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On the game-theoretic value of a linear transformation relative to a self-dual cone

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A B S T R A C T

This paper is concerned with a generalization of the concept of value of a (zero-sum) matrix game. Given a finite dimensional real inner product space V with a self-dual cone K, an element e in the interior of K, and a linear transformation L, we define the value of L by

$$v(L) := \max_{x\in\Delta}\min_{y\in\Delta}\langle L(x), y\rangle = \min_{y\in\Delta}\max_{x\in\Delta}\langle L(x), y\rangle,$$

where $\Delta = \{x \in K : \langle x, e\rangle = 1\}$. This reduces to the classical value of a square matrix when $V = R^n$, $K = R^n_+$, and e is the vector of ones. In this paper, we extend some classical results of Kaplansky and Raghavan to this general setting. In addition, for a **Z**-transformation (which is a generalization of a **Z**-matrix), we relate the value with various properties such as the positive stable property, the **S**-property, etc. We apply these results to find the values of the Lyapunov transformation L_A and the Stein transformation S_A on the cone of $n \times n$ real symmetric positive semidefinite matrices.

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1. Introduction

This paper is concerned with a generalization of the concept of value of a (zero-sum) matrix game. To explain, we consider an $n \times n$ real matrix A and the strategy set $X := \{x \in R^n_+ : \sum_1^n x_i = 1\}$, where R^n_+ denotes the nonnegative orthant in R^n. Then the value of A is given by

$$v(A) := \max_{x \in X} \min_{y \in X} \langle Ax, y\rangle = \min_{y \in X} \max_{x \in X} \langle Ax, y\rangle,$$

where $\langle Ax, y\rangle$ denotes the (usual) inner product between vectors Ax and y. Corresponding to this, there exist *optimal* strategies $\bar{x}, \bar{y} \in X$ such that

$$\langle Ax, \bar{y}\rangle \leq v(A) = \langle A\bar{x}, \bar{y}\rangle \leq \langle A\bar{x}, y\rangle \quad \forall x, y \in X.$$

The concept of value of a matrix and its applications are classical and have been well studied and documented in the game theory literature; see, for example, [12,13]. Our motivation for the generalization comes from results of Kaplansky and Raghavan. In [11], Kaplansky defines a completely mixed (matrix) game as one in which $\bar{x} > 0$ and $\bar{y} > 0$ for every pair of optimal strategies $(\bar{x}, \bar{y})$. For such a game, Kaplansky proves the uniqueness of the optimal strategy pair. In [14], Raghavan shows that for a **Z**-matrix (which is a square matrix whose off-diagonal entries are all non-positive) the game is completely mixed when the value is positive, and relates the property of value being positive to a number of equivalent properties of the matrix such as the positive stable property, the **P**-property, etc. His result, in particular, says that for a **Z**-matrix A, the value is positive if and only if there exists an $\bar{x} \in R^n$ such that

$$\bar{x} > 0 \quad \text{and} \quad A\bar{x} > 0.$$

Inequalities of the above type also appear in the study of linear continuous and discrete dynamical systems: Given an $n \times n$ real matrix A, the continuous dynamical system $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable on R^n (which means that any trajectory starting from an arbitrary point in R^n converges to the origin) if and only if there exists a real symmetric matrix $\overline{X}$ such that

$$\overline{X} > 0 \quad \text{and} \quad L_A(\overline{X}) > 0,$$

where $\overline{X} > 0$ means that $\overline{X}$ is positive definite, etc., and L_A denotes the so-called *Lyapunov transformation* defined on the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices:

$$L_A(X) := AX + XA^T \quad \big(X \in \mathcal{S}^n\big).$$

Similarly, the discrete dynamical system $x(k+1) = Ax(k)$, $k = 0, 1, \ldots,$ is asymptotically stable on R^n if and only if there exists a real symmetric matrix $\overline{X}$ such that

$$\overline{X} > 0 \quad \text{and} \quad S_A(\overline{X}) > 0,$$

where S_A denotes the so-called *Stein transformation* on $\mathcal{S}^n$:

$$S_A(X) := X - AXA^T \quad (X \in \mathcal{S}^n).$$

Motivated by the similarity between these inequalities/results, we ask if the concept of value and the related results could be extended to linear transformations such as L_A and S_A on $\mathcal{S}^n$ and, in particular, get the above dynamical system results from the value results. In this paper, we achieve this and much more: We extend the concept of value to a linear transformation relative to a self-dual cone in a finite dimensional real inner product space (of which R^n_+ and $\mathcal{S}^n_+$ are particular instances) and see its relevance in the study of so-called **Z**-transformations. To elaborate, consider a finite dimensional real inner product space $(V, \langle\cdot,\cdot\rangle)$ and a self-dual cone K in V. We fix an element e in the interior of K and let

$$\Delta := \{x \in K : \langle x, e\rangle = 1\}, \tag{1}$$

the elements of which will be called 'strategies'. Given a linear transformation L from V to V, the zero-sum game is played by two players I and II in the following way: If player I chooses strategy $x \in \Delta$ and player II chooses strategy $y \in \Delta$, then the pay-off for player I is $\langle L(x), y\rangle$ and the pay-off for player II is $-\langle L(x), y\rangle$. Since Δ is a compact convex set and L is linear, by the min–max theorem of von Neumann (see [12, Theorems 1.5.1 and 1.3.1]), there exist *optimal strategies* $\bar{x}$ for player I and $\bar{y}$ for player II such that

$$\langle L(x), \bar{y}\rangle \le \langle L(\bar{x}), \bar{y}\rangle \le \langle L(\bar{x}), y\rangle \quad \forall x, y \in \Delta. \tag{2}$$

This means that players I and II do not gain by unilaterally changing their strategies from the optimal strategies $\bar{x}$ and $\bar{y}$. We will call the number

$$v(L) := \langle L(\bar{x}), \bar{y}\rangle$$

the *value of the game*, or simply, *the value of* L. The pair $(\bar{x}, \bar{y})$ will be called an *optimal strategy pair* for L. We note that $v(L)$ is also given by [12, Theorems 1.5.1 and 1.3.1]

$$v(L) = \max_{x\in\Delta} \min_{y\in\Delta} \langle L(x), y\rangle = \min_{y\in\Delta} \max_{x\in\Delta} \langle L(x), y\rangle. \tag{3}$$

Following Kaplansky [11], we say that a linear transformation L on V (or the corresponding game) is *completely mixed* if for every optimal strategy pair $(\bar{x}, \bar{y})$ of L, $\bar{x}$ and $\bar{y}$ belong to the interior of K. As in the classical case, we show the uniqueness of the optimal strategy pair when the game is completely mixed (see Theorem 5). By extending the concept of a **Z**-matrix, but specializing a concept that is defined on any proper cone [9], we say that a linear transformation L is a **Z**-*transformation* on K if the following implication holds:

$$x \in K,\ y \in K, \quad \langle x, y\rangle = 0 \quad \Rightarrow \quad \big\langle L(x), y\big\rangle \leq 0.$$

We show that for a **Z**-transformation, the game is completely mixed when the value is positive (see Theorem 6). Easy examples (even in the classical case) show that the result fails when the value is negative. However, in this paper, we identify the following two important types of **Z**-transformations for which the game is completely mixed even when the value is negative: *Lyapunov-like* transformations defined by the condition

$$x \in K,\ y \in K, \quad \langle x, y\rangle = 0 \quad \Rightarrow \quad \big\langle L(x), y\big\rangle = 0$$

and *Stein-like* transformations which are of the form

$$L = I - \Lambda \quad \text{for some } \Lambda \in \overline{Aut(K)},$$

where I denotes the identity transformation, $Aut(K)$ denotes the set of all automorphisms of K (which are invertible linear transformations on V mapping K onto itself), and $\overline{Aut(K)}$ denotes the topological closure of $Aut(K)$.

