SCHUR IDEALS AND HOMOMORPHISMS OF THE SEMIDEFINITE CONE *

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Abstract. We consider the semidefinite cone $\mathcal{K}_n$ consisting of all $n \times n$ real symmetric positive semidefinite matrices. A set $\mathcal{I}$ in $\mathcal{K}_n$ is said to be a *Schur ideal* if it is closed under addition, multiplication by nonnegative scalars, and Schur multiplication by any element of $\mathcal{K}_n$. A Schur homomorphism of $\mathcal{K}_n$ is a mapping of $\mathcal{K}_n$ to itself that preserves addition, (nonnegative) scalar multiplication and Schur products. This paper is concerned with Schur ideals and homomorphisms of $\mathcal{K}_n$. We show that in the topology induced by the trace inner product, Schur ideals in $\mathcal{K}_n$ need not be closed, all finitely generated Schur ideals are closed, and in $\mathcal{K}_2$, a Schur ideal is closed if and only if it is a principal ideal. We also characterize Schur homomorphisms of $\mathcal{K}_n$ and, in particular, show that any Schur automorphism of $\mathcal{K}_n$ is of the form $\Phi(X) = PXP^T$ for some permutation matrix P.

Key words. Semidefinite cone, Schur ideals and homomorphisms

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1. Introduction.

Consider the Euclidean space $\mathbb{R}^n$ whose elements are regarded as column vectors. Let $\mathcal{S}^n$ denote the set of all real $n \times n$ symmetric matrices. It is known that $\mathcal{S}^n$ is a finite dimensional real Hilbert space under the trace inner product defined by

$$\langle X, Y \rangle = trace(XY) = \sum_{i,j} x_{ij} y_{ij},$$

where $X = [x_{ij}]$ and $Y = [y_{ij}]$. The space $\mathcal{S}^n$ also carries the *Schur product* (sometimes called *Hadamard product*) defined by

$$X \circ Y := [x_{ij} y_{ij}].$$

With respect to the usual addition, (real) scalar multiplication, and the Schur product, $\mathcal{S}^n$ becomes a commutative algebra. In this algebra, a nonempty set is an *ideal* if it is closed under addition, scalar multiplication, and (Schur) multiplication by any element of $\mathcal{S}^n$. Also, a (Schur) homomorphism of $\mathcal{S}^n$ is a linear transformation on $\mathcal{S}^n$ that preserves Schur products. We note that every ideal in this algebra, being a finite

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dimensional linear subspace of $\mathcal{S}^n$, is topologically closed (in the topology induced by the trace inner product) and finitely generated. Also, it is easy to describe ideals and homomorphisms of this algebra, see Sections 3 and 6.

The aim of this paper is to study Schur ideals and homomorphisms of the semidefinite cone. A matrix $A \in \mathcal{S}^n$ is said to be *positive semidefinite* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. The set of all positive semidefinite matrices in $\mathcal{S}^n$ is the *semidefinite cone* and is denoted by $\mathcal{K}_n$.

It is known that $\mathcal{K}_n$ is a closed convex cone in $\mathcal{S}^n$; this means that $\mathcal{K}_n$ is topologically closed and $X, Y \in \mathcal{K}_n, 0 \leq \lambda \in R \Rightarrow X + Y \in \mathcal{K}_n,\ \lambda X \in \mathcal{K}_n$. A famous theorem of Schur (see [5], Theorem 7.5.3) says that

$$X, Y \in \mathcal{K}_n \Rightarrow X \circ Y \in \mathcal{K}_n.$$

So $\mathcal{K}_n$ is closed under addition, multiplication by nonnegative scalars, and the Schur product. Thus, following [2], we say that $\mathcal{K}_n$ is a semi-algebra (under these operations). We call nonempty subsets of $\mathcal{K}_n$ that share these properties *sub-semi-algebras* of $\mathcal{K}_n$, and note the following examples:

Example 1. Let $\mathcal{DNN}_n$ denote the set of all $n \times n$ *doubly nonnegative* matrices – these are matrices in $\mathcal{S}^n$ that are both nonnegative and positive semidefinite. This set is a sub-semi-algebra of $\mathcal{K}_n$.

Example 2. Let $\mathcal{CP}_n$ denote the set of all *completely positive* matrices in $\mathcal{S}^n$. These are matrices of the form

$$\sum_1^N uu^T,$$

where $u \in \mathbb{R}^n_+$ (the nonnegative orthant of $\mathbb{R}^n$) and N is a natural number. Alternatively, these are matrices of the form BB^T, where B is a nonnegative (rectangular) matrix. Since

$$uu^T \circ vv^T = (u \circ v)\,(u \circ v)^T,$$

where $u \circ v$ is the componentwise product of two vectors u and v, it follows that the Schur product of two completely positive matrices is once again a completely positive matrix. Thus, $\mathcal{CP}_n$ is a sub-semi-algebra of $\mathcal{K}_n$.

