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An improved bound for the Lyapunov rank of a proper cone

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Abstract Given a proper cone K in $\mathbb{R}^n$ with its dual K^*, the complementarity set of K is $C(K) := \{(\mathbf{x}, \mathbf{s}) : \mathbf{x} \in K, \mathbf{s} \in K^*, \langle \mathbf{x}, \mathbf{s} \rangle = 0\}$. A matrix $\mathbf{A}$ on $\mathbb{R}^n$ is said to be *Lyapunov-like* on K if $\langle \mathbf{A}\mathbf{x}, \mathbf{s} \rangle = 0$ for all $(\mathbf{x}, \mathbf{s}) \in C(K)$. The set of all such matrices forms a vector space whose dimension $\beta(K)$ is called the *Lyapunov rank* of K. This number is useful in conic optimization and complementarity theory, as it relates to the number of linearly-independent bilinear relations needed to express the complementarity set. This article is a continuation of the study initiated in [6] and further pursued in [3]. By answering several questions posed in [3], we show that $\beta(K)$ is bounded above by $(n-1)^2$, thereby improving the previously known bound of $n^2 - n$. We also show that when $\beta(K) \geq n$, the complementarity set $C(K)$ can be expressed in terms of n linearly-independent Lyapunov-like matrices.

Keywords Lyapunov rank · Perfect cone

1 Introduction

Let K be a proper cone in $\mathbb{R}^n$ (that is, K is a closed convex pointed cone with nonempty interior) whose dual K^* [1] is given by

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$$K^* := \{\mathbf{y} \in \mathbb{R}^n : \forall \mathbf{x} \in K, \langle \mathbf{x}, \mathbf{y} \rangle \geq 0\}.$$

The complementarity set of K [6] is defined to be

$$C(K) := \{(\mathbf{x}, \mathbf{s}) : \mathbf{x} \in K, \mathbf{s} \in K^*, \langle \mathbf{x}, \mathbf{s} \rangle = 0\}.$$

Such a set arises in complementarity and conic optimization problems in the form of optimality conditions. In many of these problems, one solution strategy involves replacing the single equation $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ by a square system of "independent" equations. While this is not always possible, it is desirable to identify cones where this can be done. In order to quantify this, Rudolf et al. [6] introduce the concept of *bilinearity rank* of a cone. Gowda and Tao [3] showed that this rank could also be described by means of the so-called Lyapunov-like matrices on the cone, and renamed the rank as the Lyapunov rank. We say that a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is *Lyapunov-like* [2] on K if $\langle \mathbf{A}\mathbf{x}, \mathbf{s} \rangle = 0$ for all $(\mathbf{x}, \mathbf{s}) \in C(K)$. The set of all such matrices forms a vector space, denoted by $\mathrm{LL}(K)$, whose dimension $\beta(K)$ is called the *Lyapunov rank* of K. We note that all of the above concepts could be defined for a proper cone in a finite-dimensional real inner product space.

Recent papers [3], [4], and [6] contain numerous results on the Lyapunov rank. In particular, it was shown in [3] that for any proper polyhedral cone in $\mathbb{R}^n$, the Lyapunov rank can be any number between 1 and n except $n-1$. The Lyapunov ranks of moment cones, symmetric cones, l_p-cones, etc., have all been described in the above papers. In the present paper, by answering several questions posed in [3], we show that $\beta(K)$ is bounded above by $(n-1)^2$, thereby improving the previously known upper bound of $n^2 - n$. We also show that when $\beta(K) \geq n$, the complementarity set $C(K)$ can be expressed in terms of n linearly-independent Lyapunov-like matrices.

2 Preliminaries

In the Euclidean space $\mathbb{R}^n$ whose elements are regarded as column vectors, we use the notation $\mathbf{x}^T\mathbf{y}$ or $\langle \mathbf{x}, \mathbf{y} \rangle$ to denote the usual inner product on $\mathbb{R}^n$. For a set S in $\mathbb{R}^n$, $\mathrm{span}(S)$ and $\partial(S)$ respectively denote the vector (sub)space generated by S and the boundary of S. Corresponding to a nonzero vector $\mathbf{d}$ in $\mathbb{R}^n$ and a real number α, the set $\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{d} \rangle = \alpha\}$ defines a hyperplane in $\mathbb{R}^n$. It is a supporting hyperplane for a set S if a boundary point of S lies on the hyperplane and S lies on one side of the hyperplane.