While all our main results are stated for self-dual cones, illuminating examples and results are obtained for symmetric cones in Euclidean Jordan algebras. A Euclidean Jordan algebra $(V, \langle\cdot,\cdot\rangle, \circ)$ is a finite dimensional real inner product space $(V, \langle\cdot,\cdot\rangle)$ which admits a Jordan product '$\circ$' that is compatible with the inner product, see [4,8] for details. In this algebra, we let the self-dual cone K be the cone of squares $\{x \circ x : x \in V\}$. For e, we choose the unit element of the algebra. Under the assumption that $\langle x, y\rangle = tr(x \circ y)$, where the trace of an object is the sum of all its eigenvalues, we see that $\langle x, e\rangle = tr(x)$ and so $\Delta = \{x \in K : tr(x) = 1\}$. Under these canonical settings, using (3), we define the value of a linear transformation on V. Two important Euclidean Jordan algebras and their symmetric cones are given below:

◇ We get the classical concepts and results when

$$V = R^n, \qquad \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, \qquad x \circ y = x * y, \qquad K = R^n_+, \quad \text{and}$$
$$tr(x) = \sum_{i=1}^{n} x_i,$$

where '$*$' denotes the componentwise product. In this algebra, e is the vector of ones. (Note that in the classical situation, in the definition of value of a matrix A, the expression $x^T Ay$, which is $\langle x, Ay\rangle$, is used instead of $\langle Ax, y\rangle$. Our choice of the expression $\langle L(x), y\rangle$ leads to the preferred **S**-property of L instead of that of L^T, see

Theorem 2 and Proposition 1 in Section 3 below.) In this setting, **Z**-transformations reduce to **Z**-matrices and Lyapunov-like transformations become diagonal matrices.

◇ Consider the algebra of all $n \times n$ real symmetric matrices:

$$V = \mathcal{S}^n, \qquad \langle X, Y\rangle := tr(XY), \qquad X \circ Y := \frac{1}{2}(XY + YX), \qquad K = \mathcal{S}^n_+,$$
$$tr(X) = \sum_{i=1}^{n} \lambda_i(X),$$

where the trace of a real/complex matrix is the sum of its diagonal elements (or the sum of its eigenvalues). In this algebra, e is the identity matrix and K is the cone of all positive semidefinite matrices in $\mathcal{S}^n$. Also, Lyapunov-like and Stein-like transformations reduce, respectively, to *Lyapunov and Stein* transformations:

$$L_A(X) := AX + XA^T \quad \text{and} \quad S_A(X) := X - AXA^T, \tag{4}$$

where A is an $n \times n$ real matrix.

Here is a summary of our main results:

- When the game corresponding to a linear transformation is completely mixed, (a) the game has a unique optimal strategy pair and (b) the values of the transformation and its transpose are equal.
- For a **Z**-transformation, the value is positive if and only if it is positive stable (that is, all its eigenvalues have positive real parts). When the value is positive, (i) the game is completely mixed and (ii) in the case of a product cone, the value of the transformation is bounded above by the value of any principal subtransformation or the Schur complement of any principal subtransformation.
- For a Lyapunov-like transformation, the value is positive (negative) if and only if it is positive stable (respectively, negative stable); when the value is nonzero, the game is completely mixed. These results are valid, in particular, for a Lyapunov transformation L_A on $\mathcal{S}^n$.
- For a Stein-like transformation $L = I - \Lambda$, the value is positive (negative) if and only if Λ is Schur stable (respectively, inverse Schur stable); when the value is nonzero, the game is completely mixed. These results are valid, in particular, for a Stein transformation S_A on $\mathcal{S}^n$.

The paper is organized as follows. In Section 2, we recall basic concepts, definitions, and preliminary results. We will also recall some special linear transformations and state many equivalent properties of **Z**-transformations. Section 3 deals with some basic results on the value. In Section 4, we study completely mixed games. Section 5 deals with the value of a **Z**-transformation. In this section we show that for a **Z**-transformation, the value being positive is equivalent to positive stability. We also establish value results for Lyapunov-like and Stein-like transformations. Value inequalities on a product space (in terms of principal subtransformations and Schur complements) are covered in Section 6. In Section 7, we compute the values of L_A and S_A. In the concluding remarks, we indicate possible topics for further study and state a conjecture on **P**-transformations.

2. Preliminaries

2.1. Self-dual cones

In this paper, $(V, \langle\cdot,\cdot\rangle)$ denotes a finite dimensional real inner product space. For a set S in V, we denote the interior, closure, and boundary by S°, $\overline{S}$, and ∂S, respectively. Let K be a *self-dual cone* in V so that

$$K^* := \big\{x \in V : \langle x, y\rangle \geq 0 \ \forall y \in K\big\} = K.$$

We note the following consequences of the equality $K^* = K$ (see [1]):

(i) K is a closed convex cone.
(ii) $K \cap -K = \{0\}$ and $K - K = V$.
(iii) K° is nonempty.
(iv) $x \in K \Leftrightarrow \langle x, y\rangle \geq 0 \ \forall y \in K$.
(v) $0 \neq x \in K$, $y \in K^\circ \Rightarrow \langle x, y\rangle > 0$.

Henceforth, in V, we fix a self-dual cone K and an element $e \in K^\circ$.

In V, we use the notation $x \perp y$ to mean $\langle x, y\rangle = 0$ and let $e^\perp := \{x \in V : x \perp e\}$. We will use the notation

$$x \geq y \quad (\text{or } y \leq x) \quad \text{when } x - y \in K \quad \text{and} \quad x > y \quad \text{when } x - y \in K^\circ.$$

In V, we define the 'strategy set' Δ by (1). It is easy to see that Δ is a compact convex set.

We denote the space of all (continuous) linear transformations on V by $\mathcal{L}(V)$. Then the automorphism group on K is

$$Aut(K) := \big\{L \in \mathcal{L}(V) : L(K) = K\big\}.$$

Note that each $L \in Aut(K)$ is invertible, as K has nonempty interior. By $\overline{Aut(K)}$, we denote the closure of $Aut(K)$ in $\mathcal{L}(V)$.

For a linear transformation L on V, we denote the transpose by L^T. Recalling the equality $\langle L^T(x), y\rangle = \langle x, L(y)\rangle$ for all $x, y \in V$, we note (by the self-duality of K) that

$$L(K) \subseteq K \quad \Rightarrow \quad L^T(K) \subseteq K \quad \text{and} \quad L \in Aut(K) \quad \Leftrightarrow \quad L^T \in Aut(K).$$

2.2. Euclidean Jordan algebras

While [4] is our primary source on Euclidean Jordan algebras, a short summary can be found in [8]. In a Euclidean Jordan algebra, the cone of squares (called a symmetric cone) is a self-dual homogeneous cone. Examples of Euclidean Jordan algebras include R^n, $\mathcal{S}^n$ (see Section 1), the Jordan spin algebra $\mathcal{L}^n$ (whose symmetric cone is called the second order cone or Lorentz cone), the algebra $\mathcal{H}^n$ of $n \times n$ complex Hermitian matrices, the algebra $\mathcal{Q}^n$ of $n \times n$ quaternion Hermitian matrices, and the algebra $\mathcal{O}^3$ of 3×3 octonion Hermitian matrices. It is known that any nonzero Euclidean Jordan algebra is a product of those given above [4].

2.3. Some special linear transformations

Many of the linear transformations we study here have their roots in either the dynamical systems theory or complementarity theory [2]. Below, we make a list of such transformations and provide some examples. Recall that V is a finite dimensional real inner product space and K is a fixed self-dual cone in V.

For a linear transformation L on V, we say that

(1) L is an **S**-*transformation* on K (and write $L \in \mathbf{S}(K)$) if there exists a $d > 0$ such that $L(d) > 0$.

(2) L is a **Z**-*transformation* on K (and write $L \in \mathbf{Z}(K)$) if the following implication holds:

$$x \geq 0,\ y \geq 0, \quad \langle x, y \rangle = 0 \quad \Rightarrow \quad \big\langle L(x), y \big\rangle \leq 0.$$

(3) L is *Lyapunov-like* on K if L and $-L$ are **Z**-transformations on K.

(4) L is *Stein-like* on K if it is of the form $L = I - \Lambda$ for some $\Lambda \in \overline{Aut(K)}$.

(5) L is a **P**-*transformation* on a Euclidean Jordan algebra if the following implication holds:

$$\big[x \text{ and } L(x) \text{ operator commute, } x \circ L(x) \leq 0\big] \quad \Rightarrow \quad x = 0.$$

(6) $L \in \Pi(K)$ if $L(K) \subseteq K$.

(7) L is *positive (negative) stable* if the real part of any eigenvalue of L is positive (respectively, negative).

(8) L is *Schur stable* if all eigenvalues of L lie in the open unit disk of R^2; L is *inverse Schur stable* if L^{-1} exists and is Schur stable.

Note: When the context is clear, we suppress mentioning the cone K in various definitions/properties or even write V in place of K; for example, we may write '**S**-transformation on V' or just '**S**-transformation' in place of '**S**-transformation on K'.

Note that the above definitions/concepts also apply for matrices on R^n.

We first state some equivalent properties of **Z**-transformations and then provide some examples.