These two cones, along with $\mathcal{K}_n$, have become very important in optimization, particularly in conic linear programming (where one minimizes a linear function subject to linear constraints over a cone); see [1] and [4]. Thus, the problem of finding/characterizing such sub-semi-algebras of $\mathcal{K}_n$ and studying their properties becomes interesting and useful. Motivated by this, we consider special sub-semi-algebras of $\mathcal{K}_n$ called *Schur ideals*. These are *nonempty* subsets of $\mathcal{K}_n$ that are closed under addition, multiplication by nonnegative scalars, and Schur multiplication by *any* element of $\mathcal{K}_n$. A nonempty set $\mathcal{I} \subseteq \mathcal{K}_n$ is a *Schur ideal* in $\mathcal{K}_n$ if

$$X, Y \in \mathcal{I},\ Z \in \mathcal{K}_n,\ 0 \leq \lambda \in R \Rightarrow X + Y, \lambda X, X \circ Z \in \mathcal{I}.$$

We also consider related *Schur homomorphisms* of $\mathcal{K}_n$, which are mappings from $\mathcal{K}_n$ to $\mathcal{K}_n$ that preserve addition, (nonnegative) scalar multiplication, and Schur products.

Now, motivated by the problem of describing Schur ideals and homomorphisms of the semidefinite cone, we raise the following questions (using the term 'closed' to mean 'topologically closed'): (a) Is every Schur ideal in $\mathcal{K}_n$ closed? How about finitely generated ones? (b) Is every closed Schur ideal of $\mathcal{K}_n$ finitely generated? (c) What are the Schur homomorphisms/automorphisms of $\mathcal{K}_n$?
We show that:

(1) Schur ideals in $\mathcal{K}_n$ need not be closed,
(2) Finitely generated Schur ideals are always closed,
(3) In $\mathcal{K}_2$, a Schur ideal is closed if and only if it is a principal ideal, and
(4) Any Schur homomorphism of $\mathcal{K}_n$ is, up to permutation, conjugation by a 0/1-matrix with each row containing at most one 1 followed by pinching. In particular, a Schur automorphism of $\mathcal{K}_n$ (this is a bijection of $\mathcal{K}_n$ that preserves the three operations on $\mathcal{K}_n$) is of the form

$$\Phi(X) = PXP^T,$$

where P is a permutation matrix.

Section 2 covers some background material. Section 3 deals with two elementary results characterizing Schur ideals in $\mathcal{S}^n$ and $\mathcal{K}_n$. In Section 4, we give an example of a non-closed Schur ideal in any $\mathcal{K}_n$, $n \geq 2$. In Section 5, we show that all finitely generated Schur ideals in $\mathcal{K}_n$ are closed. Finally, in Section 6, we describe Schur homomorphisms/automorphisms on $\mathcal{K}_n$.

2. Preliminaries. In the Euclidean space $\mathbb{R}^n$, we let $e_1, e_2, \ldots, e_n$ denote the standard unit vectors (so that e_i has one in the i^{th} slot and zeros elsewhere). We let

$$e = e_1 + e_2 + \cdots + e_n \text{ and } \quad E := ee^T.$$

We also let E_{ij} be a symmetric matrix with ones in the (i,j) and (j,i) slots and zeros elsewhere; in particular, E_{ii} has a 1 in the (i,i) slot and zeros elsewhere. A matrix is a 0/1-matrix if each entry is either 0 or 1.

We let $\mathcal{S}^n$ denote the space of all $n \times n$ real symmetric matrices. $\mathcal{S}^n$ becomes a Hilbert space under the trace product $\langle X, Y\rangle = trace(XY) = \sum_{i,j} x_{ij} y_{ij}$, where

$X = [x_{ij}]$ and $Y = [y_{ij}]$. The corresponding norm (called the Frobenius norm) is given by $||X|| = \sqrt{\sum_{ij} x_{ij}^2}$. Thus, convergence in this space is entrywise. From now on, we say that a set in $\mathcal{S}^n$ is *closed if it is topologically closed* under the Frobenius norm. We note the following property of the inner product:

$$\langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle$$

for all X, Y and Z in $\mathcal{S}^n$. We let

$$\mathcal{K}_n := \{A \in \mathcal{S}^n : x^T A x \geq 0 \; \forall \, x \in \mathbb{R}^n\}.$$

It is known that $\mathcal{K}_n$ is a closed convex cone in $\mathcal{S}^n$. In particular, as convergence in $\mathcal{S}^n$ is entrywise, *every bounded sequence in $\mathcal{K}_n$ has a subsequence that converges to an element of $\mathcal{K}_n$.*

In $\mathcal{K}_n$, we have the Löwner ordering:

$$Y \succeq X \quad \text{or} \quad X \preceq Y \quad \text{when} \quad Y - X \in \mathcal{K}_n.$$

It is known (see [5], Corollary 7.5.4) that

$$U, V \succeq 0 \Rightarrow \langle U, V \rangle \geq 0. \tag{2.1}$$

Thus,

$$0 \preceq X \preceq Y \Rightarrow \langle X, X \rangle \leq \langle X, Y \rangle \leq \langle Y, Y \rangle \Rightarrow ||X|| \leq ||Y||. \tag{2.2}$$

It is known that *a real symmetric matrix is positive semidefinite if and only if all its principal minors are nonnegative.* In particular, for $X = [x_{ij}] \in \mathcal{K}_n$,

$$x_{ii} \geq 0 \quad \text{and} \quad x_{ii}\, x_{jj} \geq x_{ij}^2, \tag{2.3}$$

for all i and j. Thus, if a diagonal entry of X (belonging to $\mathcal{K}_n$) is zero, then all elements in the row/column containing this entry are zero. Also, if some off-diagonal entry is nonzero, then the corresponding diagonal entry is nonzero.

