has linear behavior.

Theorem 3.3. *The following statements hold:*

(i) *Let* Q *be a permutation invariant set in* $\mathcal{R}^n$ *and* $\alpha \geq 0$ *in* $\mathcal{R}$. *Then*

$$\lambda^{-1}(-Q) = -[\lambda^{-1}(Q)] \quad \textit{and} \quad \lambda^{-1}(\alpha\, Q) = \alpha\, \lambda^{-1}(Q).$$

(ii) *Let* Q_1 *and* Q_2 *be permutation invariant convex sets in* $\mathcal{R}^n$. *Then,*

$$\lambda^{-1}(Q_1 + Q_2) = \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2).$$

Proof. (i) We first observe that $-Q$ is permutation invariant. Let $x \in \lambda^{-1}(Q)$ with its spectral decomposition $x = \sum_1^n \lambda_i(x) e_i$. From $-x = \sum_1^n [-\lambda_i(x)] e_i$ we get $\lambda(-x) = [-\lambda(x)]^{\downarrow} = -[\lambda(x)^{\uparrow}]$. Since Q is permutation invariant and $\lambda(x) \in Q$, we have $\lambda(x)^{\uparrow} \in Q$ and $-[\lambda(x)^{\uparrow}] \in -Q$. Thus, $\lambda(-x) \in -Q$ and $-x \in \lambda^{-1}(-Q)$. Hence, $-[\lambda^{-1}(Q)] \subseteq \lambda^{-1}(-Q)$. Now, let $y \in \lambda^{-1}(-Q)$. Then, by the previous inclusion applied to $-Q$, $-y \in \lambda^{-1}[-(-Q)] = \lambda^{-1}(Q)$, or equivalently, $y \in -[\lambda^{-1}(Q)]$. This proves the inclusion $\lambda^{-1}(-Q) \subseteq -[\lambda^{-1}(Q)]$. Thus we have the first part of statement (i). The second part, for $\alpha = 0$ is obvious; the case $\alpha > 0$ follows easily from the positive homogeneity of λ.

(ii) Now suppose that Q_1 and Q_2 are permutation invariant convex sets in $\mathcal{R}^n$. Then $Q_1 + Q_2$ is also permutation invariant and convex. Let $x \in \lambda^{-1}(Q_1)$ and $y \in \lambda^{-1}(Q_2)$ so that $\lambda(x) \in Q_1$ and $\lambda(y) \in Q_2$. Then, by Proposition 2.5,

$$\lambda(x + y) = A\, \lambda(x) + B\, \lambda(y),$$

where A and B are doubly stochastic matrices on $\mathcal{R}^n$. By a well-known theorem of Birkhoff ([2], Theorem II.2.3), A and B are convex combinations of permutation matrices. As Q_1 and Q_2 are permutation invariant convex sets, it follows that $A\lambda(x) \in Q_1$ and $B\lambda(y) \in Q_2$. Thus, $\lambda(x+y) \in Q_1 + Q_2$. This implies that $x+y \in \lambda^{-1}(Q_1+Q_2)$. Hence,

$$\lambda^{-1}(Q_1) + \lambda^{-1}(Q_2) \subseteq \lambda^{-1}(Q_1+Q_2).$$

To see the reverse inclusion, let $z \in \lambda^{-1}(Q_1+Q_2)$ with spectral decomposition $z = \sum_1^n \lambda_i(z)e_i$. Then, $\lambda(z) \in Q_1+Q_2$. Let $\lambda(z) = u+v$, where $u \in Q_1$ and $v \in Q_2$. Define $x = \sum_1^n u_i e_i$ and $y = \sum_1^n v_i\, e_i$ so that $z = x+y$. As $\lambda(x) = u^{\downarrow} \in Q_1$ and $\lambda(y) = v^{\downarrow} \in Q_2$, we see that $x \in \lambda^{-1}(Q_1)$, $y \in \lambda^{-1}(Q_2)$. Thus, $z = x+y \in \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2)$. Hence, $\lambda^{-1}(Q_1+Q_2) \subseteq \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2)$. Thus we have (ii). □

The following is an easy consequence.

Corollary 3.4. *Let Q_1 and Q_2 be permutation invariant convex sets in $\mathcal{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{R}$. Then,*

$$\lambda^{-1}(\alpha_1\, Q_1 + \alpha_2\, Q_2) = \alpha_1\, \lambda^{-1}(Q_1) + \alpha_2\, \lambda^{-1}(Q_2).$$

Our next result deals with the dual of a spectral set. For a permutation invariant set Q in $\mathcal{R}^n$, we note that its dual (computed with respect to the usual inner product in $\mathcal{R}^n$) is also permutation invariant as

$$\langle \sigma(u), q\rangle = \langle u, \sigma^{\mathsf{T}}(q)\rangle \geq 0 \quad (\forall\ u \in Q^*,\ q \in Q, \text{ and } \sigma \in \Sigma_n).$$

The dual of a spectral set is computed with respect to the given inner product in $\mathcal{V}$.

