Linear Algebra and its Applications 498 (2016) 219–230

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Completely mixed linear games on a self-dual cone

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A R T I C L E I N F O

Article history:
Received 12 December 2014
Accepted 18 May 2015
Available online 29 May 2015
Submitted by R. Brualdi

This paper is dedicated to Hans Schneider with profound respect, admiration, and appreciation

MSC:
91A05
46N10
17C20

Keywords:
Self-dual cone
Strategy
Value
Completely mixed game
Symmetric cone
Z-transformation

A B S T R A C T

Given a finite dimensional real inner product space V with a self-dual cone K, an element e in K° (the interior of K), and a linear transformation L on V, the value of the linear game (L, e) is defined by

$$v(L,e) := \max_{x\in\Delta(e)} \min_{y\in\Delta(e)} \langle L(x), y\rangle = \min_{y\in\Delta(e)} \max_{x\in\Delta(e)} \langle L(x), y\rangle,$$

where $\Delta(e) = \{x \in K : \langle x, e\rangle = 1\}$. In [5], various properties of a linear game and its value were studied and some classical results of Kaplansky [6] and Raghavan [8] were extended to this general setting. In the present paper, we study how the value and properties change as e varies in K°. In particular, we study the structure of the set $\Omega(L)$ of all e in K° for which the game (L, e) is completely mixed and identify certain classes of transformations for which $\Omega(L)$ equals K°. We also describe necessary and sufficient conditions for a game (L, e) to be completely mixed when $v(L, e) = 0$, thereby generalizing a result of Kaplansky [6].

1. Introduction

This paper is a continuation of [5], where the concept of value of a (zero-sum) matrix game is generalized to a linear transformation defined on a self-dual cone in a real finite

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http://dx.doi.org/10.1016/j.laa.2015.05.025
0024-3795/

dimensional Hilbert space. To elaborate, consider a finite dimensional real inner product space $(V, \langle\cdot\,,\cdot\rangle)$ and a self-dual cone K in V. We fix an element e in K° (the interior of K) and let

$$\Delta(e) := \{x \in K : \langle x, e\rangle = 1\}, \tag{1}$$

the elements of which are called 'strategies'. Given a linear transformation L from V to V, the zero-sum linear game — denoted by (L, e) — is played by two players I and II in the following way: If player I chooses strategy $x \in \Delta(e)$ and player II chooses strategy $y \in \Delta(e)$, then the pay-off for player I is $\langle L(x), y\rangle$ and the pay-off for player II is $-\langle L(x), y\rangle$. Both players try to maximize their pay-offs. Since $\Delta(e)$ is a compact convex set and L is linear, by the min–max Theorem of von Neumann (see [7, Theorems 1.5.1 and 1.3.1]), there exist *optimal strategies* $\bar{x}$ for player I and $\bar{y}$ for player II which satisfy

$$\langle L(x), \bar{y}\rangle \le \langle L(\bar{x}), \bar{y}\rangle \le \langle L(\bar{x}), y\rangle \quad \forall\, x, y \in \Delta(e). \tag{2}$$

This means that players I and II do not gain by unilaterally changing their strategies from the optimal strategies $\bar{x}$ and $\bar{y}$. The number

$$v(L, e) := \langle L(\bar{x}), \bar{y}\rangle$$

is the *value of the game*, or simply, *the value of* (L, e). The pair $(\bar{x}, \bar{y})$ is called an *optimal strategy pair* for (L, e). We note that $v(L, e)$ is also given by [7, Theorems 1.5.1 and 1.3.1]

$$v(L, e) = \max_{x\in\Delta(e)} \min_{y\in\Delta(e)} \langle L(x), y\rangle = \min_{y\in\Delta(e)} \max_{x\in\Delta(e)} \langle L(x), y\rangle. \tag{3}$$

We say that the game (L, e) is *completely mixed* if for every optimal strategy pair $(\bar{x}, \bar{y})$ of (L, e), $\bar{x}$ and $\bar{y}$ belong to K°. The above concepts and definitions reduce to the classical ones when $V = R^n$ (with the usual inner product), $K = R^n_+$ (the nonnegative orthant), $L \in R^{n\times n}$ and e is the (column) vector of ones. In [5], several classical results of Kaplansky [6] and Raghavan [8] were extended to this general setting and their connections to dynamical systems were explored. As in the classical case, the uniqueness of the optimal strategy pair prevails when the game is completely mixed (see Theorem 4 in [5]). The completely mixed property was investigated for **Z**, Lyapunov-like and Stein-like transformations (see Section 2 for definitions). In particular, it was shown in [5] that the game (L, e) is completely mixed when L is a **Z**-transformation with $v(L, e) > 0$ or L is a Lyapunov/Stein-like transformation with $v(L, e) \neq 0$.