Let $diag(\mathcal{K}_n)$ denote the set of all diagonal matrices in $\mathcal{K}_n$. The interior of $\mathcal{K}_n$ is denoted by $\mathcal{K}_n^\circ$. Note that this interior consists of all *positive definite* matrices of $\mathcal{S}^n$ — these are matrices A in $\mathcal{S}^n$ satisfying $x^T A x > 0$ for all $0 \neq x \in \mathbb{R}^n$.

The *pinching operation/transformation* on $\mathcal{S}^n$ is defined as follows: By choosing a partition of $\Delta := \{1, 2, \ldots, n\}$, we write any matrix $X \in \mathcal{S}^n$ in the block form $X = [X_{\alpha\beta}]$. Then the pinching operation/transformation (corresponding to this partition) takes X to $Y = [Y_{\alpha\beta}]$, where $Y_{\alpha\beta} = X_{\alpha\beta}$ when $\alpha = \beta$ and 0 otherwise. (This means that Y is obtained from X by setting all the off-diagonal blocks to zero.). We write $Y = X_{pinch}$.

For any finite set $\{C_1, C_2, \ldots, C_N\}$ in $\mathcal{K}_n$, we let

$$I_{\{C_1,C_2,\ldots,C_N\}} := \left\{\sum C_i \circ X_i : \ X_i \in \mathcal{K}_n,\ i = 1, 2, \ldots, N\right\}$$

denote the Schur ideal generated by $\{C_1, C_2, \ldots, C_N\}$. We say that a Schur ideal of $\mathcal{K}_n$ is a *principal ideal* if it is generated by a single matrix in $\mathcal{K}_n$.

3. Two elementary results on Schur ideals. We start with a description of Schur ideals in $\mathcal{S}^n$. Let

$$\Delta := \{1, 2, 3 \ldots, n\}.$$

We say that a subset I of $\Delta \times \Delta$ is *symmetric* if $(i, j) \in I \Rightarrow (j, i) \in I$.

PROPOSITION 3.1. *Given any symmetric subset I of $\Delta \times \Delta$, the set*

$$\mathcal{I} = \{X = [x_{ij}] \in \mathcal{S}^n : x_{kl} = 0 \quad \text{for all } (k, l) \in I\}$$

is a Schur ideal of $\mathcal{S}^n$. Conversely, every Schur ideal of $\mathcal{S}^n$ arises this way.

Proof. The first part of the proposition is easy to verify. We prove the converse part. Let $\mathcal{I}$ be any Schur ideal of $\mathcal{S}^n$. If $\mathcal{I} = \{0\}$, we take $I = \Delta \times \Delta$. If $\mathcal{I} = \mathcal{S}^n$, we take $I = \emptyset$. Now assume that $\{0\} \neq \mathcal{I} \neq \mathcal{S}^n$. Define

$$I := \{(k, l) : x_{kl} = 0 \text{ for all } X = [x_{ij}] \in \mathcal{I}\}$$

and let J be the complement of I in $\Delta \times \Delta$. Let

$$\widetilde{\mathcal{I}} = \{X = [x_{ij}] \in \mathcal{S}^n : x_{kl} = 0 \ \forall\, (k, l) \in I\}.$$

Then, $\mathcal{I} \subseteq \widetilde{\mathcal{I}}$. Now for any $(k, l) \in J$, there is a $Y \in \mathcal{I}$ with $y_{kl} \neq 0$. For this (k, l), $E_{kl} = \frac{1}{y_{kl}} Y \circ E_{kl} \in \mathcal{I}$. Therefore, for any $X \in \widetilde{\mathcal{I}}$, $X = [x_{ij}] = \sum_{1 \leq i \leq j \leq n} x_{ij} E_{ij} = \sum_{1 \leq k \leq l \leq n, (k,l) \in J} x_{kl} E_{kl} \in \sum_J R\, E_{ij} \in \mathcal{I}$. It follows that $\mathcal{I} = \widetilde{\mathcal{I}}$. ∎

The following proposition gives a necessary and sufficient condition for a set to be a Schur ideal in $\mathcal{K}_n$.

PROPOSITION 3.2. *Let $\mathcal{I}$ be a set in $\mathcal{K}_n$ containing zero. Then $\mathcal{I}$ is a Schur ideal in $\mathcal{K}_n$ if and only if*

(i) $\mathcal{I}$ is a convex cone, and
(ii) $DXD \in \mathcal{I}$ for every diagonal matrix $D \in \mathcal{S}^n$ and $X \in \mathcal{I}$.

Proof. Suppose $\mathcal{I}$ is a Schur ideal in $\mathcal{K}_n$. Then for any $\lambda \geq 0$ and $X \in \mathcal{I}$, we have $\lambda X \in \mathcal{I}$. From this, it follows that $\mathcal{I}$ is a convex cone. Now take any diagonal matrix $D = diag(r_1, r_2, \ldots, r_n)$ in $\mathcal{S}^n$. Then, for $r = [r_1, r_2, \ldots, r_n]^T$, $rr^T \in \mathcal{K}_n$ and hence $DXD = X \circ rr^T \in \mathcal{I}$. This proves (ii). The converse is proved by using the fact that any matrix in $\mathcal{K}_n$ is a finite sum of matrices of the form rr^T. ■

4. A non-closed Schur ideal of $\mathcal{K}_n$. In this section, we show that in any $\mathcal{K}_n$, $n \geq 2$, there is a Schur ideal that is not closed. First, we construct such an ideal in the cone $\mathcal{K}_2$ of all 2×2 real symmetric positive semidefinite matrices.