Theorem 3.5. *Suppose $\mathcal{V}$ is either simple or carries the canonical inner product. Then, for any permutation invariant set Q in $\mathcal{R}^n$,*

$$\left[\lambda^{-1}(Q)\right]^* = \lambda^{-1}(Q^*).$$

Proof. By assumption, all primitive idempotents in $\mathcal{V}$ have the same norm (see the remarks made after the definition of canonical inner product in Section 2). Let $\omega := ||c||^2$ for any primitive idempotent c in $\mathcal{V}$. Now, let $x \in \left[\lambda^{-1}(Q)\right]^*$ and $x = \sum_i \lambda_i(x)e_i$ be its spectral decomposition. For any $q \in Q$, set $y = \sum_i q_i e_i$. As $\lambda(y) = q^{\downarrow} \in Q$, we have $y \in \lambda^{-1}(Q)$ and so $\langle x,\, y\rangle \geq 0$. Thus, we have

$$0 \leq \langle x,\, y\rangle = \left\langle \sum_{i=1}^n \lambda_i(x)e_i,\, \sum_{i=1}^n q_i e_i \right\rangle = \sum_{i=1}^n \lambda_i(x) q_i\, \|e_i\|^2 = \omega\ \langle \lambda(x),\, q\rangle\,,$$

where $\langle \lambda(x),\, q\rangle$ denotes the usual inner product between the vectors $\lambda(x)$ and q in $\mathcal{R}^n$. Since $\omega > 0$, $\langle \lambda(x),\, q\rangle \geq 0$. As q is arbitrary in Q, we have $\lambda(x) \in Q^*$ and so $x \in \lambda^{-1}(Q^*)$. This shows $\left[\lambda^{-1}(Q)\right]^* \subseteq \lambda^{-1}(Q^*)$.

To prove the reverse implication, let $x \in \lambda^{-1}(Q^*)$ so that $\lambda(x) \in Q^*$. Now, let $y \in \lambda^{-1}(Q)$ with its spectral decomposition $y = \sum_i \lambda_i(y) e_i$. We write the Peirce decomposition of x with respect to the Jordan frame $\{e_1, \ldots, e_n\}$ as

$$x = \sum_{i=1}^{n} x_i e_i + \sum_{i<j} x_{ij},$$

and define $\operatorname{diag}(x) := \sum_{i=1}^{n} x_i e_i$. Let $u := (x_1, x_2, \ldots, x_n)^{\mathsf{T}}$ in $\mathcal{R}^n$. Because of the orthogonality of the Peirce spaces, we have

$$\langle x,\, y\rangle = \left\langle \sum_{i=1}^{n} x_i e_i,\, \sum_{i=1}^{n} \lambda_i(y) e_i \right\rangle = \sum_{i=1}^{n} x_i \lambda_i(y) \left\|e_i\right\|^2 = \omega\, \langle u,\, \lambda(y)\rangle\,. \tag{5}$$

Now, it is known that $\operatorname{diag}(x) \prec x$ (see [4], Example 7); hence, $u \prec \lambda(x)$. By a well-known theorem of Hardy, Littlewood, and Pólya ([2], Theorem II.1.10), $u = A\lambda(x)$ for some doubly stochastic matrix A; furthermore, by a theorem of Birkhoff ([2], Theorem II.2.3), this A is a convex combination of permutation matrices. Thus,

$$u = \left(\sum_{k=1}^{N} \alpha_k \sigma_k\right) \lambda(x), \quad \text{where } \alpha_k > 0 \text{ with } \sum_{k=1}^{N} \alpha_k = 1, \text{ and } \sigma_k \in \Sigma_n.$$

Since Q^* is permutation invariant and convex, $u \in Q^*$.

Since $y \in \lambda^{-1}(Q) \Rightarrow \lambda(y) \in Q$, we must have $\langle u, \lambda(y)\rangle \geq 0$. Hence, from (5), we get $\langle x,\, y\rangle \geq 0$. As y is arbitrary in $\lambda^{-1}(Q)$, this proves that $x \in [\lambda^{-1}(Q)]^*$. Thus, $\lambda^{-1}(Q^*) \subseteq \left[\lambda^{-1}(Q)\right]^*$, completing the proof. □

The following example shows that the dual of a spectral set need not always be spectral.

Example 1. Consider the algebra $\mathcal{V} = \mathcal{R}^2$, where for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the Jordan and inner products are defined, respectively, by $x * y := (x_1 y_1, x_2 y_2)$ and $\langle x, y\rangle := 2x_1 y_1 + 3x_2 y_2$. Since the set (subspace) $Q := \{(x_1, x_2) : x_1 = x_2\}$ in $\mathcal{R}^2$ is permutation invariant,

$$K := \lambda^{-1}(Q) = \mathcal{R}\, e$$

is a spectral set in our algebra $\mathcal{V}$. With respect to the above inner product,

$$K^* = K^{\perp} = \{(x_1, x_2) : 2x_1 + 3x_2 = 0\}.$$

Now, $(2,-3) \sim (-3,2)$ with $(-3,2) \in K^*$ and $(2,-3) \notin K^*$. Hence, by Proposition 3.1, K^* is not spectral.