The space $\mathbb{R}^{n \times n}$ of all $n \times n$ real matrices carries the trace inner product: $\langle \mathbf{A}, \mathbf{B} \rangle := \mathrm{trace}\left(\mathbf{A}\mathbf{B}^T\right)$, where "trace" denotes the sum of all diagonal elements. Recall that a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is Lyapunov-like on a proper cone K if $\langle \mathbf{A}\mathbf{x}, \mathbf{s} \rangle = 0$ for all $(\mathbf{x}, \mathbf{s}) \in C(K)$, or equivalently, $\left\langle \mathbf{A}, \mathbf{s}\mathbf{x}^T \right\rangle = 0$ for all $(\mathbf{x}, \mathbf{s}) \in C(K)$. Thus,

$$\mathrm{LL}(K) = \left\{\mathbf{A} \in \mathbb{R}^{n \times n} : \mathbf{A} \text{ is Lyapunov-like on } K\right\}$$

is nothing but the orthogonal complement of span $\left(\left\{\mathbf{s}\mathbf{x}^T : (\mathbf{x},\mathbf{s}) \in C(K)\right\}\right)$. As $\beta(K) = \dim(\mathrm{LL}(K))$ by definition, we have the co-dimension formula [6]:

$$\beta(K) = n^2 - \dim\left(\mathrm{span}\left(\left\{\mathbf{s}\mathbf{x}^T : (\mathbf{x},\mathbf{s}) \in C(K)\right\}\right)\right). \tag{1}$$

3 Perfect cones

We say that a proper cone K in $\mathbb{R}^n$ is *perfect* [3] if its complementarity set $C(K)$ can be expressed in terms of n linearly-independent Lyapunov-like matrices; that is,

$$C(K) = \{(\mathbf{x},\mathbf{s}) : \mathbf{x} \in K, \mathbf{s} \in K^*, \langle \mathbf{A}_i\mathbf{x},\mathbf{s}\rangle = 0, i = 1,2,\dots,n\}, \tag{2}$$

where $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ are linearly-independent Lyapunov-like matrices on K.

Known examples include the nonnegative orthant $\mathbb{R}^n_+$ (which can be seen by taking $\mathbf{A}_i$ to be the diagonal matrix with 1 in its (i,i) slot and zeros elsewhere) and all symmetric cones [3]. These cones are considered "good" from a linear (conic) programming perspective as they admit polynomial algorithms. Thus, one hopes to seek similar "good" cones among perfect cones. This leads to the question of characterizing perfect cones (which was raised in [3]). To answer this, we have the following result.

Theorem 1 *For a proper cone K in $\mathbb{R}^n$, the following are equivalent:*

(i) $\beta(K) \geq n$.
(ii) *The identity matrix can be written as a linear combination of n linearly-independent Lyapunov-like matrices.*
(iii) *K is perfect.*

Proof $(i) \implies (ii)$: Suppose $m = \beta(K) \geq n$. As $\mathbf{I}$ (the identity matrix) is a nonzero Lyapunov-like matrix, we can extend $\{\mathbf{I}\}$ to a basis $\{\mathbf{I}, \mathbf{A}_2, \dots, \mathbf{A}_m\}$ of $\mathrm{LL}(K)$. Then the equality $\mathbf{I} = \mathbf{I} + 0\mathbf{A}_2 + \cdots + 0\mathbf{A}_n$ proves (ii).

$(ii) \implies (iii)$: Let $\mathbf{I}$ be a linear combination of n linearly-independent Lyapunov-like matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ and let

$$E(K) := \{(\mathbf{x},\mathbf{s}) : \mathbf{x} \in K, \mathbf{s} \in K^*, \langle \mathbf{A}_i\mathbf{x},\mathbf{s}\rangle = 0, i = 1,2,\dots,n\},$$

which is the right-hand side of (2). We show that K is perfect by proving $C(K) = E(K)$.

Suppose that $(\mathbf{x},\mathbf{s}) \in C(K)$. Then by definition, $\langle \mathbf{A}\mathbf{x},\mathbf{s}\rangle = 0$ for every Lyapunov-like matrix $\mathbf{A}$. Since each $\mathbf{A}_i$ is Lyapunov-like, we have $\langle \mathbf{A}_i\mathbf{x},\mathbf{s}\rangle = 0$ for $i = 1,2,\dots,n$ and thus $(\mathbf{x},\mathbf{s}) \in E(K)$. So, $C(K) \subseteq E(K)$.