Example 3. The set

$$\mathcal{E} = diag(\mathcal{K}_2) \cup \mathcal{K}_2^{\circ}$$

is a non-closed ideal in $\mathcal{K}_2$.
To see this, let $\lambda \geq 0$, $A \in diag(\mathcal{K}_2)$, $B \in \mathcal{K}_2^{\circ}$ and $X \in \mathcal{K}_2$. Then, $A + A \in diag(\mathcal{K}_2)$, $\mathcal{K}_2 + B \in \mathcal{K}_2^{\circ}$, $\lambda(A + B) \in \mathcal{E}$, and $A \circ X \in diag(\mathcal{K}_2)$. If X is diagonal, then $B \circ X \in diag(\mathcal{K}_2)$. Suppose X is not diagonal so that

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix},$$

with $x > 0$, $z > 0$ and $xz - y^2 \geq 0$. Now write

$$B = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

where $a > 0$, $c > 0$ and $ac - b^2 > 0$; we see that $ax > 0$, $cz > 0$ and $(ax)(cz) > (b^2)(y^2)$. This means that $B \circ X \in \mathcal{K}_2^{\circ}$. Hence in all cases, $(A + B) \circ X \in \mathcal{E}$. Thus, $\mathcal{E}$ is an ideal of $\mathcal{K}_2$.
To see that $\mathcal{E}$ is not closed, we produce a sequence in $\mathcal{E}$ whose limit is not in $\mathcal{E}$:

$$\begin{bmatrix} 1 & 1 - \frac{1}{k} \\ 1 - \frac{1}{k} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Now, in any $\mathcal{K}_n$, $n \geq 3$, we define the set of block matrices

$$\mathcal{F} = \left\{ \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \; : \; X \in \mathcal{E} \right\}.$$

This is a Schur ideal in $\mathcal{K}_n$ that is not closed.

5. Finitely generated ideals in $\mathcal{K}_n$. Example 3 raises the question of which Schur ideals are closed in $\mathcal{K}_n$. The following result partially answers this question.

THEOREM 5.1. *Every finitely generated Schur ideal in $\mathcal{K}_n$ is closed.*

The proof is based on the following two lemmas.

LEMMA 5.2. *Let $A \in \mathcal{K}_n$ with $diag(A) > 0$. If X_k is a sequence in $\mathcal{K}_n$ with $A \circ X_k$ convergent, then X_k is a bounded sequence.*

Proof. Let $X_k = \left[x_{ij}^{(k)}\right]$, $A = [a_{ij}]$ and $A \circ X_k \to Y$. Then, by componentwise convergence, for all i, $1 \le i \le n$, $a_{ii}x_{ii}^{(k)} \to y_{ii}$ and so $x_{ii}^{(k)} \to \frac{y_{ii}}{a_{ii}}$. Now, each X_k is positive semidefinite and so by (2.3),

$$x_{ii}^{(k)} x_{jj}^{(k)} \ge (x_{ij}^{(k)})^2$$

for all $i, j = 1, 2, 3, \ldots, n$. As $x_{ii}^{(k)}$ and $x_{jj}^{(k)}$ are bounded sequences, it follows that $x_{ij}^{(k)}$ is also bounded for all i, j. Hence X_k is a bounded sequence. ∎

LEMMA 5.3. *Let X_k be a sequence in $\mathcal{K}_n$ and $A \in \mathcal{K}_n$ with $A \circ X_k \to Z \in \mathcal{K}_n$. Then there is an $X \in \mathcal{K}_n$ such that $A \circ X = Z$.*

Proof. If $A = 0$, we can take $X = 0$. So, assume $A \neq 0$ and without loss of generality write $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where A_1 is positive semidefinite and $diag(A_1) > 0$. If we partition each X_k and Z conformally to A, we get

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} X_1^{(k)} & X_2^{(k)} \\ (X_2^{(k)})^T & X_3^{(k)} \end{bmatrix} \to \begin{bmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{bmatrix}.$$

Thus, $A_1 \circ X_1^{(k)} \to Z_1$, $Z_2 = 0$, and $Z_3 = 0$. Now, from Lemma 5.2, $X_1^{(k)}$ is a bounded sequence; we may assume that it converges to X_1. Then

$$\begin{bmatrix} A_1 \circ X_1^{(k)} & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} A_1 \circ X_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Z_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$A \circ \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} = Z.$$

Since $\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{K}_n$, the result follows. ∎

Proof of Theorem 5.1. Let $\mathcal{E}$ be an ideal in $\mathcal{K}_n$ generated by a finite number of matrices. For simplicity, we assume that $\mathcal{E}$ is generated by two, say, A and B, so that any element of $\mathcal{E}$ is of the form $A \circ X + B \circ Y$ for some $X, Y \in \mathcal{K}_n$.