In the present paper, we address three issues: (i) How the value and the optimal strategies change as L and e are changed, (ii) how the value changes under cone automorphisms, and (iii) how, for a given transformation L, the completely mixed property changes as e varies over the interior of K.

Addressing (i), we show that the value varies continuously and the optimal strategy set is upper semicontinuous in L and e. We also specify (upper) bounds for $v(L, e)$.

Addressing (*ii*), we show that $v(ALA^T, Ae) = v(L, e)$ for any automorphism A of K. In addition, we show that the sign of $v(ALA^T, e')$ is independent of $A \in Aut(K)$ and $e' \in K^\circ$. To address (*iii*), for a given transformation L, we define the set

$$\Omega(L) := \{e \in K^\circ : (L, e) \text{ is completely mixed}\}. \tag{4}$$

We show that $\Omega(L)$ is an open convex cone (which could be empty) and $\Omega(L) = K^\circ$ for Lyapunov/Stein-like transformations and automorphisms of K.

The above issues do not seem to have been addressed in the classical literature, perhaps because of homogeneity of the nonnegative orthant (which means that any positive vector can be mapped onto any other by an automorphism of the nonnegative orthant) thus reducing the importance of working with an arbitrary positive vector. Such a homogeneity property does not hold for a general self-dual cone unless it is a symmetric cone (which is the cone of squares in a Euclidean Jordan algebra [4]).

2. Preliminaries

We recall some notation, concepts, and results from [5]. $(V, \langle\cdot\,,\cdot\rangle)$ denotes a finite dimensional real inner product space. For $c, d \in V$, we define the linear transformation $c\,d^T$ by $(c\,d^T)(x) = \langle x, d\rangle\, c$.

For a set S in V, the interior, closure, and boundary are denoted, respectively, by S°, $\overline{S}$, and ∂S.

Let K be a *self-dual cone* in V which means that $K = K^*$, where

$$K^* := \{x \in V : \langle x, y\rangle \geq 0 \;\forall\, y \in K\}.$$

We note that K is a closed convex cone with $K \cap -K = \{0\}$ and $K^\circ \neq \emptyset$. Given K, we write

$$x \geq y \text{ (or } y \leq x) \quad \text{when} \quad x - y \in K \quad \text{and} \quad x > y \text{ (or } y < x) \quad \text{when} \quad x - y \in K^\circ.$$

We note that for $0 \neq x \geq 0, y \geq 0$ (> 0), we have $\langle x, y\rangle \geq 0$ (respectively, > 0). The set of all *automorphisms* of K (these are linear transformations which map K onto K) is denoted by $Aut(K)$. Clearly, $A \in Aut(K) \Leftrightarrow A^{-1} \in Aut(K)$ and (as K is self-dual), $A \in Aut(K) \Leftrightarrow A^T \in Aut(K)$, where A^T denotes the transpose of A.

If K (in addition to being self-dual) is also *homogeneous* (which means that for any two elements $x, y > 0$, there exists $A \in Aut(K)$ such that $A(x) = y$), then K is said to be a *symmetric cone*. Such a cone appears as cone of squares in a Euclidean Jordan algebra [4]. Examples of symmetric cones include R^n_+ (the nonnegative orthant) in R^n and S^n_+ (the cone of positive semidefinite matrices) in S^n (the space of all real $n \times n$ symmetric matrices).

Throughout this paper, we use the following notation: V denotes a finite dimensional real inner product space, K is a (fixed) self-dual cone in V, e is an arbitrary element in K° (i.e., $e > 0$) and L denotes an arbitrary linear transformation on V.

Corresponding to $e > 0$, we define $\Delta(e)$ by (1). Given a linear transformation L on V and $e > 0$, we follow the definitions and concepts given in the Introduction. The game (L, e) is said to be *completely mixed* if for every optimal strategy pair $(\bar{x}, \bar{y})$, we have $\bar{x} > 0$ and $\bar{y} > 0$.

Given a linear transformation L on V, we say that

(a) L is a **Z**-transformation (on K) if $x \in K, y \in K$, $\langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0$,
(b) L is Lyapunov-like (on K) if $x \in K, y \in K$, $\langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0$, and
(c) L is Stein-like (on K) if $L = I - \Lambda$ for some $\Lambda \in \overline{Aut(K)}$.