4. Some permutation invariant convex cones in $\mathcal{R}^n$

In this section, we describe some permutation invariant convex cones in $\mathcal{R}^n$. Given any nonempty set S in $\mathcal{R}^n$,

$$Q := \text{cone}\,(\Sigma_n(S)),$$

the convex cone generated by $\Sigma_n(S)$, is a permutation invariant convex cone. For $p \in \{1,2,\ldots,n\}$, the *rearrangement cone* Q_p^n [5] is defined by

$$Q_p^n := \{u \in \mathcal{R}^n : \ u_1^{\uparrow} + u_2^{\uparrow} + \cdots + u_p^{\uparrow} \geq 0\},$$

where we note that $u_1^{\uparrow} + u_2^{\uparrow} + \cdots + u_p^{\uparrow}$ is the sum of the smallest p entries of $u \in \mathcal{R}^n$.

Proposition 4.1. *The following statements hold:*

(a) Q_p^n *is a permutation invariant polyhedral (closed convex) solid cone; it is pointed when* $1 \leq p \leq n-1$.
(b) $Q_1^n \subseteq Q_2^n \subseteq \cdots \subseteq Q_n^n$.
(c) Q_p^n *is isomorphic to* Q_{n-p}^n *for* $1 \leq p \leq n-1$.

Proof. (a) For any nonempty index set $I \subseteq \{1,2,\ldots,n\}$, let $|I|$ denote the cardinality of I and e_I be the vector with $(e_I)_i = 1$ for $i \in I$ and $(e_I)_i = 0$ otherwise. Then,

$$u_1^{\uparrow} + u_2^{\uparrow} + \cdots + u_p^{\uparrow} = \min_{|I|=p} \langle u,\, e_I \rangle\,,$$

hence,

$$\begin{aligned} Q_p^n &= \{u \in \mathcal{R}^n : \ u_1^{\uparrow} + u_2^{\uparrow} + \cdots + u_p^{\uparrow} \geq 0\} \\ &= \{u \in \mathcal{R}^n : \ \min_{|I|=p} \langle u, e_I \rangle \geq 0\} \\ &= \{u \in \mathcal{R}^n : \ \langle u,\, e_I \rangle \geq 0 \text{ for every } e_I \text{ with } |I| = p\}. \end{aligned}$$

Since Q_p^n is now the intersection of a finite number of closed half-spaces, it is a polyhedral closed convex cone. Since $\sigma(u)^{\uparrow} = u^{\uparrow}$ for any permutation matrix $\sigma \in \Sigma_n$, we see that Q_p^n is permutation invariant. That Q_p^n has nonempty interior follows from the inclusion $\mathcal{R}_+^n \subseteq Q_p^n$. Now let $1 \leq p \leq n-1$. To see the pointedness, suppose both u and $-u$ are in Q_p^n. Then, the inequalities $\min_{|I|=p} \langle u,\, e_I \rangle \geq 0$ and $\min_{|I|=p} \langle -u,\, e_I \rangle \geq 0$ imply that $\langle u,\, e_I \rangle = 0$ whenever $|I| = p$. This means that sum of any p entries of u is zero.

Hence, as $p \leq n-1$, all entries of u are equal, and so, $u = 0$. Thus, Q_p^n is pointed when $1 \leq p \leq n-1$.

(b) Let $1 \leq p \leq n-1$. If $u \in Q_p^n$, then $u_1^{\uparrow} + u_2^{\uparrow} + \cdots + u_p^{\uparrow} \geq 0$. Since $0 \leq u_p^{\uparrow} \leq u_{p+1}^{\uparrow}$, we get $u_1^{\uparrow} + u_2^{\uparrow} + \cdots + u_p^{\uparrow} + u_{p+1}^{\uparrow} \geq 0$. Thus, $u \in Q_{p+1}^n$.

(c) We let $1 \leq p \leq n-1$. Let $\mathbf{1}$ denote the vector in $\mathcal{R}^n$ with all entries one, $N := \mathbf{1}\,\mathbf{1}^{\mathsf{T}}$ (the $n \times n$ matrix with all entries one), and I denote the identity matrix. Defining

$$M_p := \frac{1}{p} N - I,$$

we easily verify that $M_p\, M_{n-p} = I$; hence M_p is invertible. Now, let $u \in Q_{n-p}^n$; this means that the sum of $n-p$ smallest entries of u is nonnegative, that is, $\sum_{p+1}^{n} u_i^{\downarrow} \geq 0$. Then, from $M_p u = \frac{\mathrm{tr}(u)}{p}\,\mathbf{1} - u$ and $(-u)^{\uparrow} = -\,u^{\downarrow}$ we have $(M_p u)^{\uparrow} = \frac{\mathrm{tr}(u)}{p}\,\mathbf{1} - u^{\downarrow}$. Hence,

$$\sum_{i=1}^{p} (M_p u)_i^{\uparrow} = \mathrm{tr}(u) - \sum_{1}^{p} u_i^{\downarrow} = \sum_{p+1}^{n} u_i^{\downarrow} \geq 0.$$

Thus $M_p u \in Q_p^n$, and so $M_p(Q_{n-p}^n) \subseteq Q_p^n$. Similarly, $M_{n-p}(Q_p^n) \subseteq Q_{n-p}^n$. Hence, $M_p(Q_{n-p}^n) = Q_p^n$. □