To show that $\mathcal{E}$ is closed, let X_k and Y_k be sequence of matrices in $\mathcal{K}_n$ such that $A \circ X_k + B \circ Y_k \to Z$. We show that $Z = A \circ X + B \circ Y$ for some $X, Y \in \mathcal{K}_n$. Now, with respect to the Löwner ordering, $0 \preceq A \circ X_k \preceq A \circ X_k + B \circ Y_k$. From (2.2), we get $||A \circ X_k \| \leq \| A \circ X_k + B \circ Y_k \|$. Thus, the sequence $A \circ X_k$ is bounded. We may assume (by going through a subsequence, if necessary) that $A \circ X_k \to U$. By Lemma 5.3, we may write $U = A \circ X$ for some $X \in \mathcal{K}_n$. Similarly, we may assume that $B \circ Y_k \to V$, where $V = B \circ Y$ for some $Y \in \mathcal{K}_n$. It follows that $Z = A \circ X + B \circ Y$ with $X, Y \in \mathcal{K}_n$. Thus, $\mathcal{E}$ is closed. ∎

Remarks. By modifying the above proof, one can show the following: If $\mathcal{I}$ and $\mathcal{J}$ are two closed ideals in $\mathcal{K}_n$, then so is $\mathcal{I} + \mathcal{J}$. In particular, if $\mathcal{I}$ is a closed ideal in $\mathcal{K}_n$ and

$$\mathcal{I}^\perp := \{Y \in \mathcal{K}_n : \langle Y, X \rangle = 0 \; \forall \; X \in \mathcal{I}\},$$

then $\mathcal{I} + \mathcal{I}^\perp$ is a closed ideal in $\mathcal{K}_n$. We note, however, that $\mathcal{I}^\perp$ may be just $\{0\}$, as in the case of the ideal consisting of all diagonal matrices in $\mathcal{K}_n$.

Remarks. Every finitely generated ideal in $\mathcal{K}_n$ is closed, but we do not know if the converse holds. In the following theorem, we show that the converse does hold for $\mathcal{K}_2$.

THEOREM 5.4. *Every closed ideal in $\mathcal{K}_2$ is a principal ideal. Conversely, every principal ideal in $\mathcal{K}_2$ is closed.*

We begin with a lemma.

LEMMA 5.5. *Let $\mathcal{E}$ be a closed ideal in $\mathcal{K}_2$ that contains a matrix with an off-diagonal entry nonzero. Let*

$$u := \sup \left\{ \frac{\beta}{\sqrt{\alpha\gamma}} : \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \in \mathcal{E} \quad \text{and} \quad \alpha, \beta, \gamma > 0 \right\}. \tag{5.1}$$

Then, $\begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix} \in \mathcal{E}$. *Moreover,*

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathcal{E} \Rightarrow \begin{bmatrix} a & \frac{b}{u} \\ \frac{b}{u} & c \end{bmatrix} \in \mathcal{K}_2.$$

Proof. Let $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \in \mathcal{E}$ with $\beta \neq 0$; we may, if necessary, consider the Schur product of this matrix with the positive semidefinite matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and assume that $\beta > 0$. Then α, β, and γ are positive. Since $\alpha\gamma \geq \beta^2$, it follows that u in (5.1) is well defined. Now consider a sequence $\begin{bmatrix} \alpha_k & \beta_k \\ \beta_k & \gamma_k \end{bmatrix} \in \mathcal{E}$ such that $\alpha_k, \beta_k, \gamma_k > 0$ and $\frac{\beta_k}{\sqrt{\alpha_k\gamma_k}} \to u$ as $k \to \infty$. Since $\begin{bmatrix} \frac{1}{\alpha_k} & \frac{1}{\sqrt{\alpha_k\gamma_k}} \\ \frac{1}{\sqrt{\alpha_k\gamma_k}} & \frac{1}{\gamma_k} \end{bmatrix}$ is positive semidefinite (as it is symmetric and all its principal minors are nonnegative) and $\mathcal{E}$ is an ideal, it follows that

$$\begin{bmatrix} \alpha_k & \beta_k \\ \beta_k & \gamma_k \end{bmatrix} \circ \begin{bmatrix} \frac{1}{\alpha_k} & \frac{1}{\sqrt{\alpha_k\gamma_k}} \\ \frac{1}{\sqrt{\alpha_k\gamma_k}} & \frac{1}{\gamma_k} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\beta_k}{\sqrt{\alpha_k\gamma_k}} \\ \frac{\beta_k}{\sqrt{\alpha_k\gamma_k}} & 1 \end{bmatrix} \in \mathcal{E}.$$

Since $\mathcal{E}$ is closed, letting $k \to \infty$, we have

$$\begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix} \in \mathcal{E}.$$

Now for the second part of the lemma. The implication is obvious if $b = 0$. If $b \neq 0$, then $a, c > 0$ and $\begin{bmatrix} a & |b| \\ |b| & c \end{bmatrix} \in \mathcal{E}$ and so (by definition of u), $u \geq \frac{|b|}{\sqrt{ac}}$. This shows that the symmetric matrix $\begin{bmatrix} a & \frac{b}{u} \\ \frac{b}{u} & c \end{bmatrix}$ belongs to $\mathcal{K}_2$ as its principal minors are nonnegative. ■

Proof of Theorem 5.4. If $\mathcal{E}$ is a principal ideal, then it is closed by Theorem 5.1. For the converse, let $\mathcal{E}$ be a closed ideal in $\mathcal{K}_2$. Suppose first that $\mathcal{E}$ contains a matrix with a nonzero off-diagonal entry. Then from the preceding lemma there exists a nonzero number u with $U := \begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix} \in \mathcal{E}$. Now, the principal ideal generated by U is contained in $\mathcal{E}$. The identity