Theorem 1. *(See [9, Theorems 6 and 7].) Suppose L is a $\mathbf{Z}$-transformation on K. Then the following are equivalent:*

(1) L is an $\mathbf{S}$-transformation on K.
(2) L is invertible with $L^{-1}(K) \subseteq K$, or equivalently, $L^{-1}(K^\circ) \subseteq K^\circ$.
(3) L is positive stable.
(5) All real eigenvalues of L are positive.
(6) L^T is an $\mathbf{S}$-transformation on K.
(7) L^T is invertible with $(L^T)^{-1}(K) \subseteq K$.
(8) For every $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, K, q)$ has a solution, that is, there exists x such that

$$x \geq 0, \quad L(x) + q \geq 0, \quad \text{and} \quad \langle x, L(x) + q \rangle = 0.$$

Equivalent properties of Lyapunov-like transformations: For a linear transformation L on V, the following are equivalent [10, Theorem 4]:

(i) L is Lyapunov-like on K.
(ii) $e^{tL}(K) \subseteq K$ for all $t \in R$.
(iii) $L \in \mathit{Lie}(\mathit{Aut}(K))$, that is, L is an element of the Lie algebra of the automorphism group of K.

Equivalent properties of L_A [6]: Let $V = \mathcal{S}^n$. For any real $n \times n$ matrix A, consider the Lyapunov transformation L_A defined in (4). It is known (see [3]) that on $\mathcal{S}^n$, a linear transformation is Lyapunov-like if and only if it is of the form L_A for some A. The following are equivalent for L_A:

(i) The dynamical system $\frac{dx}{dt} + Ax = 0$ is asymptotically stable in R^n;
(ii) There exists a positive definite matrix D in $\mathcal{S}^n$ such that $AD + DA^T$ is positive definite;
(iii) L_A is an **S**-transformation on $\mathcal{S}^n_+$;
(iv) L_A is positive stable;
(v) A is positive stable;
(vi) L_A is a **P**-transformation.

Equivalent properties of S_A [5]: Let $V = \mathcal{S}^n$. For any real $n \times n$ matrix A, consider the Stein transformation S_A defined in (4). Then S_A is a **Z**-transformation on $\mathcal{S}^n_+$. Moreover, since every automorphism of $\mathcal{S}^n_+$ is given by $\Lambda(X) = BXB^T$ ($X \in \mathcal{S}^n$) for some real $n \times n$ invertible matrix B [15], it follows that on $\mathcal{S}^n$, a linear transformation is Stein-like if and only if it is of the form S_A for some real $n \times n$ matrix A. For S_A, the following are equivalent:

(i) The discrete dynamical system $x(k+1) = Ax(k)$, $k = 0, 1, 2, \ldots$, is asymptotically stable in R^n;
(ii) There is a positive definite matrix D in $\mathcal{S}^n$ such that $D - ADA^T$ is positive definite;
(iii) S_A is an **S**-transformation;
(iv) S_A is positive stable;
(v) A is Schur stable;
(vi) S_A is a **P**-transformation.

Here are some more examples of **Z**-transformations.

- Every Lyapunov-like transformation is a **Z**-transformation.
- For any $r \in R$ and $\Lambda \in \Pi(K)$, $L = rI - \Lambda$ is a **Z**-transformation. In particular, every Stein-like transformation is a **Z**-transformation.
- If $c, d > 0$, then (it is easy to see that) $rI - cd^T$ and $(I + cd^T)^{-1}$ are **Z**-transformations on K, where $r \in R$ and $(cd^T)(x) := \langle d, x \rangle\, c$.
- Let V be a Euclidean Jordan algebra with corresponding symmetric cone K. For $a \in V$, let L_a be defined by $L_a(x) = a \circ x$; let D be a derivation on V, that is, $D(x \circ y) = D(x) \circ y + x \circ D(y)$ for all $x, y \in V$. Then $L := L_a + D$ is Lyapunov-like on K. In fact (see [17]), every Lyapunov-like transformation on K arises this way.
- Let V be a Euclidean Jordan algebra with corresponding symmetric cone K. For $a \in V$, let P_a be defined by $P_a(x) = 2a \circ (a \circ x) - a^2 \circ x$; let Γ be an algebra automorphism on V, that is, $\Gamma(x \circ y) = \Gamma(x) \circ \Gamma(y)$ for all $x, y \in V$. Then, $L := I - P_a\Gamma$ is a Stein-like transformation on K. In fact, on a simple Euclidean Jordan algebra (see [4, Theorem III.5.1]), every Stein-like transformation on K arises this way.

3. The value of a linear transformation; some general results

Let L be a linear transformation on V. Corresponding to this (and the fixed self-dual cone K and $e \in K^\circ$), we define the value $v(L)$ by (3) and consider an optimal strategy pair $(\bar{x}, \bar{y})$ satisfying (2).

Theorem 2. *The saddle point inequalities (2) imply*

$$L^T(\bar{y}) \leq v\, e \leq L(\bar{x}), \tag{5}$$

where v is the value of L. Conversely, if (5) holds for some $v \in R$ and $\bar{x}, \bar{y} \in \Delta$, then v is the value of L and $(\bar{x}, \bar{y})$ is an optimal strategy pair for L.

Proof. From the definition of value, we have the saddle-point inequalities (2). We see that $\langle L(\bar{x})-v(L)\, e, y\rangle \geq 0$ holds for all $y \in \Delta$ and (by scaling) for all $y \in K$. As K is self-dual, $L(\bar{x}) - v(L)\, e \geq 0$, from which we get $v(L)\, e \leq L(\bar{x})$. Similarly, $L^T(\bar{y}) \leq v(L)\, e$. Now suppose v is a real number that satisfies the inequalities (5). Then $\langle x, L^T(\bar{y})\rangle \leq v\langle x, e\rangle$ and $v\langle e, y\rangle \leq \langle L(\bar{x}), y\rangle$ for all $x, y \in \Delta$. These yield

$$\big\langle L(x), \bar{y}\big\rangle \leq v \leq \big\langle L(\bar{x}), y\big\rangle \quad \forall x, y \in \Delta.$$

Upon putting $x = \bar{x}$ and $y = \bar{y}$, we see that $v = v(L)$ and $(\bar{x}, \bar{y})$ is an optimal strategy pair for L. □

Remarks. Based on the above theorem, the following are easy to prove:

(a) $v(\lambda L) = \lambda\, v(L)$ for $\lambda \geq 0$.
(b) If $(\bar{x}, \bar{y})$ is an optimal strategy pair for L, then $(\bar{y}, \bar{x})$ is an optimal strategy pair for $-L^T$. Moreover, $v(-L^T) = -v(L)$.
(c) If $(\bar{x}, \bar{y})$ is an optimal strategy pair for L, then for any real number λ, $(\bar{x}, \bar{y})$ is an optimal strategy pair for $L+\lambda ee^T$, where $ee^T(x) := \langle x, e\rangle\, e$. Moreover, $v(L+\lambda\, ee^T) = v(L) + \lambda$.
(d) Suppose $A \in Aut(K)$ and let $(\bar{x}, \bar{y})$ be an optimal strategy pair for L relative to the chosen interior point e. Define $\widetilde{L} = ALA^T$, $\widetilde{e} = Ae$, $\widetilde{x} = (A^{-1})^T\bar{x}$, and $\widetilde{y} = (A^{-1})^T\bar{y}$. Then $(\widetilde{x}, \widetilde{y}\,)$ is an optimal strategy pair for $\widetilde{L}$ relative to the interior point $\widetilde{e}$. Moreover, the value of $\widetilde{L}$ relative to $\widetilde{e}$ is the same as the value of L relative to e.

Theorem 3. *If $(\bar{x}, \bar{y})$ is an optimal strategy pair for L and v denotes the value of L, then*

$$0 \leq \bar{x} \perp v\, e - L^T(\bar{y}) \geq 0 \quad \text{and} \quad 0 \leq \bar{y} \perp L(\bar{x}) - v\, e \geq 0.$$

In addition,

$$L(\bar{x}) = v\, e \quad \text{when } \bar{y} > 0 \quad \text{and} \quad L^T(\bar{y}) = v\, e \quad \text{when } \bar{x} > 0. \tag{6}$$

When V is a Euclidean Jordan algebra, $\bar{x}$ and $L^T(\bar{y})$ operator commute, and $\bar{y}$ and $L(\bar{x})$ operator commute.

Proof. The nonnegativity and orthogonality relations follow easily. Now suppose $\bar{y} > 0$, that is, $\bar{y} \in K^\circ$. As $L(\bar{x}) - v\, e \geq 0$, it follows from the above orthogonality relations that $L(\bar{x}) - v\, e = 0$, i.e., $L(\bar{x}) = v\, e$. Similarly, when $\bar{x} > 0$, we have $L^T(\bar{y}) = v\, e$. Finally, when V is a Euclidean Jordan algebra, the operator commutativity relations follow from [8, Proposition 6], where it is shown that when $0 \leq x \perp y \geq 0$, the elements x and y operator commute. (Recall that now, e is the unit element in V.) □

Remarks. In the classical case (for the algebra $V = R^n$), optimal strategies (operator) commute. This may fail in the general case, see the numerical example given in Section 7.