The negative of a **Z**-transformation is the so-called cross-positive matrix/transformation introduced in [10]. Lyapunov/Stein-like transformations are particular instances of **Z**-transformations. On R^n_+, **Z**-transformations are just **Z**-matrices (those with off-diagonal entries nonpositive) and Lyapunov-like transformations are nothing but diagonal matrices. On S^n_+, Lyapunov-like transformations reduce to Lyapunov transformations which are of the form L_A, where $A \in R^{n\times n}$ and $L_A(X) := AX + XA^T$ $(X \in \mathcal{S}^n)$ [3]; since every automorphism of $\mathcal{S}^n_+$ is given by $\Lambda(X) = BXB^T$ $(X \in \mathcal{S}^n)$ for some real $n \times n$ invertible matrix B [9], Stein transformations are of the form S_A, where $A \in R^{n\times n}$ and $S_A(X) := X - AXA^T$. (We remark that these transformations appear in continuous and discrete dynamical systems.)

For any real number α, $sgn\,\alpha$ is $\frac{\alpha}{|\alpha|}$ if $\alpha \neq 0$ or zero otherwise.

For ease of reference, we recall the following results from [5]. These refer to the game (L, e).

Theorem 1.

(1) *Suppose there exist vectors $x, y \in \Delta(e)$ and a number v such that $L^T(y) \leq v\,e \leq L(x)$. Then $v = v(L, e)$ and (x, y) is an optimal strategy pair for (L, e). Furthermore, if there exists an optimal strategy pair $(\bar{x}, \bar{y})$ with $\bar{x} > 0$ and $\bar{y} > 0$, then $L(x) = v(L, e)\,e$ and $L^T(y) = v(L, e)\,e$.*
(2) *$v(L, e) > 0$ if and only if there exists $d > 0$ such that $L(d) > 0$.*
(3) *If (L, e) is completely mixed, then the optimal strategy pair $(\bar{x}, \bar{y})$ is unique; also, $L(\bar{x}) = v(L, e)\,e$ and $L^T(\bar{y}) = v(L, e)\,e$.*
(4) *When (L, e) is completely mixed, $v(L, e) \neq 0$ if and only if L is invertible.*
(5) *If L is a **Z**-transformation and $v(L, e) > 0$, then (L, e) is completely mixed.*
(6) *If L is Lyapunov/Stein-like and $v(L, e) \neq 0$, then (L, e) is completely mixed.*

From item (1) of the above theorem, we see that

$$v(-L^T, e) = -v(L, e)$$

and (x, y) is an optimal strategy pair of (L, e) if and only if (y, x) is an optimal pair of $(-L^T, e)$. The following is also an easy consequence of item (1).

Theorem 2. *Suppose that L is invertible, and for some $e > 0$, we have $L^{-1}(e) > 0$ and $(L^T)^{-1}(e) > 0$. Then (L, e) is completely mixed with $(\bar{x}, \bar{y})$ as the unique optimal pair, where*

$$\bar{x} := \frac{L^{-1}(e)}{\langle L^{-1}(e), e\rangle} \quad \text{and} \quad \bar{y} := \frac{(L^T)^{-1}(e)}{\langle L^{-1}(e), e\rangle}.$$

3. Some basic results

Let $\mathcal{B}(V, V)$ denote the set of all (bounded) linear transformations from V to V. We consider two mappings on $\mathcal{B}(V, V) \times K^\circ$: $(L, e) \mapsto v(L, e)$ and $(L, e) \mapsto \Phi(L, e)$, where

$$\Phi(L, e) := \{x : (x, y) \,\text{is an optimal strategy pair for}\, (L, e)\}.$$

Theorem 3. *On $\mathcal{B}(V, V) \times K^\circ$, $v(L, e)$ is continuous and $\Phi(L, e)$ is upper semicontinuous. Consequently, if some game (L, e) is completely mixed, then so is any (L', e') near (L, e).*

Proof. Assuming $v(L, e)$ is not continuous, let $(L_n, e_n) \to (L, e)$ and $\varepsilon > 0$ such that

$$|v(L_n, e_n) - v(L, e)| \geq \varepsilon \;\forall\; n. \tag{5}$$

Let (x_n, y_n) be an optimal strategy pair for (L_n, e_n). As V is finite dimensional, the conditions $x_n, y_n \in K$, $\langle x_n, e_n\rangle = 1 = \langle y_n, e_n\rangle$ for all n, and $e_n \to e \in K^\circ$ imply that x_n and y_n are bounded. We may assume, by considering subsequences, that $x_{n_k} \to x \in K$ and $y_{n_k} \to y \in K$. Taking limits in

$$(L_{n_k})^T(y_{n_k}) \leq v(L_{n_k}, e_{n_k})\, e_{n_k} = \langle L_{n_k}(x_{n_k}), y_{n_k}\rangle\, e_{n_k} \leq L_{n_k}(x_{n_k})$$

we see that $L^T(y) \leq v\, e \leq L(x)$, where $v = \langle L(x), y\rangle = \lim v(L_{n_k}, e_{n_k})$. Since $x, y \in K$ and $\langle x, e\rangle = 1 = \langle y, e\rangle$, it follows (see Theorem 1) that $v = v(L, e)$; hence $v(L_{n_k}, e_{n_k}) \to v(L, e)$. This clearly contradicts (5). Hence we have the continuity of the value.