To see the reverse inclusion, let $(\mathbf{x},\mathbf{s}) \in E(K)$ so that $\langle \mathbf{A}_i\mathbf{x},\mathbf{s}\rangle = 0$ for $i = 1,2,\dots,n$. Since $\mathbf{I}$ can be written as a linear combination of the $\mathbf{A}_i$, we have $\langle \mathbf{I}\mathbf{x},\mathbf{s}\rangle = 0$. In other words, $(\mathbf{x},\mathbf{s}) \in C(K)$. So we have $E(K) \subseteq C(K)$ as well.

It follows that $C(K) = E(K)$ and that K is perfect.

The implication $(iii) \implies (i)$ is obvious from the definition of a perfect cone, because there exist at least n linearly-independent Lyapunov-like matrices on K.

$\square$

Remark 1 When $\beta(K) = n$, $\mathbf{I}$ is a linear combination of *any* n linearly-independent Lyapunov-like matrices (as there are exactly n objects in any basis of $\mathrm{LL}(K)$). This need not happen when $\beta(K) > n$: while $\mathbf{I}$ can always be expressed as a linear combination of some n linearly-independent Lyapunov-like matrices, one could construct a basis of $\mathrm{LL}(K)$ such that $\mathbf{I}$ can never be expressed as a linear combination of n linearly-independent Lyapunov-like matrices from this basis. This can be seen as follows. With $m = \beta(K) > n$, consider a basis of $\mathrm{LL}(K)$ of the form $\{\mathbf{I}, \mathbf{A}_2, \ldots, \mathbf{A}_m\}$. Then

$$\{\mathbf{I} + \mathbf{A}_2 + \cdots + \mathbf{A}_m, \mathbf{A}_2, \ldots, \mathbf{A}_m\}$$

is also basis such that $\mathbf{I}$ is not a linear combination of n objects in this basis.

This raises an interesting problem: Given a basis of $\mathrm{LL}(K)$ whose rank is more than n, find a necessary and sufficient condition for the identity matrix to be a linear combination of n elements of the basis.

4 An improved upper bound for $\beta(K)$

As mentioned previously, it was shown in [3] that $\beta(K) \leq n^2 - n$. The problem of improving this bound (or potentially determining the best bound) was raised in [3]. Towards this, we prove the following result.

Theorem 2 *For every proper cone K in $\mathbb{R}^n$ with $n \geq 3$,*

$$1 \leq \beta(K) \leq (n-1)^2. \tag{3}$$

The proof of this theorem relies on the following lemmas.

Lemma 1 *Let K be a proper cone in $\mathbb{R}^n$. Then, for any nonzero $\mathbf{x} \in \partial(K)$, there exists a nonzero $\mathbf{s} \in \partial(K^*)$ such that $(\mathbf{x}, \mathbf{s}) \in C(K)$. Similarly, for any $\mathbf{s} \in \partial(K^*)$, there exists a nonzero $\mathbf{x} \in \partial(K)$ such that $(\mathbf{x}, \mathbf{s}) \in C(K)$.*

This is a known result: see [7], Lemma 3.

While the following result may be known, for lack of precise reference, we offer a proof.

Lemma 2 *Suppose K is a proper cone in $\mathbb{R}^n$, with $n \geq 2$, whose boundary is contained in a finite union of proper subspaces. Then, K is polyhedral.*

Proof Without loss of generality, we assume that the proper subspaces are of dimension $n-1$. Among these $(n-1)$-dimensional subspaces, some may be non-supporting hyperplanes. We prove the result by induction on the number such hyperplanes. Let $\partial(K) \subseteq \cup_1^N H_i$, where

$$H_i := \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{z}_i \rangle = 0\}$$

with $\mathbf{z}_i \neq \mathbf{0}$ for all i. Denote the left and right half-planes of H_i by $L_i = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{z}_i \rangle \leq 0\}$ and $R_i = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{z}_i \rangle \geq 0\}$ respectively.

We first consider the base case when there are no non-supporting hyperplanes among these H_i. This means that every H_i is a supporting hyperplane. Without loss of generality, let $K \subseteq L_i$ for all i. Now consider the polyhedral cone $C := \cap_1^N L_i$ which contains K. We claim that $K = C$. Assume the contrary that $K \neq C$. As $n \geq 2$, every interior point of a proper cone is the convex combination of two boundary points. We may therefore assume that there is a $\mathbf{p} \in \partial(C)$ such that $\mathbf{p} \notin K$. Let $\mathbf{q} \in \operatorname{int}(K) \subseteq \operatorname{int}(C)$. From convex analysis [5] we know that the line segment $(\mathbf{p}, \mathbf{q}] \subset \operatorname{int}(C)$ and there exists $\mathbf{r} \in (\mathbf{p}, \mathbf{q}]$ which lies on the boundary of K.