Proposition 1. *The following statements hold:*

(1) If L is copositive, that is, $\langle L(x), x\rangle \geq 0$ for all $x \geq 0$, then $v(L) \geq 0$.
(2) If $L \in \Pi(K)$, then $v(L) \geq 0$.
(3) If L is monotone, that is, $\langle L(x), x\rangle \geq 0$ for all $x \in V$, then $v(L) \geq 0$.
(4) If L is skew-symmetric, that is, $\langle L(x), x\rangle = 0$ for all x, then $v(L) = 0$.
(5) $v(L) > 0$ if and only if L is an $\mathbf{S}$-transformation on K.
(6) If V is a Euclidean Jordan algebra and L is a $\mathbf{P}$-transformation on V, then $v(L) > 0$.
(7) The value $v(L)$, as a function of L, is continuous.
(8) Suppose L is invertible and $L^{-1}(K) \subseteq K$. Then

$$v(L) = \frac{1}{\langle L^{-1}(e), e\rangle} = v\left(L^T\right)$$

and $(\bar{x}, \bar{y})$ is an optimal strategy pair, where $\bar{x} = v(L)L^{-1}(e)$ and $\bar{y} = v(L)(L^T)^{-1}(e)$.

Proof. We use Theorem 2.

(1) From $L^T(\bar{y}) \leq v(L)\, e$, we get $0 \leq \langle L(\bar{y}), \bar{y}\rangle \leq v(L)\langle e, \bar{y}\rangle = v(L)$.

Items (2)–(4) follow from item (1).

(5) Suppose $v(L) > 0$. Then, using Theorem 2, $L(\bar{x}) \geq v(L)e > 0$. As $\bar{x} \geq 0$, we can perturb $\bar{x}$ to get an element $d > 0$ such that $L(d) > 0$. Thus L is an $\mathbf{S}$-transformation. Conversely, suppose L is an $\mathbf{S}$-transformation so that there is a $d > 0$ with $L(d) > 0$. From $L^T(\bar{y}) \leq v(L)\, e$, we see that $0 < \langle \bar{y}, L(d)\rangle = \langle L^T(\bar{y}), d\rangle \leq v(L)\langle e, d\rangle$. As $\langle e, d\rangle > 0$, we get $v(L) > 0$.

(6) Suppose that V is a Euclidean Jordan algebra and L is a $\mathbf{P}$-transformation on V. Then it follows from [8, Theorem 12], that for every $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, K, q)$ (as defined in Theorem 1) has a solution; in particular, $\mathrm{LCP}(L, K, -e)$ has a solution so that for some $x \geq 0$, $L(x) - e \geq 0$. Perturbing x we get a $d > 0$ such that $L(d) > 0$. Thus, L is an $\mathbf{S}$-transformation. By item (5), $v(L) > 0$.

The statement (7) follows easily from the continuity of L and compactness of Δ.

(8) As K is self-dual, the inclusion $L^{-1}(K) \subseteq K$ implies $(L^T)^{-1}(K) \subseteq K$. We also have $L^{-1}(K^\circ) \subseteq K^\circ$ and $(L^T)^{-1}(K^\circ) \subseteq K^\circ$. Since $e \in K^\circ$, we have $u := L^{-1} e > 0$ and $w := (L^T)^{-1} e > 0$. As $\langle u, e\rangle = \langle w, e\rangle$, putting $\alpha := \frac{1}{\langle u,e\rangle} = \frac{1}{\langle w,e\rangle}$, we see that $\bar{x} := \alpha\, u = \frac{1}{\langle u,e\rangle}\, u \in \Delta$ and $\bar{y} = \alpha\, w = \frac{1}{\langle w,e\rangle}\, w \in \Delta$. That these are optimal strategies and α is the value of L follows from Theorem 2. $\square$

4. Completely mixed games

In what follows, we extend some results of Kaplansky [11] by modifying his arguments.

First, we state a simple lemma.

Lemma 1. *Let $0 < \bar{y} \in \Delta$ and $u \in V$.*

(a) If $u \neq \bar{y}$ and $\langle u, e\rangle = 1$, then there exist $t > 0$ and $s < 0$ in R such that $(1+t)\bar{y} - tu \in \partial K$ and $(1+s)\bar{y} - su \in \partial K$.
(b) If $u \neq 0$ and $\langle u, e\rangle = 0$, then there exist $t > 0$ and $s < 0$ in R such that $\bar{y} - tu \in \partial K$ and $\bar{y} - su \in \partial K$.

Proof. Suppose that $u \neq \bar{y}$ and $\langle u, e\rangle = 1$. We consider the ray $(1+t)\bar{y} - tu$ as t varies over $[0, \infty)$. If $(1+t)\bar{y} - tu \in K$ for all $t \in [0, \infty)$, then $\bar{y} - u = \lim_{t\to\infty} \frac{1}{t}((1+t)\bar{y} - tu) \in K$. But then $0 \leq \langle \bar{y} - u, e\rangle = 1 - 1 = 0$ implies that $\bar{y} = u$. As this cannot happen, the ray $(1+t)\bar{y} - tu$, which starts in K° (for $t = 0$) must eventually go out of K. By considering the supremum of all $t > 0$ for which $(1+t)\bar{y} - tu \in K$, we get a $t > 0$ for which $(1+t)\bar{y} - tu \in \partial K$. Similarly, the existence of s is proved by considering the ray $(1+s)\bar{y} - su$ over $(-\infty, 0]$. Statements in (b) are proved in a similar way. □

Theorem 4. *Consider a linear transformation L on V with $v(L) = 0$. Suppose for every optimal strategy pair $(\bar{x}, \bar{y})$ of L, we have $\bar{y} > 0$. Then the following statements hold:*

(i) $Ker(L^T) \cap e^\perp = \{0\}$.
(ii) $\dim(Ker(L^T)) = 1$ and $\dim(Ker(L)) = 1$.
(iii) For every optimal strategy pair $(\bar{x}, \bar{y})$, we have $L(\bar{x}) = 0$ and $L^T(\bar{y}) = 0$.
(iv) There is only one optimal strategy pair $(\bar{x}, \bar{y})$; moreover, $\bar{x} > 0$ and $\bar{y} > 0$.

Proof. Take any optimal strategy pair $(\bar{x}, \bar{y})$ of L. As $v(L) = 0$ and $\bar{y} > 0$, we have $L(\bar{x}) = 0$ from the complementarity relations in Theorem 3. As $\bar{x} \neq 0$, $Ker(L) \neq \{0\}$. Since V is finite dimensional, $Ker(L^T) \neq \{0\}$.

(i) Suppose $L^T(u) = 0$ and $\langle u, e\rangle = 0$. If $u \neq 0$, by the above lemma, we can find a $t \in R$ such that $y := \bar{y} - tu \in \partial K$. Since $L^T(y) = L^T(\bar{y}) \leq 0$ and $\langle y, e\rangle = 1$, we see that $(\bar{x}, y)$ is an optimal strategy pair with $y \not> 0$. Hence $u = 0$ proving (i).

(ii) As $Ker(L^T) \neq \{0\}$ and $Ker(L^T) \cap e^\perp = \{0\}$, we see that $\dim(Ker(L^T)) = 1$. This also yields $\dim(Ker(L)) = 1$.

(iii) We already know that $L^T(\bar{y}) \leq 0 = L(\bar{x})$. We now show that $L^T(\bar{y}) = 0$. Let $L^T(u) = 0$, where $u \neq 0$. As $\langle u, e\rangle \neq 0$ (from (i)), we may assume that $\langle u, e\rangle = 1$. We claim that $u = \bar{y}$ and conclude $L^T(\bar{y}) = 0$. Suppose $u \neq \bar{y}$. Then from the above lemma, there exists $t > 0$ such that $y = (1+t)\bar{y} - tu \in \partial K$. Clearly, for this y, $L^T(y) \leq 0$ and $\langle y, e\rangle = 1$. Thus, $(\bar{x}, y)$ is an optimal strategy pair with $y \not> 0$, contradicting our assumption. Hence, $u = \bar{y}$ and $L^T(\bar{y}) = 0$.