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & u \\ u & 1 \end{bmatrix} \circ \begin{bmatrix} a & \frac{b}{u} \\ \frac{b}{u} & c \end{bmatrix}$$

shows that every element of $\mathcal{E}$ is in this principal ideal. Hence, $\mathcal{E}$ is a principal ideal generated by U.
Now suppose that every off-diagonal entry of every matrix in $\mathcal{E}$ is zero. Then $\mathcal{E}$ consists of diagonal matrices. In this case, $\mathcal{E}$ is generated by one of the following:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, we have shown that $\mathcal{E}$ is a principal ideal. ■

6. Schur homomorphisms of $\mathcal{K}_n$. In this section, we characterize Schur homomorphisms of $\mathcal{K}_n$. A map $f : \mathcal{K}_n \to \mathcal{K}_n$ is a *Schur homomorphism* if for all $X, Y \in \mathcal{K}_n$

and $\lambda \geq 0$ in R, we have

$$f(X+Y) = f(X) + f(Y),\ f(\lambda X) = \lambda f(X),\ \text{and}\ f(X \circ Y) = f(X) \circ f(Y).$$

These are special cases of Schur multiplicative maps (which preserve the Schur products) studied in [3].

To motivate our results, we consider an $n \times n$ 0/1-matrix Q with each row containing at most one 1. Then, the mapping $X \mapsto QXQ^T$ is a Schur homomorphism of $\mathcal{K}_n$. The process of pinching (corresponding to a partition of Δ) followed by a permutation continues to retain the Schur homomorphism property, that is, the mapping $X \mapsto P(QXQ^T)_{pinch}P^T$ is a Schur homomorphism of $\mathcal{K}_n$, where P denotes a permutation matrix. We now show that every Schur homomorphism arises this way.

Let f be a Schur homomorphism on $\mathcal{K}_n$. Then f can be extended to $\mathcal{S}^n$ in the following way. As $\mathcal{S}^n = \mathcal{K}_n - \mathcal{K}_n$, any element X of $\mathcal{S}^n$ can be written as $X = A - B$ with $A, B \in \mathcal{K}_n$. Then we define

$$F(X) := f(A) - f(B). \tag{6.1}$$

Note that $F(X)$ is well defined: If $X = A - B = C - D$ with $A, B, C, D \in \mathcal{K}_n$, then $A + D = B + C$ and so $f(A) + f(D) = f(B) + f(C)$ yielding $F(X) = f(A) - f(B) = f(C) - f(D)$. Then, $F : \mathcal{S}^n \to \mathcal{S}^n$ is a Schur homomorphism of $\mathcal{S}^n$, that is, F is linear and for all $X, Y \in \mathcal{S}^n$, $F(X \circ Y) = F(X) \circ F(Y)$.

We now characterize Schur homomorphisms of $\mathcal{S}^n$, such as F. First, a definition. We say that a mapping $\phi : \Delta \times \Delta \to \Delta \times \Delta$ is *symmetric* if $\phi(i,j) = \phi(j,i)$ for all (i,j); for notational convenience, we sometimes write $\phi(ij)$ in place of $\phi(i,j)$. Corresponding to such a mapping, we define $\Phi : \mathcal{S}^n \to \mathcal{S}^n$ by

$$\Phi(X) = Y \quad \text{where} \quad y_{ij} = x_{\phi(ij)}. \tag{6.2}$$

Theorem 6.1. *Given any symmetric mapping ϕ on $\Delta \times \Delta$ and a symmetric 0/1-matrix Z, the mapping $F : \mathcal{S}^n \to \mathcal{S}^n$ defined by*

$$F(X) = Z \circ \Phi(X) \quad (X \in \mathcal{S}^n)$$

is a Schur homomorphism on $\mathcal{S}^n$. Conversely, every Schur homomorphism of $\mathcal{S}^n$ arises in this way.

Proof. The first part in the theorem is easily verified. We prove the converse. Consider a Schur homomorphism F on $\mathcal{S}^n$. In order to simplify the proof and make it more transparent, we think of matrices X and Y as vectors x and y in R^d, where $d = \frac{n^2+n}{2}$. Then $F(X)$ can be regarded as a linear transformation $f(x) = Ax$ on R^d

(for some square matrix A) satisfying $A(x \circ y) = A(x) \circ A(y)$. This equation splits into d independent equations of the form $a^T(x \circ y) = (a^T x)(a^T y)$, where a is any row vector of A. From this we see that a is a 0/1-vector with at most one 1. Thus, A is a 0/1-matrix with at most one 1 in each row. Reverting back to F, we get the stated result. ∎

Remarks. The preceding theorem is analogous to Theorem 1.1 in [3], proved for Schur multiplicative maps on $R^{m\times n}$ under a 'nonsingularity' assumption, but without any linearity assumptions.