Our next example is a *permutation invariant non-polyhedral self-dual cone.*

Example 2. In $\mathcal{R}^n$ $(n \geq 3)$, let

$$Q = \left\{ u : \mathrm{tr}(u) \geq \sqrt{\frac{n}{2}}\, ||u|| \right\},$$

where $||u||$ denotes the 2-norm of the vector u in $\mathcal{R}^n$. As trace and the 2-norm are permutation invariant, we see that the above set is permutation invariant. We show that Q is the image of the Lorentz cone $\mathcal{L}_+^n$ (the symmetric cone in $\mathcal{L}^n$) under an orthogonal matrix. Recall

$$\mathcal{L}_+^n := \left\{ u = (u_1, u_2, \ldots, u_n)^{\mathsf{T}} : u_1 \geq \left[\sum_{2}^{n} u_i^2 \right]^{\frac{1}{2}} \right\} = \left\{ u \in \mathcal{R}^n : \langle u, c_1 \rangle \geq \frac{1}{\sqrt{2}} ||u|| ||c_1|| \right\},$$

where c_1 denotes the vector in $\mathcal{R}^n$ with one in the first slot and zeros elsewhere. It is easily seen that $Q = U(\mathcal{L}_+^n)$, where U is an orthogonal matrix with $Uc_1 = \frac{1}{\sqrt{n}}\,\mathbf{1}$. Since $\mathcal{L}_+^n$ is a non-polyhedral self-dual cone and U is orthogonal, it follows that Q is non-polyhedral and self-dual.

5. Spectral cones in Euclidean Jordan algebras

Recall that a spectral cone, by definition, is of the form $K = \lambda^{-1}(Q)$, for some permutation invariant convex cone Q in $\mathcal{R}^n$. It is easy to verify, by the positive homogeneity

of λ, that the above K is a cone and, by Proposition 3.2, convex. Thus, *every spectral cone is a convex cone.* The metaformula (4) shows that *the closure/interior of a spectral cone is again a spectral cone.* For a list of spectral cones in $\mathcal{S}^n$, see [6].

The following result characterizes spectral cones among convex cones in $\mathcal{V}$.

Theorem 5.1. *For a convex cone K in $\mathcal{V}$, consider the following statements:*

(a) *K is a spectral cone.*
(b) *If $x \prec y$ and $y \in K$, then $x \in K$.*
(c) *If $x \sim y$ and $y \in K$, then $x \in K$.*
(d) *For every doubly stochastic map Ψ on $\mathcal{V}$, $\Psi(K) \subseteq K$.*
(e) *For every algebra automorphism ϕ on $\mathcal{V}$, $\phi(K) \subseteq K$.*

Then, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ and $(b) \Rightarrow (d) \Rightarrow (e)$. Moreover, when $\mathcal{V}$ is essentially simple, all the above statements are equivalent.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$: When (a) holds, K is a spectral convex set. For such a set, (b) and (c) hold by Proposition 3.2.

$(c) \Rightarrow (a)$: When (c) holds, from Proposition 3.2, we can write $K = \lambda^{-1}(Q)$, where Q is permutation invariant and convex. Since K is given to be a cone, we see that Q (which equals $\Sigma_n(\lambda(K))$ by (3)) is a cone. Thus, K is a spectral cone.

$(b) \Rightarrow (d)$: Assume that K is spectral cone and let Ψ be a doubly stochastic map on $\mathcal{V}$. For $y \in K$, let $x = \Psi(y)$. Then by Proposition 2.7, $x \prec y$. It follows that $x \in K$.

$(d) \Rightarrow (e)$: This is obvious, as every algebra automorphism is doubly stochastic.

Now we prove $(e) \Rightarrow (b)$ assuming $\mathcal{V}$ is essentially simple. Then, Proposition 3.2 shows that K is a spectral set that corresponds to a permutation invariant convex set Q in $\mathcal{R}^n$. As K is also a cone, we see that Q is a cone. Hence, K is a spectral cone. □

Remark. When $\mathcal{V}$ is not essentially simple, the reverse implications in the above theorem may not hold. For instance, consider $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$, $K = \mathcal{R}_+ \times \mathcal{S}^2$, and

$$x = \Big(-1, \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] \Big), \; y = \Big(1, \left[\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right] \Big).$$

From Proposition 2.6, we see that K is invariant under automorphisms of $\mathcal{V}$. However, K is not a spectral set as $x \sim y$, $y \in K$, and $x \notin K$. Thus, (e) holds, but not (a).

The above example shows that the *Cartesian product of two spectral cones need not be spectral.* However, as we see below, certain projections of a spectral cone are spectral.