(iv) Suppose $(\bar{z}, \bar{w})$ is another optimal strategy pair. By item (iii), $L(\bar{x}) = L(\bar{z}) = 0$. Since $\dim(Ker(L)) = 1$, $\bar{z}$ must be a multiple of $\bar{x}$. As $\langle \bar{z}, e\rangle = 1 = \langle \bar{x}, e\rangle$, we see that this multiple is one and so $\bar{z} = \bar{x}$. In a similar way, we can show that $\bar{w} = \bar{y}$. Thus, we have proved the uniqueness of the optimal pair.

By assumption, $\bar{y} > 0$. We now show that $\bar{x} > 0$. Suppose, if possible, $\bar{x} \in \partial K$. As K is self-dual, by the supporting hyperplane theorem [2, Theorem 2.7.5], we can find a unit vector $c \in K$ such that $\langle \bar{x}, c\rangle = 0$. With $c_1 = c$, let $\{c_1, c_2, \ldots, c_n\}$ be an orthonormal basis in V, where $n = \dim(V)$. Let $A = [a_{ij}]$ be the matrix representation of L with respect to this basis and $adj(A)$ be the adjoint matrix of A. (Recall that $adj(A)$ is the transpose of the cofactor matrix of A.) From the equation $A\, adj(A) = det(A)\, I$ and the fact that $det(A) = det(L) = 0$, we see that $A\, adj(A) = 0$. This means that $Ap = 0$ for every column p of $adj(A)$. Letting $a = [a_1, a_2, \ldots, a_n]^T$ be the coordinate vector of $\bar{x}$ with respect to the chosen basis, we have $a_1 = \langle \bar{x}, c_1\rangle = \langle \bar{x}, c\rangle = 0$. As $L(\bar{x}) = 0$ implies $Aa = 0$ and $\dim(Ker\, A) = 1$, we see that each column p of $adj(A)$ is a multiple of a. Thus, the first coordinate of any such p is zero, which implies that the first row of $adj(A)$, namely, $[A_{11}, A_{21}, \ldots, A_{n1}]$ is zero. This shows that the matrix obtained by deleting the first row of A^T has rank less than $n-1$. Consequently, this $(n-1) \times n$ matrix will have at least two independent coordinate vectors in its kernel. As one of these vectors can be taken to be the coordinate vector of $\bar{y}$, we let b be the other coordinate vector so that $A^T b$ has all coordinates except the first one zero. (Note that the first coordinate of $A^T b$ is nonzero, else, $A^T b = 0$ would imply that b is a multiple of the coordinate vector of $\bar{y}$ with respect to the chosen basis.) Letting $y = \sum_1^n b_i c_i$, we see that $0 \neq y \neq \bar{y}$ and $\langle L^T(y), c_j\rangle = (A^T b)_j = 0$ for $j = 2, 3, \ldots, n$. Hence, $L^T(y) = \alpha\, c_1$ for some nonzero $\alpha \in R$. Now, by the above lemma, depending on whether $\langle y, e\rangle$ is nonzero (in which case, we may assume $\langle y, e\rangle = 1$) or zero, we form $w = (1+t)\bar{y} - ty$ or $w = \bar{y} - ty$ for an appropriate t (positive or negative to make $t\,\alpha > 0$) with $w \in \partial K$, $\langle w, e\rangle = 1$, and $L^T(w) = -t\alpha c_1 \leq 0$. This means that we have a new optimal strategy pair $(\bar{x}, w)$ for L contradicting the uniqueness of the optimal pair. Thus, $\bar{x} > 0$. This completes the proof. □

Recall that L is said to be *completely mixed* if for every optimal strategy pair $(\bar{x}, \bar{y})$ of L, we have $\bar{x} > 0$ and $\bar{y} > 0$.

Theorem 5. *For a linear transformation L on V, the following are equivalent:*

(a) For every optimal strategy pair $(\bar{x}, \bar{y})$ of L, $\bar{y} > 0$.
(b) L is completely mixed.

Moreover, when L is completely mixed, the following statements hold:

(i) $v(L^T) = v(L)$.
(ii) L^T is also completely mixed.
(iii) $Ker(L) \cap e^{\perp} = \{0\}$.
(iv) $v(L) = 0$ if and only if L is not invertible. When $v(L) = 0$, $\dim Ker(L) = 1$.
(v) $v(L) \neq 0$ if and only if L is invertible. When L is invertible, $v(L) = \frac{1}{\langle L^{-1}(e), e\rangle}$.
(vi) L has a unique optimal strategy pair; If $v(L) \neq 0$, it is given by $(\bar{x}, \bar{y})$, where $\bar{x} = v(L)\, L^{-1}(e)$ and $\bar{y} = v(L)\, (L^T)^{-1}(e)$.

Proof. When $v(L) = 0$, the equivalence of (a) and (b) comes from the previous result. For $v(L) \neq 0$, we work with $\widetilde{L} := L - v(L)ee^T$. Then $v(\widetilde{L}) = 0$ and as observed in remarks following Theorem 2, any optimal strategy pair $(\bar{x}, \bar{y})$ of L is an optimal strategy pair of $\widetilde{L}$ and conversely. Thus, in this case also, (a) and (b) are equivalent. Now assume that L is completely mixed and fix an optimal strategy pair $(\bar{x}, \bar{y})$ of L.

(i) Since $\bar{x} > 0$ and $\bar{y} > 0$, from (6),

$$L(\bar{x}) = v(L)\, e \quad \text{and} \quad L^T(\bar{y}) = v(L)\, e.$$

Rewriting these as $(L^T)^T(\bar{x}) = v(L)\, e$ and $(L^T)(\bar{y}) = v(L)\, e$, and using Theorem 2, we see that $v(L^T) = v(L)$. This proves (i).

(ii) Let $v := v(L) = v(L^T)$. Suppose $(\bar{z}, \bar{w})$ is an optimal pair for L^T so that $L(\bar{w}) \leq v\, e \leq L^T(\bar{z})$. Then, $L^T(\bar{z}) - v\, e \geq 0$, and

$$0 \leq \left\langle L^T(\bar{z}) - v\, e, \bar{x} \right\rangle = \left\langle \bar{z}, L(\bar{x}) \right\rangle - v = \langle \bar{z}, v\, e \rangle - v = 0.$$

(Note that $\bar{x}, \bar{z} \in \Delta$.) Since $\bar{x} > 0$, we must have $L^T(\bar{z}) - v\, e = 0$, that is, $L^T(\bar{z}) = v\, e$. Similarly, $L(\bar{w}) = v\, e$. But this means that $(\bar{w}, \bar{z})$ is an optimal pair for L; Since L is completely mixed, we must have $\bar{w} > 0$ and $\bar{z} > 0$. Thus, L^T is completely mixed.

(iii) Suppose there is a nonzero u such that $L(u) = 0 = \langle e, u \rangle$. By the above lemma, we can find some real number t such that $\bar{u} := \bar{x} - tu \in \partial K$. Since $\langle \bar{x} - tu, e \rangle = 1$ and $L(\bar{u}) = v(L)\, e$, it follows that $(\bar{u}, \bar{y})$ is also an optimal strategy pair, with $\bar{u} \not> 0$. This cannot happen as L is completely mixed. Thus, $Ker(L) \cap e^{\perp} = \{0\}$.

(iv) Suppose $v(L) = 0$. Then from the previous result, L is not invertible and $\dim(Ker(L)) = 1$. Now suppose L is not invertible; let $L(u) = 0$ for some nonzero u. From item (iii), we may assume that $\langle u, e \rangle = 1$. Then

$$L^T(\bar{y}) = v\, e \quad \Rightarrow \quad v = v\, \langle e, u \rangle = \left\langle L^T(\bar{y}), u \right\rangle = \left\langle \bar{y}, L(u) \right\rangle = 0.$$

(v) From (iv) it follows that $v(L) \neq 0$ if and only if L is invertible. Suppose that L is invertible. From $L(\bar{x}) = v\, e$ and $L^T(\bar{y}) = v\, e$, we have

$$\bar{x} = L^{-1}(v\, e) \quad \text{and} \quad \bar{y} = \left(L^T\right)^{-1}(v\, e).$$

Taking the inner product of these expressions with e, we see that $v = \frac{1}{\langle L^{-1}(e), e \rangle} \neq 0$.

(vi) When $v(L) = 0$, the uniqueness of the optimal strategy pair comes from the previous result. When $v(L) \neq 0$, L is invertible and (as in the proof of item (v)), $\bar{x} = v(L)\, L^{-1}(e)$ and $\bar{y} = v(L)\, (L^T)^{-1}(e)$. □

Remark. We recall that $(\bar{x}, \bar{y})$ is an optimal strategy pair for L if and only if $(\bar{y}, \bar{x})$ is an optimal strategy pair for $-L^T$. Thus, in item (a) of the above theorem, one could replace the condition $\bar{y} > 0$ by the condition $\bar{x} > 0$ and get the completely mixed property of L.