Before we present our general result on the Schur homomorphisms of $\mathcal{K}_n$, we introduce some notation and prove a lemma. For $1 \le m \le n$, let $\alpha := \{1, 2, \ldots, m\}$. Corresponding to a symmetric mapping $\psi : \alpha \times \alpha \to \Delta \times \Delta$, we define $\Psi : \mathcal{S}^n \to \mathcal{S}^m$ by

$$\Psi(X) = Y, \quad \text{where} \quad y_{ij} = x_{\psi(ij)}. \tag{6.3}$$

Lemma 6.2. *Consider ψ and Ψ defined in (6.3). Then $\Psi(\mathcal{K}_n) \subseteq \mathcal{K}_m$ if and only if there exists an $m \times n$ 0/1-matrix Q such that each row of Q contains exactly one 1 and $\Psi(X) = QXQ^T$ for all $X \in \mathcal{K}_n$.*

Proof. Since the 'if' part is obvious, we prove the 'only if' part. Suppose $\Psi(\mathcal{K}_n) \subseteq \mathcal{K}_m$. We prove the following statements:

- (a) If $\psi(i,i) = (k,l)$, then $k = l$.
- (b) If $i \ne j$, $\psi(i,i) = (k,k)$ and $\psi(j,j) = (l,l)$, then $\psi(i,j) \in \{(k,l),(l,k)\}$. (This implies that $y_{ij} = x_{\psi(ij)} = x_{kl}$.)

(a) If possible, let $\psi(i,i) = (k,l)$ with $k \ne l$. Then there exists a matrix $X \in \mathcal{K}_n$ such that $x_{kl} = -1$. But then, for $Y = \Psi(X)$, $y_{ii} = x_{\psi(ii)} = x_{kl} = -1$. This cannot happen as Y, being positive semidefinite, has all nonnegative diagonal entries. Hence $k = l$.

(b) Let $i \ne j$, $\psi(i,i) = (k,k)$ and $\psi(j,j) = (l,l)$. We consider two cases.

Case 1: $k = l$. Without loss of generality, let $k = l = 1$. Suppose $\psi(i,j) \ne (1,1)$. Consider $u = [1, 2, 3, \ldots, n]^T$ and $X = uu^T \in \mathcal{K}_n$. Then every entry in X other than the $(1,1)$ is bigger than one. Moreover, $y_{ii} = y_{jj} = x_{11} = 1$ and $y_{ij} = x_{\psi(ij)} > 1$. As $Y = \Psi(X) \in \mathcal{K}_m$, these values of Y violate (2.3). Hence, $\psi(i,j) = (1,1) = (k,l)$.

Case 2: $k \ne l$. Without loss of generality (using (a)), let $\psi(i,i) = (1,1)$ and $\psi(j,j) = (2,2)$. We need to show that $\psi(i,j) \in \{(1,2),(2,1)\}$. Assuming this does not hold, we consider the following 2×2 principal submatrix of Y:

$$\begin{bmatrix} y_{ii} & y_{ij} \\ y_{ij} & y_{jj} \end{bmatrix} = \begin{bmatrix} x_{\psi(ii)} & x_{\psi(ij)} \\ x_{\psi(ij)} & x_{\psi(jj)} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{\psi(ij)} \\ x_{\psi(ij)} & x_{22} \end{bmatrix}. \tag{6.4}$$

Case 2.1: Suppose $\psi(i,j) = (1,1)$. Then, taking $u = [2,1,0,\ldots,0]^T$, $X = uu^T$, we see that the principal submatrix (6.4) of Y violates (2.3). Hence this case cannot arise.
Case 2.2: Suppose $\psi(i,j) = (2,2)$. Then, taking $u = [1,2,0,\ldots,0]$, $X = uu^T$, we see that the principal submatrix (6.4) of Y violates (2.3). Hence this case cannot arise.
Case 2.3: Suppose $\psi(i,j) \notin \{(1,1),(2,2),(1,2),(2,1)\}$. Then, taking $X = uu^T$, where $u = [1,1,2,2,\ldots,2]^T$, we see that even in this case, the principal submatrix (6.4) of Y violates (2.3). We conclude that $\psi(i,j) \in \{(1,2),(2,1)\}$. Thus, we have (b).
Having proved Items (a) and (b), we now define the $m \times n$ 0/1-matrix $Q = [q_{ij}]$ as follows: For any pair (i,k) with $i \in \{1,2,\ldots,m\}$ and $k \in \{1,2,\ldots,n\}$,

$$q_{ik} = 1 \quad \text{when} \quad \psi(i,i) = (k,k) \quad \text{and} \quad q_{ik} = 0 \quad \text{otherwise.}$$

Then each row of Q contains exactly one 1 and $\Psi(X) = QXQ^T$ for all $X \in \mathcal{S}^n$. ∎

We now come to the description of Schur homomorphisms of $\mathcal{K}_n$.

Theorem 6.3. *Let Q be an $n \times n$ 0/1 matrix with each row containing at most one 1 and P be an $n \times n$ permutation matrix. Given a (specific) pinching operation,*

$$f(X) := P(QXQ^T)_{pinch}P^T \quad (X \in \mathcal{S}^n)$$

defines a Schur homomorphism of $\mathcal{K}_n$. Conversely, every Schur homomorphism of $\mathcal{K}_n$ arises this way.