Now to see the upper semicontinuity property of $\Phi(L, e)$, suppose U is an open set containing $\Phi(L, e)$ and for some sequence $(L_n, e_n) \to (L, e)$, $\Phi(L_n, e_n) \nsubseteq U$. Then, for each n, there exists an optimal strategy pair (x_n, y_n) of (L_n, e_n) with $x_n \in U^c$. Then, as in the above proof, by working with appropriate subsequences, we produce an optimal strategy pair (x, y) of (L, e) with $x \in U^c$ (note that U^c is closed). This contradicts the assumption that $\Phi(L, e) \subseteq U$. Hence we have the upper semicontinuity of Φ.

Now suppose (L, e) is completely mixed. Then $\Phi(L, e) \subseteq K^\circ$. By the upper semicontinuity of Φ, for all (L', e') near (L, e), $\Phi(L', e') \subseteq K^\circ$. This means that $x' > 0$ for every optimal strategy pair (x', y') of (L', e'). By Theorem 5 in [5], we see that (L', e') is completely mixed. $\square$

We remark that as a consequence of the above result, the multivalued mapping which takes (L, e) to its set of all optimal strategy pairs is also upper semicontinuous.

The following result shows how the value and optimal strategies change under automorphisms of K.

Theorem 4. *For any $A \in Aut(K)$ and $e > 0$, $v(L, e) = v(ALA^T, Ae)$. Furthermore, (L, e) is completely mixed if and only if (ALA^T, Ae) is completely mixed.*

Proof. Let (x, y) be an optimal strategy pair for (L, e). Let $A \in Aut(K)$ and define $p := (A^{-1})^T x$ and $q := (A^{-1})^T y$. Now, $Ae > 0$ and

$$\langle x, e\rangle = 1 = \langle y, e\rangle \Rightarrow \langle p, Ae\rangle = 1 = \langle q, Ae\rangle.$$

Also, as $u > v$ ($u \geq v$) if and only if $Au > Av$ (respectively, $Au \geq Av$), we have

$$\begin{aligned} L^T(y) \leq v(L, e)\, e \leq L(x) &\Rightarrow AL^T(y) \leq v(L, e)\, Ae \leq AL(x) \\ &\Rightarrow AL^T A^T(q) \leq v(L, e)\, Ae \leq ALA^T(p). \end{aligned}$$

From Theorem 1, item (1), we see that (p, q) is an optimal strategy pair for (ALA^T, Ae) and $v(L, e) = v(ALA^T, Ae)$. Since A and A^{-1} are automorphisms, we see that $x > 0$ ($y > 0$) if and only if $p > 0$ (respectively, $q > 0$). From this, we get the invariance of the completely mixed property. $\square$

Suppose K is a symmetric cone and $e, e' > 0$. Then, there exists $A \in Aut(K)$ such that $Ae' = e$. It follows from the previous result that the games (ALA^T, e) and (L, e') are equivalent in the sense they have the same value and the there is a linear correspondence between the optimal strategy pairs. Thus, one can fix a particular $e \in K^\circ$ to analyze a linear game over a symmetric cone. However, in a general self-dual cone this may not be possible. In spite of this, we have the following result.

Theorem 5. *Let L be a linear transformation on V. Then for any $A \in Aut(K)$ and $e, e' > 0$, we have*

$$sgn\, v(ALA^T, e') = sgn\, v(L, e).$$

In particular, $sgn\, v(L, e)$ is a constant on K°.

Proof. Suppose $v(L, e) > 0$. From Theorem 1, item (2), this is equivalent to the existence of a $d > 0$ such that $L(d) > 0$. As the latter condition is independent of e, we see that $v(L, e') > 0$ for all $e' > 0$. Furthermore, for any $A \in Aut(K)$, we have $ALA^T(d) > 0$; thus, $v(ALA^T, e') > 0$ for all e'. When $v(L, e) < 0$, we work with $-L^T$ (so that $v(-L^T, e) > 0$) and conclude that for all $A \in Aut(K)$ and $e' > 0$, $v(ALA^T, e') < 0$. Finally, when $v(L, e) = 0$ for some e, by what has been proved, $v(ALA^T, e')$ cannot be positive or negative for any $A \in Aut(K)$ and $e' > 0$; hence $v(ALA^T, e')$ must be zero. Thus in all cases, $sgn\, v(ALA^T, e') = sgn\, v(L, e)$. $\square$

4. A generalization of Kaplansky's result

In the classical setting (i.e., $V = R^n$, $K = R^n_+$, $L = A \in R^{n\times n}$, and e is the column vector of ones), Kaplansky [6] shows that when $v(A, e) = 0$, the game (A, e) is completely mixed if and only if the rank of A is $n - 1$ and all cofactors of A are nonzero and have the same sign. In what follows, we generalize this result.