5. The value of a Z-transformation

The following extends the results of [11] and [14].

Theorem 6. *Suppose L is a $\mathbf{Z}$-transformation on K. Then $v(L) > 0$ if and only if L is positive stable (equivalently, an $\mathbf{S}$-transformation). When $v(L) > 0$,*

(i) L is completely mixed,
(ii) $v(L) = \frac{1}{\langle L^{-1}(e), e\rangle} = v(L^T)$, and
(iii) $(\bar{x}, \bar{y})$ is the unique optimal strategy pair, where $\bar{x} := v(L)\, L^{-1}(e)$ and $\bar{y} := v(L)\, (L^T)^{-1}(e)$.

Proof. We have already observed in Proposition 1 that $v(L) > 0$ if and only if L is an $\mathbf{S}$-transformation. But, by Theorem 1, L is an $\mathbf{S}$-transformation if and only if it is positive stable.

Thus, $v(L) > 0$ if and only if L is positive stable.

To see items (i)–(iii), assume that $v = v(L) > 0$. Then L is positive stable. By Theorem 1, L^{-1} exists and $L^{-1}(K) \subseteq K$. By continuity, $L^{-1}(K^\circ) \subseteq K^\circ$. In particular, $L^{-1}(e) > 0$. Now, let $(\bar{x}, \bar{y})$ be any optimal strategy pair so that $L^T(\bar{y}) \leq ve \leq L(\bar{x})$. Then $\bar{x} \geq v\, L^{-1}(e) > 0$ as $v > 0$ and $L^{-1}(e) > 0$. We now claim that $\bar{y} > 0$. (This comes from the previous Remark/Theorem, but we provide an alternative, perhaps, simpler proof.) Suppose, if possible, $\bar{y} \in \partial K$. As K is self-dual, by the supporting hyperplane theorem [2, Theorem 2.7.5], we can find $0 \neq z \in K$ such that $\langle z, \bar{y}\rangle = 0$. By the $\mathbf{Z}$-property of L, we have $\langle z, L^T(\bar{y})\rangle = \langle L(z), \bar{y}\rangle \leq 0$. However, as $\bar{x} > 0$, by (6), $L^T(\bar{y}) = v\, e$. Hence $\langle z, v\, e\rangle \leq 0$. But this cannot happen as $v > 0$ and $0 \leq z \neq 0$. Thus, $\bar{y} > 0$. This proves that L is completely mixed. Items (ii) and (iii) follow from Theorem 5. $\square$

Remarks. The above theorem may not hold if $v(L) < 0$. For example, consider the $\mathbf{Z}$-matrix

$$A = \begin{bmatrix} 1 & -5 & -15 \\ -1 & 2 & -3 \\ -12 & -15 & 1 \end{bmatrix}.$$

Using Matlab and Theorem 2, we can verify the following: For A, $v(A) = -6.1724$ with optimal strategies $\bar{x} = (0.55172, 0, 0.44828)$ and $\bar{y} = (0.44828, 0, 0.55172)$. However, for A^T, $v(A^T) = -2.4666$ with optimal strategies $\bar{x} = (0, 0.86667, 0.13333)$ and $\bar{y} = (0.26667, 0, 0.73333)$.

Theorem 7. *Suppose L is Lyapunov-like on V. Then*

(a) $v(L) > 0$ if and only if L is positive stable.
(b) $v(L) < 0$ if and only if L is negative stable.

(c) When $v(L) \neq 0$, L is completely mixed.
(d) $v(L) = v(L^T)$.

Proof. Recall that when L is Lyapunov-like, the transformations L, $-L$, and $-L^T$ are **Z**-transformations. Thus, item (a) follows from the previous theorem. Also, when $v(L) > 0$, items (i)–(iii) of Theorem 6 hold. We now come to item (b). As $-v(L) = v(-L^T)$, $v(L) < 0$ if and only if $v(-L^T) > 0$. Since $-L^T$ is a **Z**-transformation, by the above theorem, $v(L) < 0$ if and only if $-L^T$ is positive stable, or equivalently (as L is a real linear transformation), L is negative stable.

(c) When $v(L) \neq 0$, we can apply the above theorem to L or to $-L^T$ to see that L or $-L^T$ is completely mixed. Now, from the remarks made after Theorem 2 we conclude that L is completely mixed in both cases.

(d) When $v(L) \neq 0$, by item (c), L is completely mixed and hence $v(L) = v(L^T)$. Since the case $v(L) = 0$ and $v(L^T) \neq 0$ cannot arise (else, $v(L) = v((L^T)^T) \neq 0$), we see that $v(L) = v(L^T)$ in all cases. □

Remark. Item (b) of the above corollary does not extend to **Z**-transformations. For example, consider $V = R^2$ and the **Z**-matrix

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

Since a **Z**-matrix has positive value if and only if it is a **P**-matrix, and A is not a **P**-matrix, we see that $v(A) \leq 0$. The value cannot be zero, as there is no nonzero nonnegative vector $\bar{x}$ satisfying $0 \leq A\bar{x}$. We conclude that $v(A) < 0$. Yet, A is not negative stable since the eigenvalues of A are 3 and -1.

We now consider Stein-like transformations. Schneider [15, Lemma 1] has shown that for $L = I - \Lambda$ with $\Lambda(K) \subseteq K$,

$$L^{-1}(K) \subseteq K \quad \text{if and only if} \quad \rho(\Lambda) < 1.$$

In view of Theorem 1 this means that:

> *For a Stein-like transformation $L = I - \Lambda$, L is positive stable if and only if Λ is Schur stable.*

We will use this result to prove the following.

Theorem 8. *For a Stein-like transformation $L = I - \Lambda$, the following hold:*

(i) $v(L) > 0$ if and only if Λ is Schur stable.
(ii) $v(L) < 0$ if and only if Λ is inverse Schur stable (and $\Lambda \in Aut(K)$).

(iii) *When* $v(L) \neq 0$, L *is completely mixed.*
(iv) $v(L) = v(L^T)$.

Proof. (i) As L is a **Z**-transformation, Theorem 6 shows that $v(L) > 0$ if and only if L is positive stable, or equivalently, $v(L) > 0$ if and only if Λ is Schur stable.

(ii) Suppose $v(L) < 0$. Then $v(-L^T) = -v(L) > 0$. This means, by Proposition 1, that $-L^T$ is an **S**-transformation on K. Thus, there exists a $d > 0$ in V such that $-L^T(d) > 0$. This, upon simplification, leads to $\Lambda^T(d) - d > 0$ and to $\Lambda^T(d) > 0$. As $\Lambda \in \overline{Aut(K)}$ and K is self-dual, we easily see that $\Lambda^T \in \overline{Aut(K)}$. Since we also have $\Lambda^T(d) > 0$, where $d > 0$, from Lemma 2.7 in [7], $\Lambda^T \in Aut(K)$. By the self-duality of K, $\Lambda \in Aut(K)$. Now, $(\Lambda^T)^{-1}(K^\circ) \subseteq K^\circ$ and so $\Lambda^T(d) - d > 0 \Rightarrow d - (\Lambda^T)^{-1}(d) > 0$. Putting $\widetilde{L} := I - (\Lambda^T)^{-1}$, we see that $\widetilde{L}$ (which is a **Z**-transformation) is also an **S**-transformation. Thus, $\widetilde{L}$ is positive stable, or equivalently, $(\Lambda^T)^{-1}$ is Schur stable. This means that Λ is inverse Schur stable. To see the converse, assume that Λ is inverse Schur stable. By reversing some of the arguments above, we see that $d - (\Lambda^T)^{-1}(d) > 0$ for some $d > 0$. As $\Lambda \in \overline{Aut(K)}$, $\Lambda(K) \subseteq K$ and, by the self-duality of K, $\Lambda^T(K) \subseteq K$. In addition, by the invertibility of Λ^T, we have $\Lambda^T(K^\circ) \subseteq K^\circ$. Thus, $d - (\Lambda^T)^{-1}(d) > 0 \Rightarrow \Lambda^T(d) - d > 0$. This means that $-L^T$ is an **S**-transformation. By Proposition 1, $v(-L^T) > 0$ and so $v(L) = -v(-L^T) < 0$. Thus we have (ii).