Proof. We have shown that $f(X) := P(QXQ^T)_{pinch}P^T \quad (X \in \mathcal{S}^n)$ defines a Schur homomorphism. We prove the converse. Suppose f is a nonzero Schur homomorphism of $\mathcal{K}_n$. We extend f to F on $\mathcal{S}^n$ via (6.1). By Theorem 6.1, there exists a nonzero 0/1 symmetric matrix Z and a symmetric mapping ϕ such that

$$F(X) = Z \circ \Phi(X) \quad \forall\ X \in \mathcal{S}^n,$$

where Φ is defined by (6.2). Since $\Phi(E) = E$ (for any mapping ϕ), we see that $Z = Z \circ E = Z \circ \Phi(E) = F(E) = f(E) \in \mathcal{K}_n$. Thus, the 0/1 symmetric matrix Z is also positive semidefinite. Now by Proposition 2 in [6], we see that up to a permutation, Z is of the form

$$Z = \begin{bmatrix} Z_1 & 0 & 0 & \cdots & 0 \\ 0 & Z_2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & Z_k & 0 \\ 0 & \cdots 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.5}$$

where $k \leq n$ and each Z_i is a matrix of ones. We assume without loss of generality that Z is of the form (6.5) (this induces our permutation P) and $F(X) = Z \circ \Phi(X)$ for all X. For each $i = 1, 2, \ldots, k$, let α_i be a subset of $\{1, 2, \ldots, n\}$ such that $Z_{\alpha_i\alpha_i} = Z_i$. We let $m_i := |\alpha_i|$ (the number of elements in α_i). These sets are disjoint. Now

$$(F(X))_{\alpha_i\alpha_i} = Z_i \circ (\Phi(X))_{\alpha_i\alpha_i} = (\Phi(X))_{\alpha_i\alpha_i}.$$

We know that for each $X \in \mathcal{K}_n$, $F(X) = f(X) \in \mathcal{K}_n$. Thus, for each $i = 1, 2, \ldots, k$, $(\Phi(X))_{\alpha_i\alpha_i} \in \mathcal{K}_{m_i}$. Now we define $\psi_i : \alpha_i \times \alpha_i \to \Delta \times \Delta$ by

$$\psi_i(k, l) := \phi(k, l).$$

Then, for the corresponding mapping Ψ_i we have $\Psi_i(\mathcal{K}_n) \subseteq \mathcal{K}_{m_i}$. By Lemma 6.2, for each $i = 1, 2, \ldots, k$, there exists an $m_i \times n$ 0/1-matrix Q_i (with exactly one 1 in each row) such that $\Psi_i(X) = Q_i X Q_i^T$. Now, let Q_{k+1} be the zero matrix and define the $n \times n$ 0/1-matrix Q with blocks $Q_1, Q_2, \ldots, Q_{k+1}$. A computation shows that QXQ^T has diagonal blocks $Q_i X Q_i^T$, $i = 1, 2, \ldots, k+1$. By setting the off-diagonal blocks to zero (pinching), We see that $F(X) = (QXQ^T)_{pinch}$. By taking into consideration the permutation used to rearrange rows and columns of Z, we finally arrive at $f(X) = P(QXQ^T)_{pinch}P^T$. ■

The following result characterizes Schur automorphisms of $\mathcal{K}_n$.

Corollary 6.4. *Let $f : \mathcal{K}_n \to \mathcal{K}_n$ be a Schur homomorphism. Then the following are equivalent:*

(i) f is one-to-one (or onto) on $\mathcal{K}_n$.
(ii) There exists a permutation matrix P such that $f(X) = PXP^T$ for all $X \in \mathcal{K}_n$.
(iii) f is a Schur automorphism of $\mathcal{K}_n$.

Proof. Let $f : \mathcal{K}_n \to \mathcal{K}_n$ be a Schur homomorphism. Then, Theorem 6.1 ensures that its extension F to $\mathcal{S}^n$ can be written in the form $F(X) = Z \circ \Phi(X)$ for all $X \in \mathcal{S}^n$.
$(i) \Rightarrow (ii)$: Suppose that f is one-to-one. Then, F is also one-to-one. (When f is onto, that is, $f(\mathcal{K}_n) = \mathcal{K}_n$, its extension F satisfies $F(\mathcal{K}_n) = \mathcal{K}_n$. Since F is linear and its range contains an open set, namely, the interior of $\mathcal{K}_n$, F is also onto and hence one-to-one.) We now show that $Z = E$ (the matrix of all ones). Suppose that Z has a zero entry, say $z_{ij} = 0$. Then, for any X, $\langle F(X), E_{ij}\rangle = 0$. Thus, $\langle X, F^T(E_{ij})\rangle = 0$ for all $X \in \mathcal{S}^n$. This implies that $F^T(E_{ij}) = 0$. As F is one-to-one (hence invertible), F^T is also one-to-one. Thus, $E_{ij} = 0$, which is a contradiction. Hence, Z has no zero entries. As Z is a 0/1-matrix, we have $Z = E$. Now, this implies that $F(X) = E \circ \Phi(X) = \Phi(X)$ so that $\Phi(= F)$ is a one-to-one Schur homomorphism on $\mathcal{S}^n$ such that $\Phi(\mathcal{K}_n) \subseteq \mathcal{K}_n$. By Lemma 6.2, there exists a (square) 0/1-matrix Q with each row containing exactly one 1 such that $F(X) = QXQ^T$ for all $X \in \mathcal{S}^n$. As F is one-to-one, it is necessarily invertible. This implies (for example, by looking at $F(X) = I$) that Q is invertible. This means that Q cannot have a pair of identical

rows. Thus, Q has exactly one 1 in each column/row. Hence, Q is a permutation matrix.
The implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ are easy to verify or obvious. ■

Concluding Remarks. Schur homomorphisms of $\mathcal{K}_n$ are completely characterized, but the problem of describing all closed Schur ideals of $\mathcal{K}_n$ remains open. In particular, it is not known if every closed Schur ideal of $\mathcal{K}_n$ is finitely generated.

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