Theorem 6. *Suppose $v(L, e) = 0$. Then (L, e) is completely mixed if and only if the following conditions hold:*

(i) $dim\, ker(L) = 1$.
(ii) *There exist vectors $\bar{x}, \bar{y} > 0$ such that for $S := \bar{x}\, \bar{y}^T$ we have $LS = SL = 0$.*

Moreover, when conditions (i) and (ii) hold, the above S is unique up to a scalar multiple: any linear transformation M on V with $LM = ML = 0$ is a scalar multiple of S.

Proof. We first suppose (L, e) is completely mixed. Under the (given) assumption that $v(L, e) = 0$, it has been proved in Theorem 4, [5] that $dim\, ker(L) = 1$. Now, let $(\bar{x}, \bar{y})$ be the unique optimal pair with $\bar{x} > 0$ and $\bar{y} > 0$. As $v(L, e) = 0$, we have from Theorem 1, item (3), $L(\bar{x}) = 0 = L^T(\bar{y})$. Putting $S := \bar{x}\, \bar{y}^T$ we see that $LS = SL = 0$. Thus, we have items (i) and (ii). Now suppose that these conditions hold for some $\bar{x}, \bar{y} > 0$. From (ii) we have $(LS)(\bar{y}) = 0$ and $(L^T S^T)(\bar{x}) = 0$. These imply that $L(\bar{x}) = 0$ and $L^T(\bar{y}) = 0$. Hence $(\bar{x}, \bar{y})$ is an optimal strategy pair for (L, e). Now suppose (x, y) is another optimal strategy pair so that (from Theorem 1, item (1)) $L(x) = 0 = L^T(y)$ with $x, y \in \Delta(e)$. Since $dim\, ker(L) = 1$, x is a scalar multiple of $\bar{x}$, and because these vectors are in $\Delta(e)$, $x = \bar{x}$. Also, from $dim\, ker(L) = 1$ we see that $dim\, ker(L^T) = 1$ (as L is a linear transformation from V to V). By noting that both y and $\bar{y}$ are in $ker(L^T) \cap \Delta(e)$, we deduce that $y = \bar{y}$. Hence, $(\bar{x}, \bar{y})$ is the only optimal strategy pair for (L, e) with $\bar{x} > 0$ and $\bar{y} > 0$. We see that (L, e) is completely mixed. Now for the additional statement: Suppose that conditions (i) and (ii) hold, and M is a linear transformation on V with $LM = ML = 0$. Then for any $x \in V$, $L(M(x)) = 0$. As $dim\, ker(L) = 1$, $M(x)$ is a scalar multiple of $\bar{x}$. We may write $M(x) = \lambda(x)\, \bar{x}$, where $\lambda(x)$ is linear (functional) in x. By the Riesz representation theorem, we may write $\lambda(x) = \langle x, a\rangle$ for some $a \in V$. Thus, $M(x) = \langle x, a\rangle\, \bar{x}$ for all $x \in V$. Similarly, by considering $L^T(M^T(x)) = 0$, we may write $M^T(x) = \langle x, b\rangle\, \bar{y}$ for some $b \in V$. As $\langle M^T(x), y\rangle = \langle x, M(y)\rangle$ for all $x, y \in V$, we see that

$$\langle x, b\rangle \langle \bar{y}, y\rangle = \langle y, a\rangle\, \langle x, \bar{x}\rangle.$$

Upon putting $x = \bar{x}$, we get

$$\langle y, a\rangle = \frac{\langle \bar{x}, b\rangle \, \langle y, \bar{y}\rangle}{||\bar{x}||^2}.$$

As this holds for all $y \in V$, we see that $a = \alpha\, \bar{y}$ for some real α. Then, $M(x) = \alpha \,\langle x, \bar{y}\rangle\, \bar{x}$ for all $x \in V$. Thus, $M = \alpha\, \bar{x}\, \bar{y}^T = \alpha\, S$. $\square$

We specialize the above result by taking $V = R^n$ and $L = A \in R^{n\times n}$. We use the notation *adj* A for the adjoint of A (that is, the transpose of the cofactor matrix).