Corollary 5.2. *Suppose $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_N$ are essentially simple algebras and $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_N$. Let $\Pi_j : \mathcal{V} \to \mathcal{V}_j$ denote the projection map and K be a spectral cone in $\mathcal{V}$. Then $\Pi_j(K)$ is a spectral cone in $\mathcal{V}_j$ for all $j = 1, 2 \ldots, N$.*

Proof. By the linearity of Π_j, $\Pi_j(K)$ is a convex cone in $\mathcal{V}_j$. To show that $\Pi_j(K)$ is a spectral cone in the essentially simple algebra $\mathcal{V}_j$, we show that it is invariant under automorphisms of $\mathcal{V}_j$ and apply Theorem 5.1. Now, without loss of generality, let $j = 1$, $x_1 \in \Pi_1(K)$ and $\phi_1 \in \mathrm{Aut}(\mathcal{V}_1)$. Then, there exists $x_i \in \mathcal{V}_i$, $i = 2, 3, \ldots, N$ such that $x := (x_1, x_2, \ldots, x_N) \in K$. Let ϕ_i be the identity transformation on $\mathcal{V}_i$, $i = 2, \ldots, N$. Then, $\phi := (\phi_1, \phi_2, \phi_3, \ldots, \phi_N) \in \mathrm{Aut}(\mathcal{V})$, where the action of ϕ on $\mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_N$ is given by $\phi(z_1, z_2, \ldots, z_n) := (\phi(z_1), \phi(z_2), \ldots, \phi(z_n))$. Since $\phi(x) \in K$ by Theorem 5.1, we see that $\phi_1(x_1) \in \Pi_1(K)$. This completes the proof. □

Remarks. (1) The automorphisms of $\mathcal{R}^n$ are just permutation matrices. Since $\mathcal{R}^n$ is also essentially simple, spectral cones in $\mathcal{R}^n$ are just permutation invariant convex cones.

(2) Let $\mathcal{V}$ be essentially simple. For any $z \in \mathcal{V}$, consider

$$K = \mathrm{cone}\{\phi(z) : \phi \in \mathrm{Aut}(\mathcal{V})\},$$

the convex cone generated by $\{\phi(z) : \phi \in \mathrm{Aut}(\mathcal{V})\}$. Then, by Theorem 5.1, K is a spectral cone.

Theorem 5.3. *Let K_1, K_2, and K be spectral cones in $\mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathcal{R}$. Then, the following hold:*

(a) *$\alpha_1 K_1 + \alpha_2 K_2$, $\overline{K}$, and K° are spectral cones.*
(b) *When $\mathcal{V}$ is simple or carries the canonical inner product, K^* and $K^\perp$ are spectral cones.*

Proof. Let $K_i = \lambda^{-1}(Q_i)$, where Q_i is a permutation invariant convex cone in $\mathcal{R}^n$ for $i = 1, 2$. Then, $\alpha_1 Q_1 + \alpha_2 Q_2$ is a permutation invariant convex cone in $\mathcal{R}^n$ and from Corollary 3.4, $\alpha_1 K_1 + \alpha_2 K_2 = \lambda^{-1}(\alpha_1 Q_1 + \alpha_2 Q_2)$. Thus, $\alpha_1 K_1 + \alpha_2 K_2$ is a spectral cone. Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. It is easy to see that $\overline{Q}$, Q°, Q^* and $Q^\perp$ (defined with respect to the usual inner product in $\mathcal{R}^n$) are permutation invariant convex cones in $\mathcal{R}^n$. (Note that $Q^\perp = Q^* \cap -Q^*$.) That $\overline{K}$ and K° are spectral cones follow from the metaformula (4). Thus we have (a).

To see (b) suppose that $\mathcal{V}$ is simple or carries the canonical inner product. Then, Theorem 3.5 shows that K^* is a spectral cone. Finally, the equality

$$K^\perp = K^* \cap -K^* = \lambda^{-1}(Q^*) \cap \lambda^{-1}(-Q^*) = \lambda^{-1}(Q^* \cap -Q^*) = \lambda^{-1}(Q^\perp) \tag{6}$$

shows that $K^\perp$ is a spectral cone. □

Remark. In [6], Lemma 3, it is shown that the dual of a spectral cone in $\mathcal{S}^n$ is a spectral cone. Item (b) in the above theorem is a generalization. However, it does not hold in a general algebra: The set K in Example 1 is a spectral cone (actually, a subspace) while K^* (which is $K^\perp$) is not even a spectral set.

6. A dimensionality result

Inspired by a result of Lahtonen and its proof in [8], we present the following dimensionality result for spectral cones. Recall that the dimension of a convex cone K is the dimension of the subspace $K - K$. Throughout this section, for the Euclidean Jordan algebra $\mathcal{V}$, we let

$$m := \dim(\mathcal{V}).$$

Theorem 6.1. *For any spectral cone K in $\mathcal{V}$,*

$$\dim(K) \in \{0, 1, m-1, m\}.$$

Proof. We assume without loss of generality that K is nonempty and that $\mathcal{V}$ carries the canonical inner product so that the norm of any primitive idempotent is one. (See the remarks made after Proposition 2.2.) Now we show that either every element of K is a scalar multiple of e or every element of $K^\perp$ is a scalar multiple of e. Assuming the contrary, let $x \in K$ and $y \in K^\perp$ have at least two distinct eigenvalues. Writing their spectral decompositions,

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \quad \text{and} \quad y = y_1e'_1 + y_2e'_2 + \cdots + y_ne'_n,$$

we assume, without loss of generality, $x_1 \neq x_2$ and $y_1 \neq y_2$. Now, define $\overline{x}$, $\overline{\overline{x}}$ by

$$\overline{x} := x_1e'_1 + x_2e'_2 + x_3e'_3 + \cdots + x_ne'_n,$$
$$\overline{\overline{x}} := x_2e'_1 + x_1e'_2 + x_3e'_3 + \cdots + x_ne'_n.$$