(iii) Now suppose that $v(L) \neq 0$. When $v(L) > 0$, by Theorem 6, L is completely mixed; so suppose $v(L) < 0$, in which case, Λ is inverse Schur stable and $\Lambda \in Aut(K)$. Let $(\bar{x}, \bar{y})$ be an optimal strategy pair for L so that $L^T(\bar{y}) \leq v(L)\, e \leq L(\bar{x})$. Now $(\Lambda^T)^{-1} \in Aut(K)$ and so

$$\bar{y} - \Lambda^T(\bar{y}) \leq v(L)\, e \quad \Rightarrow \quad \left(\Lambda^T\right)^{-1}(\bar{y}) - \bar{y} \leq v(L) \left(\Lambda^T\right)^{-1}(e).$$

This implies

$$\bar{y} \geq \left(\Lambda^T\right)^{-1}(\bar{y}) - v(L) \left(\Lambda^T\right)^{-1}(e) > 0$$

as $v(L) < 0$ and $(\Lambda^T)^{-1}(e) > 0$. By Theorem 3, $L(\bar{x}) = v(L)\, e$. This gives $\bar{x} - \Lambda(\bar{x}) = v(L)\, e$ and since $\Lambda^{-1}(K) \subseteq K$, we have $\bar{x} = \Lambda^{-1}(\bar{x}) - v(L)\, \Lambda^{-1}(e) > 0$. Thus, we have shown that $\bar{x} > 0$ and $\bar{y} > 0$. Hence L is completely mixed.

(iv) When $v(L) \neq 0$, L is completely mixed and so $v(L) = v(L^T)$. When $v(L) = 0$, we must have $v(L^T) = 0$; else, $v(L) = v((L^T)^T) \neq 0$. Thus, $v(L) = v(L^T)$ in all cases. □

6. Value inequalities on product spaces

Our next set of results deals with the value of a **Z**-transformation defined on a product inner product space. Consider finite dimensional real inner product spaces V_i,

$i = 1, 2, \dots, l$. We let $V = V_1 \times V_2 \times \cdots \times V_l$ and define an inner product on V as follows: For any two elements $x = (x_1, x_2, \dots, x_l)$ and $y = (y_1, y_2, \dots, y_l)$ in V,

$$\langle x, y \rangle = \sum_1^l \langle x_i, y_i \rangle.$$

For each i, let K_i denote a self-dual cone in V_i and $e_i \in K_i^\circ$. We let $K = K_1 \times K_2 \times \cdots \times K_l$ and $e = (e_1, e_2, \dots, e_l)$. Clearly, K is self-dual in V and $e \in K^\circ$. Let $P_i : V \to V_i$, which takes $x = (x_1, x_2, \dots, x_l)$ to x_i, denote the projection map on the ith coordinate space. Given a linear transformation $L : V \to V$ and indices $i, j \in \{1, 2, \dots, l\}$, we define subtransformations $L_{ij} : V_j \to V_i$ by $L_{ij}(x_j) = (P_i L)(0, 0, \dots, x_j, 0, 0, \dots, 0)$. The subtransformations L_{ii}, $i = 1, 2, \dots, l$, will be called *principal subtransformations* of L. For example, if $V = V_1 \times V_2$, then L takes the block form

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{7}$$

where $A = L_{11}$, $B = L_{12}$, $C = L_{21}$, and $D = L_{22}$.

Theorem 9. *Suppose $V = V_1 \times V_2 \times \cdots \times V_l$ and L is a* **Z***-transformation on $K_1 \times K_2 \times \cdots \times K_l$. Then each L_{ii} is a* **Z***-transformation on K_i. Moreover, if $v(L) > 0$, then $v(L_{ii}) > 0$ for all $i = 1, 2, \dots, l$, and*

$$\frac{1}{v(L)} \geq \sum_1^l \frac{1}{v(L_{ii})}.$$

Proof. For simplicity, we assume that $V = V_1 \times V_2$ and L has the block form given in (7). It is easy to show that A is a **Z**-transformation on K_1 and D is a **Z**-transformation on K_2. Now suppose that $v(L) > 0$. Then by Theorem 6, L is an **S**-transformation on K. We now quote Theorem 2 in [16] to conclude that A and D are also **S**-transformations; by Theorem 6, $v(A) > 0$ and $v(D) > 0$. Now we prove the stated inequality. Let $x = (x_1, x_2) = L^{-1}(e)$ so that from Theorem 6,

$$\frac{1}{v(L)} = \left\langle L^{-1}(e), e \right\rangle = \langle x, e \rangle = \langle x_1, e_1 \rangle + \langle x_2, e_2 \rangle. \tag{8}$$

From the block representation of L, we have

$$Ax_1 + Bx_2 = e_1 \quad \text{and} \quad Cx_1 + Dx_2 = e_2.$$

As A and D are invertible (being positive stable), we can write

$$x_1 = A^{-1}e_1 - A^{-1}Bx_2 \quad \text{and} \quad x_2 = D^{-1}e_2 - D^{-1}Cx_1.$$

Now, since A and D are **Z**-transformations which are also positive stable, from Theorem 1, $A^{-1}(K_1) \subseteq K_1$ and $D^{-1}(K_2) \subseteq K_2$. Also, as $L^{-1}(K) \subseteq K$, we see that $x \in K$ and so $x_i \in K_i$, $i = 1, 2$. Using Proposition 2 in [16], which says that $-B(K_2) \subseteq K_1$ and $-C(K_1) \subseteq K_2$, we conclude that $-A^{-1}Bx_2 \geq 0$ in V_1 and $-D^{-1}Cx_1 \geq 0$ in V_2. Thus, $x_1 \geq A^{-1}e_1$ in V_1 and $x_2 \geq D^{-1}e_2$ in V_2. From these, we get

$$\langle x_1, e_1 \rangle \geq \langle A^{-1}e_1, e_1 \rangle = \frac{1}{v(A)} \quad \text{and} \quad \langle x_2, e_2 \rangle \geq \langle D^{-1}e_2, e_2 \rangle = \frac{1}{v(D)}.$$

Now, from (8), we get the inequality

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(D)}.$$

The general inequality stated in the theorem is proved by induction. □

Remark. The above proof reveals an important special case: If the off-diagonal blocks B and C are zero, then

$$\frac{1}{v(L)} = \frac{1}{v(A)} + \frac{1}{v(D)}.$$

In the general case, if all the off-diagonal subtransformations L_{ij} $(i \neq j)$ are zero, then

$$\frac{1}{v(L)} = \sum_{1}^{l} \frac{1}{v(L_{ii})}.$$

An expression of this form appears in [11].

Corollary 1. *Suppose $V = V_1 \times V_2 \times \cdots \times V_l$ and L is Lyapunov-like on $K_1 \times K_2 \times \cdots \times K_l$. Then the off-diagonal subtransformations L_{ij} (for $i \neq j$) are zero and the principal subtransformations L_{ii}, $i = 1, 2, \ldots, l$, are Lyapunov-like. Moreover, the following statements hold:*

(i) $v(L) > 0$ if and only if $v(L_{ii}) > 0$ for all i, in which case,

$$\frac{1}{v(L)} = \sum_{1}^{l} \frac{1}{v(L_{ii})}. \tag{9}$$

(ii) $v(L) < 0$ if and only if $v(L_{ii}) < 0$ for all i. In this case, (9) holds.
(iii) $v(L) = 0$ if and only if there exist i and j such that $v(L_{ii}) \leq 0$ and $v(L_{jj}) \geq 0$.
(iv) When $l \geq 2$, L is completely mixed if and only if $v(L) \neq 0$.

Proof. We first show that all off-diagonal subtransformations L_{ij} (for $i \neq j$) are zero. For simplicity, we let $i = 2$ and $j = 1$ and show that $L_{21} = 0$. Consider any $0 \leq x_1 \in V_1$. Then for any $0 \leq y_2 \in V_2$, we have $0 \leq (x_1, 0, 0, \dots, 0) \perp (0, y_2, 0, \dots, 0) \geq 0$ in V. By the Lyapunov-like property of L, $\langle L(x_1, 0, \dots, 0), (0, y_2, 0, \dots, 0)\rangle = 0$. As this equality holds for all $y_2 \in K_2$ and $K_2 - K_2 = V_2$, it also holds for all $y_2 \in V_2$. It follows that the second component of $L(x_1, 0, \dots, 0)$ is zero. This proves that $L_{21}(x_1) = P_2 L(x_1, 0) = 0$. Since x_1 is an arbitrary element of K_1 and $K_1 - K_1 = V_1$, we see that $L_{21} = 0$. In a similar way, we show that all off-diagonal subtransformations of L are zero. This shows that for any $x = (x_1, x_2, \dots, x_l)$,

$$L(x) = \big(L_{11}(x_1), L_{22}(x_2), \dots, L_{ll}(x_l)\big). \tag{10}$$

Now, the verification that each L_{ii} is Lyapunov-like on V_i is easy and will be omitted.