Corollary 1. *Consider R^n with the usual inner product. Let K be a self-dual cone in R^n, $A \in R^{n\times n}$, and $e > 0$ (that is, $e \in K^\circ$). Suppose $v(A, e) = 0$. Then (A, e) is completely mixed if and only if the following conditions hold:*

(a) *$dim\, ker(A) = 1$ (equivalently, rank of A is $n - 1$).*
(b) *$adj\,(A)$ is a nonzero scalar multiple of a matrix of the form $\bar{x}\, \bar{y}^T$ for some $\bar{x}, \bar{y} > 0$.*

Proof. We first suppose (A, e) is completely mixed. By the above theorem, $dim\, ker(A) = 1$ and $AS = SA = 0$ for some $S := \bar{x}\, \bar{y}^T$ with $\bar{x}, \bar{y} > 0$. Since $A\,(adj\, A) = (adj\, A)\, A = (\det A)\, I = 0$, we can let $M = adj\, A$ in the above theorem to see that *adj* A is a scalar multiple of S. As rank of A is $n - 1$, *adj* A cannot be zero. Hence this scalar is also nonzero. Thus, we have (b). On the other hand, when conditions (a) and (b) hold, because $A\,(adj\, A) = (adj\, A)\, A = (\det A)\, I = 0$, conditions (i) and (ii) of the above theorem hold; thus, (A, e) is completely mixed. $\square$

Remarks. By letting $K = R^n_+$ in the above corollary, we can state condition (b) in an equivalent form: All entries of *adj* A are nonzero and have the same sign. In this way, we recover the above mentioned result of Kaplansky.

5. The structure of $\Omega(L)$

In this section, we describe the structure of $\Omega(L)$ given by (4). First, we start with some examples that show that $\Omega(L)$ can be empty or a proper subset of K°.

Example. Let $V = R^2$, $K = R^2_+$, and e be the vector of ones. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}.$$

With e_1 and e_2 denoting the standard basis vectors in R^2, we see that $A^T e_1 \le 0\, e \le Ae_2$; hence by Theorem 1, item (1), (e_2, e_1) is an optimal strategy pair for (A, e) and $v(A, e) = 0$. By Theorem 5, $v(A, d) = 0$ for all $d \in K^\circ$. If (A, d) is completely mixed for some $d > 0$, then for any optimal strategy pair $(\bar{x}, \bar{y})$, we have $A^T\bar{y} = 0 = A\bar{x}$. Since A is invertible, we must have $\bar{x} = 0$, leading to a contradiction. Thus, (A, d) is not completely mixed for any $d > 0$. Hence $\Omega(A)$ is empty. We note that A is a diagonal matrix, hence Lyapunov-like on R^2_+.

Now $B^{-1}e > 0$ and $(B^T)^{-1}(e) > 0$. Thus, by Theorem 2, (B, e) is completely mixed. Hence $\Omega(B) \neq \emptyset$. Let $d = [1, 2]^T$ and suppose that (B, d) is completely mixed. Then, $v(B, d)B^{-1}d > 0$. However, this cannot hold as $v(B, d) > 0$ (since B is a matrix with positive entries) and $B^{-1}d$ has a zero component. Thus, $\Omega(B)$ is a nonempty proper subset of K°.

Finally, we note that C (which is $-B^T$) is a **Z**-matrix and $v(C, d) = -v(B, d) < 0$. Thus, by Theorem 5, $v(C, e') < 0$ for all $e' \in K^\circ$. Yet $\emptyset \neq \Omega(C) = \Omega(B) \neq K^\circ$.

Theorem 7. *Suppose $e^* > 0$ and (L, e^*) is completely mixed. Then the following hold:*

(a) *If $v(L, e^*) = 0$, then $\Omega(L) = K^\circ$.*
(b) *If $v(L, e^*) > 0$, then $\Omega(L) = K^\circ \cap L(K^\circ) \cap L^T(K^\circ)$.*
(c) *If $v(L, e^*) < 0$, then $\Omega(L) = K^\circ \cap L(-K^\circ) \cap L^T(-K^\circ)$.*

Thus, for any L, $\Omega(L)$ is either empty or an open convex cone in K°.

Proof. (a) It is given that (L, e^*) is completely mixed with $v(L, e^*) = 0$. Then, by Theorem 5, $v(L, e) = 0$ for all $e > 0$. Now, conditions (i) and (ii) of Theorem 6, which hold for (L, e^*), also hold for (L, e). Thus, (L, e) is completely mixed for all $e \in K^\circ$. Hence, $\Omega(L) = K^\circ$.