By our assumption on the boundary points of K, $\mathbf{r}$ belongs to, say, H_1. As $K \subseteq L_i$ for all i, we see that $\mathbf{r} \in \partial(C)$. This clearly contradicts the fact that $(\mathbf{p}, \mathbf{q}] \in \operatorname{int}(C)$. Thus, $K = C$, proving the polyhedrality of K in the base case.

Now suppose that among H_i, $i = 1, 2, \ldots, N$, there is a non-supporting hyperplane, say, H_N. This means that the sets $K \cap \operatorname{int}(L_N)$ and $K \cap \operatorname{int}(R_N)$ are both nonempty. Let $K_1 := K \cap L_N$ and $K_2 := K \cap R_N$. Clearly, both K_1 and K_2 are proper cones in $\mathbb{R}^n$ and $K = K_1 \cup K_2$. Now, as $\partial(K_1) \subseteq \partial(K) \cup \partial(L_N)$, we see that the boundary of K_1 is contained in the union of a finite number of $n-1$ dimensional subspaces, but now with a fewer non-supporting hyperplanes (H_N has become a supporting hyperplane for K_1). By our induction hypothesis, K_1 is polyhedral. Similarly, K_2 is polyhedral. Let $K_1 = \operatorname{convcone}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l\})$ be the convex conic hull of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l\}$, and let $K_2 = \operatorname{convcone}(\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\})$. As $K = K_1 \cup K_2$ is convex, we have

$$K = \operatorname{convcone}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}).$$

Hence, K is polyhedral.

□

Proof of Theorem 2 Let K be a proper cone in $\mathbb{R}^n$ with $n \geq 3$.

When K is polyhedral, Gowda and Tao [3] have shown that $\beta(K) \leq n$. As $n \leq (n-1)^2$ for $n \geq 3$, we have (3) in this case. Hence from now on, *we assume that K is non-polyhedral and that $n \geq 3$.* Our proof consists of showing that there are at least $2n-1$ linearly-independent matrices of the form $\mathbf{s}\mathbf{x}^T$ with $(\mathbf{x}, \mathbf{s}) \in C(K)$; then, the codimension formula (1) yields (3).

Now, since K is proper, its interior is non-empty. Thus, $\operatorname{span}(\partial(K)) = \operatorname{span}(K) = \mathbb{R}^n$. Hence, there exist n linearly-independent vectors on the boundary of K. There also exists an isomorphism of $\mathbb{R}^n$ mapping these n

vectors to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Since the Lyapunov rank is invariant under an isomorphism, without loss of generality we can suppose that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ lie on the boundary of K.

Since each $\mathbf{e}_i$ lies on the boundary of K, by Lemma 1, there exists a nonzero $\mathbf{s}_i$ such that $(\mathbf{e}_i, \mathbf{s}_i) \in C(K)$.

Our goal now is to find $(\mathbf{v}_i, \mathbf{w}_i) \in C(K)$, $i = 1, 2, \ldots, n-1$ so that the set

$$\left\{\mathbf{s}_i \mathbf{e}_i^T : i = 1, 2, \ldots, n\right\} \cup \left\{\mathbf{w}_i \mathbf{v}_i^T : i = 1, 2, \ldots, n-1\right\} \tag{4}$$

is linearly-independent in $\mathbb{R}^{n \times n}$, thus giving us the desired $2n-1$ linearly-independent matrices of the form $\mathbf{s}\mathbf{x}^T$ with $(\mathbf{x}, \mathbf{s}) \in C(K)$.

Note that $\left\{\mathbf{s}_i \mathbf{e}_i^T : i = 1, 2, \ldots, n\right\}$ is a linearly-independent set in $\mathbb{R}^{n \times n}$, as the ith column of $\mathbf{s}_i \mathbf{e}_i^T$ is $\mathbf{s}_i$ ($\neq \mathbf{0}$) and the other columns are zero.