We note that $\overline{x} \sim x$ and $\overline{\overline{x}} \sim x$. Since K is a spectral set and $x \in K$, by Theorem 5.1, $\overline{x}$, $\overline{\overline{x}} \in K$; hence, $\overline{x}$, $\overline{\overline{x}} \perp y$. Then,

$$\begin{aligned} 0 &= \langle \overline{x}, y\rangle - \langle \overline{\overline{x}}, y\rangle = \langle \overline{x} - \overline{\overline{x}}, y\rangle \\ &= \langle (x_1 - x_2)(e'_1 - e'_2), y_1e'_1 + y_2e'_2 + \cdots + y_ne'_n\rangle \\ &= (x_1 - x_2)y_1 - (x_1 - x_2)y_2 = (x_1 - x_2)(y_1 - y_2) \neq 0. \end{aligned}$$

We reach a contradiction. This shows that either K or $K^\perp$ must contain just multiples of e. Thus, the dimension of K or $K^\perp$ is at most 1. As $\dim(K)+\dim(K^\perp) = \dim(\mathcal{V}) = m$, we deduce that the possible values for $\dim(K)$ are 0, 1, $m-1$, and m. □

It is easy to see that a nonempty (spectral) cone has dimension zero if and only if it is $\{0\}$. Also, it is of dimension m if and only if it has nonempty interior. We now describe nonempty spectral cones in $\mathcal{V}$ with dimensions 1 and $m-1$.

Theorem 6.2. *For a nonempty spectral cone K in $\mathcal{V}$, the following statements hold.*

(*a*) $\dim(K) = 1$ *if and only if* K *is a nonzero cone contained in* $\mathcal{R}\, e$.
(*b*) $\dim(K) = m - 1$ *if and only if* $K = \{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$.

We note that in $\mathcal{V}$, there are five nonzero cones contained in $\mathcal{R}\, e$, namely, $\mathcal{R}_+\, e$, $\mathcal{R}_{++}\, e$, $-\mathcal{R}_+\, e$, $-\mathcal{R}_{++}\, e$, and $\mathcal{R}\, e$.

Proof. (*a*) As the 'if' part is obvious, we prove the 'only if' part. Suppose $\dim(K) = 1$. Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. We claim that Q has dimension one. If, on the contrary, u, v are linearly independent vectors in Q, then for any Jordan frame $\{e_1, e_2, \ldots, e_n\}$, $x := \sum_1^n u_i e_i$ and $y := \sum_1^n v_i e_i$ are two linearly independent elements in K contradicting $\dim(K) = 1$. Thus, Q has dimension one; let Q be contained in the span of a nonzero vector w. As Q is permutation invariant, this w must be a multiple of $\mathbf{1}$. Then $K = \lambda^{-1}(Q)$ is contained in $\mathcal{R}\, e$, as $\{e\} = \lambda^{-1}(\{\mathbf{1}\})$.

(*b*) If $K = \{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$, then it has dimension $m - 1$ as $x \mapsto \mathrm{tr}(x)$ from $\mathcal{V}$ to $\mathcal{R}$ is linear with null space K.

Now suppose (without loss of generality) $\mathcal{V}$ carries the canonical inner product and K has dimension $m - 1$. Letting $K = \lambda^{-1}(Q)$, we see, from (6), $\lambda^{-1}(Q^\perp) = K^\perp = (K - K)^\perp$ has dimension one. As in Item (*a*), we can show that $Q^\perp$ is spanned by the vector $\mathbf{1}$ in $\mathcal{R}^n$; hence, $\mathrm{tr}(v) = \langle v, \mathbf{1}\rangle = 0$ for all $v \in Q$. This means that $Q \subseteq M$, where $M = \{v \in \mathcal{R}^n : \mathrm{tr}(v) = 0\}$. Now, take any vector $v \in M$ and a nonzero $u \in Q$. As M and Q are permutation invariant, we may assume that the entries of u and v are decreasing. Note that $\sum_{i=1}^n v_i = \sum_{i=1}^n u_i = 0$. Now, from Proposition 2.1, $v \prec \alpha\, u$ for some $\alpha > 0$. Since Q is a permutation invariant convex cone, $\alpha\, u \in Q$ and by Proposition 3.2, $v \in Q$. This proves that $Q = M$. From this, we get $K = \lambda^{-1}(Q) = \{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$. This proves the result. □

Corollary 6.3. *Suppose K is a spectral cone in $\mathcal{V}$ such that $\{0\} \subseteq K \subseteq \{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$. Then, either $K = \{0\}$ or $K = \{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$.*

Proof. For the specified K, let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. Then,

$$\{0\} \subseteq \lambda^{-1}(Q) \subseteq \lambda^{-1}(M),$$

where $M = \{v \in \mathcal{R}^n : \mathrm{tr}(v) = 0\}$. Now, from (3), $P = \Sigma_n(\lambda(\lambda^{-1}(P)))$ for any permutation invariant set P in $\mathcal{R}^n$; thus,

$$\{0\} \subseteq Q \subseteq M.$$

The proof of Item (*b*) in the above theorem shows that either $Q = \{0\}$ or $Q = M$. Then, K, which is $\lambda^{-1}(Q)$, is either $\{0\}$ or $\{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$. □