(b) Suppose $v(L, e^*) > 0$ and let $e \in \Omega(L)$. From Theorem 1, item (4), L is invertible. If (x, y) is the unique optimal pair of (L, e), then $v(L, e)L^{-1}(e) > 0$ and $v(L, e)(L^T)^{-1}(e) > 0$. Since $v(L, e^*) > 0 \Rightarrow v(L, e) > 0$ (by Theorem 5), we must have $z := L^{-1}(e) > 0$ and $w := (L^T)^{-1}(e) > 0$. Thus, $e = L(z)$, $e = L^T(w)$, where $z, w \in K^\circ$. Hence $e \in K^\circ \cap L(K^\circ) \cap L^T(K^\circ)$. Since e is any element of $\Omega(L)$, we must have $\Omega(L) \subseteq K^\circ \cap L(K^\circ) \cap L^T(K^\circ)$. To prove the reverse inclusion, let $e \in K^\circ \cap L(K^\circ) \cap L^T(K^\circ)$. Then, $e > 0$ with $L^{-1}(e) > 0$ and $(L^T)^{-1}(e) > 0$. From Theorem 2, we see that $e \in \Omega(L)$. Thus, $K^\circ \cap L(K^\circ) \cap L^T(K^\circ) \subseteq \Omega(L)$ and the stated equality in (b) follows.

(c) When $v(L, e^*) < 0$, we have $v(-L^T, e^*) > 0$. Since $\Omega(L) = \Omega(-L^T)$, we can apply item (b) to $-L^T$ and get the stated equality of sets. $\square$

Theorem 8. *Suppose L is invertible. Then (L, e) is completely mixed for all $e > 0$ if and only if either $L^{-1}(K) \subseteq K$ or $(-L)^{-1}(K) \subseteq K$.*

Proof. Suppose (L, e) is completely mixed for all $e > 0$ so that $\Omega(L) = K^\circ$. As L is invertible we have (see Theorem 1, item (4)) $v(L, e) > 0$ for all e or $v(L, e) < 0$ for all e. From items (b) and (c) of Theorem 7, either $K^\circ = K^\circ \cap L(K^\circ) \cap L^T(K^\circ)$ or

$K^\circ = K^\circ \cap L(-K^\circ) \cap L^T(-K^\circ)$. From these, we see that $K^\circ \subseteq L(K^\circ) \subseteq L(K)$ or $K^\circ \subseteq -L(K^\circ) \subseteq -L(K)$. By taking closures, we get $K \subseteq L(K)$ or $K \subseteq -L(K)$. Finally, we get $L^{-1}(K) \subseteq K$ or $(-L)^{-1}(K) \subseteq K$. Now for the 'if' part: Suppose $L^{-1}(K) \subseteq K$. As K is self-dual, we get $(L^T)^{-1}(K) \subseteq K$. As these inclusions hold with K° in place of K, for any $e \in K^\circ$, we have $L^{-1}(e) > 0$ and $(L^T)^{-1}(e) > 0$. By Theorem 2, (L, e) is completely mixed. When $(-L)^{-1}(K) \subseteq K$, we have $(-L^T)^{-1}(K) \subseteq K$ and $\Omega(L) = \Omega(-L^T) = K^\circ$. Thus, even in this case, (L, e) is completely mixed for all $e > 0$. □

Corollary 2. *Suppose L is invertible. Then the following are equivalent:*

(i) $\pm L \in Aut(K)$.
(ii) *(L, e) and (L^{-1}, e) are completely mixed for all $e > 0$.*

Proof. $(i) \Rightarrow (ii)$: Let $L \in Aut(K)$. Then, $L(K) \subseteq K$ and $L^{-1}(K) \subseteq K$. An application of the above theorem shows that (L, e) and (L^{-1}, e) are completely mixed for all $e > 0$. Now suppose $-L \in Aut(K)$. Then $-L^T \in Aut(K)$ (as K is self-dual). It follows that $(-L^T, e)$ and $((-L^T)^{-1}, e)$ are completely mixed for all $e > 0$. Equivalently, (L, e) and (L^{-1}, e) are completely mixed for all $e > 0$.

$(ii) \Rightarrow (i)$: From the above theorem, we see that $L^{-1}(K) \subseteq K$ or $(-L)^{-1}(K) \subseteq K$ and $L(K) \subseteq K$ or $(-L)(K) \subseteq K$. As $K \cap -K = \{0\}$, we cannot have $L^{-1}(K) \subseteq K$ and $(-L)(K) \subseteq K$ (likewise, we cannot have $(-L)^{-1}(K) \subseteq K$ and $L(K) \subseteq K$). Thus, we must have $L^{-1}(K) \subseteq K$ and $L(K) \subseteq K$ or $(-L)^{-1}(K) \subseteq K$ and $(-L)(K) \subseteq K$. These show that $\pm L \in Aut(K)$. □

Theorem 9. *Under each of the following conditions, (L, e) is completely mixed for all $e > 0$.*

(1) *L is a **Z**-transformation and $v(L, e^*) > 0$ for some $e^* > 0$.*
(2) *L is Lyapunov/Stein-like on K and (L, e^*) is completely mixed for some $e^* > 0$.*

Proof. Let $e > 0$.
(1) Suppose $v(L, e^*) > 0$. Then, from Theorem 5, $v(L, e) > 0$. When L is a **Z**-transformation, by Theorem 1, item (5), (L, e) is completely mixed.