To construct $(\mathbf{v}_i, \mathbf{w}_i) \in C(K)$, we proceed as follows. K is a non-polyhedral proper cone, so K^* is also a non-polyhedral proper cone. By Lemma 2, the boundary of K^* is not contained in $\cup_{i=1}^n \operatorname{span}(\{\mathbf{s}_i\})$. We can therefore find a nonzero $\mathbf{w}_1$ on the boundary of K^* with

$$\mathbf{w}_1 \notin \bigcup_{i=1}^{n} \operatorname{span}(\{\mathbf{s}_i\}).$$

By an application of Lemma 1, we can then find a nonzero $\mathbf{v}_1 \in \partial(K)$ such that $(\mathbf{v}_1, \mathbf{w}_1) \in C(K)$. We claim that the set

$$\left\{\mathbf{s}_i \mathbf{e}_i^T : i = 1, 2, \ldots, n\right\} \cup \left\{\mathbf{w}_1 \mathbf{v}_1^T\right\}$$

is linearly-independent in $\mathbb{R}^{n \times n}$. If this were not true, as the first set above is linearly-independent, $\mathbf{w}_1 \mathbf{v}_1^T$ must be a linear combination of $\mathbf{s}_i \mathbf{e}_i^T$, $i = 1, 2, \ldots, n$. Since any linear combination of $\mathbf{s}_i \mathbf{e}_i^T$, $i = 1, 2, \ldots, n$, is a matrix whose columns are multiples of $\mathbf{s}_i$, this would mean that $\mathbf{w}_1$ is a multiple of some $\mathbf{s}_i$, contradicting our choice of $\mathbf{w}_1$. We thus have our claim.

To proceed with the induction, assume the existence of $(\mathbf{v}_i, \mathbf{w}_i) \in C(K)$, $i = 1, 2, \ldots, m$ for $1 \leq m < n-1$ such that the set

$$\left\{\mathbf{s}_i \mathbf{e}_i^T : i = 1, 2, \ldots, n\right\} \cup \left\{\mathbf{w}_i \mathbf{v}_i^T : i = 1, 2, \ldots, m\right\}$$

is linearly-independent in $\mathbb{R}^{n \times n}$. We will construct a new pair $(\mathbf{v}_{m+1}, \mathbf{w}_{m+1})$ with the same property. Let $H_i := \operatorname{span}(\{\mathbf{s}_i, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\})$. As K^* is a non-polyhedral proper cone, its boundary cannot be contained in the union $\cup_{i=1}^n H_i$ by Lemma 2. Thus, there exists a vector $\mathbf{w}_{m+1} \in \partial(K^*)$ such that

$$\mathbf{w}_{m+1} \notin \bigcup_{i=1}^{n} H_i.$$

(Note that this argument works only when $m < n-1$. When $m = n-1$, each H_i is equal to $\mathbb{R}^n$, and clearly no such $\mathbf{w}_{m+1}$ can be found: the subspaces H_i are not proper, and the lemma does not apply.)

Now by an application of Lemma 1, we obtain a $\mathbf{v}_{m+1}$ corresponding to $\mathbf{w}_{m+1}$ such that $(\mathbf{v}_{m+1}, \mathbf{w}_{m+1}) \in C(K)$.

We next claim that

$$\left\{\mathbf{s}_i\mathbf{e}_i^T : i = 1, 2, \ldots, n\right\} \cup \left\{\mathbf{w}_i\mathbf{v}_i^T : i = 1, 2, \ldots, m+1\right\}$$

is linearly-independent in $\mathbb{R}^{n\times n}$. By our assumption on the m elements $\{\mathbf{w}_i\}$, we need only show that $\mathbf{w}_{m+1}\mathbf{v}_{m+1}^T$ is not a linear combination of matrices in $\left\{\mathbf{s}_i\mathbf{e}_i^T : i = 1, 2, \ldots, n\right\} \cup \left\{\mathbf{w}_i\mathbf{v}_i^T : i = 1, 2, \ldots, m\right\}$. Assuming the contrary, by considering the columns of these matrices, we see that $\mathbf{w}_{m+1}$ is a linear combination of vectors in $\{\mathbf{s}_j, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ for some j, which contradicts the choice of $\mathbf{w}_{m+1}$. We thus have our claim.

Now, by successively taking $m = 1, 2, \ldots, n-2$, we generate the linearly-independent set (4). As this set contains $2n-1$ elements, by the codimension formula (1), we see that $\beta(K) \leq (n-1)^2$.

□

Remark 2 The bound $\beta(K) \leq (n-1)^2$ is sharp for $n = 3$. For example, if K is the second-order cone $\mathcal{L}_+^n$ in $\mathbb{R}^n$, then $\beta(K) = \frac{n^2-n+2}{2}$. Thus, for this cone, when $n = 3$, $\beta(K) = 4 = (n-1)^2$. It is not clear if the bound $(n-1)^2$ continues to be sharp for $n > 3$.

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