Remark. *Suppose K is a nonempty spectral cone different from $\{0\}$ and $\{x \in \mathcal{V} : \mathrm{tr}(x) = 0\}$. Then, either $e \in K$ or $-e \in K$.* This is seen as follows. Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. Then, Q is nonempty and different from $\{0\}$ and $\{u \in \mathcal{R}^n : \mathrm{tr}(u) = 0\}$. Let u be a nonzero element of Q with $\mathrm{tr}(u) \neq 0$. As Q is permutation invariant, from (1),

$$(n-1)!\,\mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u) \in Q.$$

Since Q is a cone, we see that either $\mathbf{1} \in Q$ or $-\mathbf{1} \in Q$. From this, we see that either $e \in K$ or $-e \in K$.

7. Pointed/solid spectral cones

The following result is a generalization of a similar result stated in the setting of $\mathcal{V} = \mathcal{S}^n$ (see [5], Theorem 3.3).

Theorem 7.1. *Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. Then*

(*i*) *K is pointed if and only if Q is pointed,*
(*ii*) *K is solid if and only if Q is solid.*

Proof. (*i*) From Theorem 3.3, $K \cap -K = \lambda^{-1}(Q \cap -Q)$. Thus, $K \cap -K \subseteq \{0\}$ if and only if $Q \cap -Q \subseteq \{0\}$. This proves (*i*).

(*ii*) This follows from the fact that $K^\circ = \lambda^{-1}(Q^\circ)$, see (4). □

In the next several results, we characterize pointedness and solidness of spectral cones in $\mathcal{R}^n$ and $\mathcal{V}$.

Lemma 7.2. *Let Q be a nonempty nonzero permutation invariant convex cone in $\mathcal{R}^n$. Then Q is pointed if and only if exactly one of the following conditions holds:*

(*i*) *$\mathrm{tr}(u) > 0$ for all nonzero $u \in Q$ (and $\mathbf{1} \in Q$).*
(*ii*) *$\mathrm{tr}(u) < 0$ for all nonzero $u \in Q$ (and $-\mathbf{1} \in Q$).*

Proof. Suppose Q is pointed. We first show that $\mathrm{tr}(u) \neq 0$ for every nonzero $u \in Q$. Suppose, on the contrary, there is a nonzero vector $u \in Q$ such that $\mathrm{tr}(u) = 0$. Then by (1), $0 = (n-1)!\,\mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u)$. This implies

$$v := \sum_{\sigma \in \Sigma_n,\, \sigma \neq I} \sigma(u) = \left[\sum_{\sigma \in \Sigma_n} \sigma(u)\right] - I(u) = -I(u) = -u,$$

where I is the identity matrix. However, as both u and v are in Q, we reach a contradiction to the pointedness of Q. Now, if there exist nonzero $u, v \in Q$ such that $\mathrm{tr}(u) > 0$ and $\mathrm{tr}(v) < 0$, then for a suitable convex combination w of u and v, we have $\mathrm{tr}(w) = 0$. As Q is convex, $w \in Q$. From what has been proved earlier, $w = 0$. This contradicts the pointedness of the convex cone Q. Hence, either (a) $\mathrm{tr}(u) > 0$ for all nonzero $u \in Q$ or (b) $\mathrm{tr}(u) < 0$ for all nonzero $u \in Q$. Now, in the first case, by (1),

$$(n-1)!\ \mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u) \in Q.$$

It follows (by scaling) that $\mathbf{1} \in Q$. This is Item (i). Similarly, when (b) holds, we get Item (ii). By the pointedness of Q, both (i) and (ii) cannot hold simultaneously.

To see the reverse implication, suppose without loss of generality (i) holds so that $\mathrm{tr}(u) > 0$ for every nonzero $u \in Q$. By the linearity of the trace function, $-u$ cannot be in Q for any nonzero $u \in Q$. Thus, Q is pointed. □

Theorem 7.3. *Let K be a nonempty, nonzero spectral cone in $\mathcal{V}$. Then K is pointed if and only if exactly one of the following conditions holds:*

(i) $\mathrm{tr}(x) > 0$ *for all nonzero* $x \in K$ *(and* $e \in K$*).*
(ii) $\mathrm{tr}(x) < 0$ *for all nonzero* $x \in K$ *(and* $-e \in K$*).*

Proof. Let $K = \lambda^{-1}(Q)$ where Q is a permutation invariant convex cone. Then, by an earlier result, K is pointed if and only if Q is pointed. Thus, by the above lemma, K is pointed if and only if either $\mathrm{tr}(u) > 0$ for all $u \in Q$ (and $\mathbf{1} \in Q$) or $\mathrm{tr}(u) < 0$ for all $u \in Q$ (and $-\mathbf{1} \in Q$). As $\mathrm{tr}(\lambda(x)) = \mathrm{tr}(x)$, and $\pm\mathbf{1} \in Q$ if and only if $\pm e \in K$, we get the stated results from $K = \lambda^{-1}(Q)$. □

We say that a vector in $\mathcal{R}^n$ is a *nonconstant vector* if it is not a multiple of $\mathbf{1}$.