(2) Now suppose L is Lyapunov-like or Stein-like. If $v(L, e^*) = 0$, then by item (a) in Theorem 7, (L, e) is completely mixed. If $v(L, e^*) \neq 0$, then $v(L, e) \neq 0$ by Theorem 5. Now Theorem 1, item (6) shows that (L, e) is completely mixed. □

6. Some bounds on the value

In this section, we take up the issue of describing bounds for the value. This is motivated by results in [2]. With $\sigma(L)$ denoting the spectrum of L, let

$$\delta(e) := \max\{\langle x, y\rangle : x, y \in \Delta(e)\},\ \rho(L) := \max\{|\lambda| : \lambda \in \sigma(L)\},\ \text{and}$$
$$\tau(L) := \min\{Re(\lambda) : \lambda \in \sigma(L)\}.$$

We note that (as K is self-dual) $\delta(e) \geq 0$.

Theorem 10. *Suppose λ is a nonnegative real eigenvalue of L^T with a corresponding eigenvector $u \in \Delta(e)$. Then*

$$v(L, e) \leq \delta(e)\lambda. \tag{6}$$

We also have the following statements:

(i) *If $L(K) \subseteq K$, then $0 \leq v(L, e)) \leq \delta(e)\,\rho(L)$.*
(ii) *If $L^{-1}(K) \subseteq K$, then $0 < v(L, e) \leq \delta(e)\,\frac{1}{\rho(L^{-1})}$.*
(iii) *If L is a **Z**-transformation and $\tau(L) \geq 0$, then $0 \leq v(L, e) \leq \delta(e)\,\tau(L)$.*

Proof. Let $(\bar{x}, \bar{y})$ be an optimal strategy pair for (L, e). Then $v(L, e)\,e \leq L(\bar{x})$ implies

$$v(L, e) = v(L, e)\,\langle e, u\rangle \leq \langle L(\bar{x}), u\rangle = \langle \bar{x}, L^T(u)\rangle = \langle \bar{x}, \lambda\, u\rangle \leq \delta(e)\,\lambda,$$

where the last inequality follows from the definition of $\delta(e)$.

(i) Suppose $L(K) \subseteq K$. Then, $v(L, e) = \langle L(\bar{x}), \bar{y}\rangle \geq 0$. Now, as K is self-dual, $L^T(K) \subseteq K$. So, from the Krein–Rutman theorem [1], there exists a nonzero $u \in K$ (by scaling we can assume that $u \in \Delta(e)$) such that $L^T(u) = \rho(L)\,u$. From (6), we have $v(L, e) \leq \delta(e)\,\rho(L)$.

(ii) Suppose $L^{-1}(K) \subseteq K$ so that $L^{-1}(K^\circ) \subseteq K^\circ$ and $K^\circ \subseteq L(K^\circ)$. From the last inclusion, we see the existence of a $d > 0$ with $L(d) > 0$. Thus, by Theorem 1, item (2), $v(L, e) > 0$. Now, $L^{-1}(K) \subseteq K$ implies that $(L^T)^{-1}(K) \subseteq K$; by the Krein–Rutman Theorem [1], there exists a nonzero u in K (we can assume that $u \in \Delta(e)$) such that $(L^T)^{-1}(u) = \rho(L^{-1})u$. Thus, $L^T(u) = \frac{1}{\rho(L^{-1})}u$. From (6), we get $v(L, e) \leq \delta(e)\,\frac{1}{\rho(L^{-1})}$.

(iii) Now suppose that L is a **Z**-transformation on K. As K is self-dual, L^T is a **Z**-transformation on K. Then, by Theorem 6 in [10], there exist nonzero vectors $u, w \in K$ (without loss of generality, $u, w \in \Delta(e)$) such that $L^T(u) = \tau(L^T)u$ and $L(w) = \tau(L)w$. As $\tau(L^T) = \tau(L)$, when $\tau(L) \geq 0$, we apply (6) to get $v(L, e) \leq \delta(e)\,\tau(L)$. Also, $0 \leq \langle L(w), \bar{y}\rangle = \langle w, L^T(\bar{y})\rangle \leq v(L, e)\langle w, e\rangle = v(L, e)$. □

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