Lemma 7.4. *Let Q be a permutation invariant convex cone in $\mathcal{R}^n$. Then the following are equivalent:*

(a) *Q is solid.*
(b) $\mathbf{1} \in Q^\circ$ *or* $-\mathbf{1} \in Q^\circ$.
(c) *When $n \geq 2$, Q has a nonconstant vector, and either* $\mathbf{1} \in Q$ *or* $-\mathbf{1} \in Q$.

Proof. $(a) \Rightarrow (b)$: Suppose Q is solid, that is, $Q^\circ \neq \emptyset$. As Q° cannot be contained in the subspace $\{u \in \mathcal{R}^n : \mathrm{tr}(u) = 0\}$, there exists a nonzero vector $u \in Q^\circ$ with $\mathrm{tr}(u) \neq 0$. As Q° is permutation invariant, $\sigma(u) \in Q^\circ$ for every $\sigma \in \Sigma_n$. Now, as Q° is a convex cone, by (1),

$$(n-1)!\ \mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u) \in Q^\circ.$$

Since $\mathrm{tr}(u) \neq 0$, by scaling, we get (b).

$(b) \Rightarrow (c)$: This is obvious.

$(c) \Rightarrow (a)$: Let $n \geq 2$. Assume that there is a nonconstant vector $u = (u_1, u_2, \ldots, u_n)^{\mathsf{T}}$ in Q and (without loss of generality) $\mathbf{1} \in Q$. Suppose, if possible, Q is not solid. Then $Q - Q \neq \mathcal{R}^n$ and so there exists a nonzero $v \in (Q-Q)^{\perp} = Q^{\perp}$. As $\mathbf{1} \in Q$, we must have $\sum_i v_i = \langle \mathbf{1}, v \rangle = 0$, where $v = (v_1, v_2, v_2 \ldots, v_n)^{\mathsf{T}}$. Now, as u has at least two distinct components, we may assume (by permuting the coordinates) that $u_1 \neq u_2$. Let $\bar{u}$ be the vector in Q obtained from u by interchanging u_1 and u_2. Since $u, \bar{u} \perp v$, we get

$$\langle u,\, v \rangle = u_1 v_1 + u_2 v_2 + \sum_{i=3}^{n} u_i v_i = 0, \text{ and}$$

$$\langle \bar{u},\, v \rangle = u_2 v_1 + u_1 v_2 + \sum_{i=3}^{n} u_i v_i = 0.$$

From these, we get $0 = \langle u - \bar{u}, v \rangle = (u_1 - u_2)(v_1 - v_2)$. As $u_1 - u_2 \neq 0$, we must have $v_1 = v_2$. Now, we can permute u so that the resulting vector has different entries in i and j slots, $i \neq j$. Then, the above argument can be repeated to get $v_i = v_j$. Hence, v is a multiple of $\mathbf{1}$. However, as $\sum_i v_i = 0$, we must have $v = 0$, a contradiction. Thus (a) holds. □

Theorem 7.5. *Let K be a spectral cone in $\mathcal{V}$. Then the following are equivalent:*

(a) *K is solid.*
(b) *$e \in K^{\circ}$ or $-e \in K^{\circ}$.*
(c) *When $n \geq 2$, K has a vector which is not a multiple of e, and either $e \in K$ or $-e \in K$.*

Proof. Let $K = \lambda^{-1}(Q)$ where Q is a permutation invariant convex cone. As $K^{\circ} = \lambda^{-1}(Q^{\circ})$ and $\lambda(e) = \mathbf{1}$, we see that the items listed above are equivalent to the similar ones in the previous lemma. This completes the proof. □

References

[1] M. Baes, Convexity and differentiability of spectral functions on Euclidean Jordan algebra, Linear Algebra Appl. 422 (2007) 664–700.
[2] R. Bhatia, Matrix Analysis, Grad. Texts in Math., Springer-Verlag, New York, 1997.
[3] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Clarendon Press, Oxford, 1994.
[4] M.S. Gowda, Positive and doubly stochastic maps, and majorization in Euclidean Jordan algebras, Linear Algebra Appl. (2016), http://dx.doi.org/10.1016/j.laa.2016.02.024, in press.
[5] R. Henrion, A. Seeger, Inradius and circumradius of various convex cones arising in applications, Set Valued Anal. 18 (2010) 483–511.
[6] A. Iusem, A. Seeger, Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory, Comput. Appl. Math. 26 (2007) 191–214.
[7] J. Jeong, M.S. Gowda, Spectral sets and functions on Euclidean Jordan algebras, research report, Department of Mathematics and Statistics, University of Maryland, Baltimore County, January 2016, available at www.math.umbc.edu/~gowda/tech-reports/index.html.

[8] J. Lahtonen, http://math.stackexchange.com/questions/75427.
[9] A.W. Marshall, I. Olkin, B.C. Arnold, Inequalities: Theory of Majorization and Its Applications, Springer, 2010.
[10] H. Ramírez, A. Seeger, D. Sossa, Commutation principle for variational Problems on Euclidean Jordan algebras, SIAM J. Optim. 23 (2013) 687–694.
[11] D. Sun, J. Sun, Löwner's operator and spectral functions in Euclidean Jordan algebras, Math. Oper. Res. 33 (2008) 421–445.