(i) Suppose that $v(L) > 0$. Then, by Theorem 6, L is an **S**-transformation: there exists $d = (d_1, d_2, \dots, d_l) > 0$ such that $L(d) > 0$. This means that $d_i > 0$ and $L_{ii}(d_i) > 0$ for $i = 1, 2, \dots, l$. We see that each (Lyapunov-like transformation) L_{ii} is also an **S**-transformation. Thus, $v(L_{ii}) > 0$ for all i by Theorem 6. Conversely, if $v(L_{ii}) > 0$ for all i, we can find $d_i > 0$ in V_i such that $L_{ii}(d_i) > 0$ for $i = 1, 2, \dots, l$. Then, $d = (d_1, d_2, \dots, d_l) > 0$ and $L(d) > 0$. This means that L is an **S**-transformation and so by Theorem 6, $v(L) > 0$.

The second part of (i) comes from the previous Remark.

(ii) This can be handled by considering $-L^T$ (which is a Lyapunov-like transformation) and using $v(-L^T) = -v(L)$.

Item (iii) follows immediately from items (i) and (ii).

(iv) If $v(L) \neq 0$, then L is completely mixed by Theorem 7. Now suppose $l \geq 2$ and L is completely mixed. If possible, let $v(L) = 0$. Then by Theorem 5, $\dim(Ker(L)) = 1$ and there exists a pair $(\bar{x}, \bar{y})$ such that $\bar{x} > 0$, $\bar{y} > 0$, and $L(\bar{x}) = 0 = L^T(\bar{y})$. Writing $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_l)$, where each $\bar{x}_i > 0$ in K_i, we see from (10) that $L_{ii}(\bar{x}_i) = 0$ for all i. This implies that for each $i = 1, 2, \dots, l$, the (nonzero) vector $(0, 0, \dots, \bar{x}_i, 0, \dots, 0)$ belongs to $Ker(L)$. As these vectors are linearly independent, we see that $\dim(Ker(L)) \geq l \geq 2$, contradicting the fact that $\dim(Ker(L)) = 1$. Thus, $v(L) \neq 0$ and the proof is complete. □

Our next result deals with the value of a Schur complement. To describe this, consider a linear transformation L defined on $V = V_1 \times V_2$ and let L be given in the block form (7). If A is invertible, we define the Schur complement of L with respect to A by:

$$L/A := D - CA^{-1}B.$$

Theorem 10. *Suppose L is a* **Z***-transformation on $V = K_1 \times K_2$. If $v(L) > 0$, then $v(A) > 0$ and $v(L/A) > 0$, and moreover,*

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(L/A)}.$$

Proof. Suppose that $v(L) > 0$. Then by Theorem 6, L is an **S**-transformation. By Theorem 2 in [16], A is a **Z**- and **S**-transformation on K_1 and L/A is a **Z**- and **S**-transformation on K_2. Thus, by Theorem 6, $v(A) > 0$ and $v(L/A) > 0$. Now to prove the stated inequality. Let $x = (x_1, x_2) = L^{-1}(e)$ so that $L(x) = e$. From the block form of L,

$$Ax_1 + Bx_2 = e_1 \quad \text{and} \quad Cx_1 + Dx_2 = e_2.$$

As A is invertible (being positive stable), $x_1 = A^{-1}e_1 - A^{-1}Bx_2$. Putting this in the second equation above and simplifying, we get

$$x_2 = (L/A)^{-1}e_2 - (L/A)^{-1}CA^{-1}e_1.$$

Applying Theorem 1 to A and L/A, we get

$$A^{-1}(K_1) \subseteq K_1 \quad \text{and} \quad (L/A)^{-1}(K_2) \subseteq K_2.$$

From Proposition 2 in [16], we also have $-B(K_2) \subseteq K_1$ and $-C(K_1) \subseteq K_2$. Thus, $-A^{-1}Bx_2 \geq 0$ in V_1 and $-(L/A)^{-1}CA^{-1}e_1 \geq 0$ in V_2. Hence

$$x_1 \geq A^{-1}e_1 \quad \text{and} \quad x_2 \geq (L/A)^{-1}e_2.$$

Using (8), we get $\langle L^{-1}e, e\rangle = \langle x_1, e_1\rangle + \langle x_2, e_2\rangle \geq \langle A^{-1}e_1, e_1\rangle + \langle (L/A)^{-1}e_2, e_2\rangle$, and

$$\frac{1}{v(L)} \geq \frac{1}{v(A)} + \frac{1}{v(L/A)}.$$

This completes the proof. □

7. Value computations for L_A and S_A

In this section, we describe/compute the values of L_A and S_A on $\mathcal{S}^n$. Recall that objects of $\mathcal{S}^n_+$ are (symmetric and) positive semidefinite, while those in its interior are positive definite. We use capital letters for objects/matrices in $\mathcal{S}^n$ and continue to write $X \geq 0$ ($X > 0$) for matrices in $\mathcal{S}^n_+$ (respectively, $(\mathcal{S}^n_+)^\circ$).

For any $n \times n$ real matrix A, consider the Lyapunov transformation L_A on $\mathcal{S}^n$ defined in (4). We have already observed that this is a Lyapunov-like transformation on $\mathcal{S}^n$. In view of the properties of L_A stated in Section 2.3 and Theorem 7, we have the following result.

Theorem 11. *For L_A, the following statements hold:*

(i) $v(L_A) > 0$ if and only if A is positive stable.
(ii) $v(L_A) < 0$ if and only if A is negative stable.

(iii) *When* $v(L_A) \neq 0$, L_A *is completely mixed.*
(iv) $v(L_A) = v(L_{A^T})$.

Now we compute $v(L_A)$. Assume that A is either positive stable or negative stable. (In the other case, $v(L_A) = 0$.) Then by Theorem 7,

$$v(L_A) = \frac{1}{\langle (L_A)^{-1}(I), I \rangle}.$$

Let $X := (L_A)^{-1}(I)$ so that $AX + XA^T = I$. Note that either X is positive definite (when A is positive stable) or negative definite (when A is negative stable). Let $X = UDU^T$, where U is an orthogonal matrix and D is a diagonal matrix with all positive/negative entries. Then $AX + XA^T = I$ becomes $BD + DB^T = I$, where $B := U^T AU$. Comparing the diagonal entries in $BD + DB^T$ and I, we get $2d_i b_{ii} = 1$ for all $i = 1, 2, \ldots, n$. Now

$$\langle X, I \rangle = tr(X) = tr(D) = \sum_1^n d_i = \sum_1^n \frac{1}{2b_{ii}}.$$

Thus,

$$v(L_A) = \frac{2}{\sum_1^n \frac{1}{b_{ii}}}.$$

As a further illustration of positive stable case, let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then, A is positive stable and $L_A(X) = I$, $(L_A)^T(Y) = I$ have solutions

$$X = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

We see that $v(L_A) = \frac{1}{tr(X)} = \frac{2}{5}$ and

$$\overline{X} = v(L_A)\, X \quad \text{and} \quad \overline{Y} = v(L_A)\, Y$$

are optimal strategies. Note that these two optimal strategies do not (operator) commute.

For any $n \times n$ real matrix A, consider the Stein transformation S_A on $\mathcal{S}^n$, $S_A(X) = X - AXA^T$. The following result comes from Theorem 8 and the equivalent properties of S_A stated in Section 2.3.

Theorem 12. *For* S_A*, the following statements hold:*

(i) $v(S_A) > 0$ *if and only if* A *is Schur stable.*
(ii) $v(S_A) < 0$ *if and only if* A *is inverse Schur stable.*

(iii) When $v(S_A) \neq 0$*,* S_A *is completely mixed.*
(iv) $v(S_A) = v(S_{A^T})$.

We now compute $v(S_A)$ when A is Schur stable. Since $\frac{1}{v(S_A)} = \langle X, I\rangle$, where $X = (S_A)^{-1}(I)$, we proceed as follows. We solve the equation $X - AXA^T = I$ for a positive definite X, write $X = UDU^T$ with a diagonal matrix D and an orthogonal matrix U, and define $B := U^T AU$. Then $D - BDB^T = I$ and D is a diagonal matrix with diagonal (vector) $d > 0$. By considering the diagonals in $D - BDB^T$ and I, we see that $(I - B \circ B)d = e$, where $B \circ B$ is the Schur (or Hadamard) product of B with itself, and e is the vector of ones in R^n. Writing $M = I - B \circ B$, we see that M is a **Z**-matrix with $Md > 0$. Hence M is a positive stable matrix and $\frac{1}{v(M)} = \langle d, e\rangle$ in R^n. But $\langle d, e\rangle = \langle D, I\rangle = \langle X, I\rangle = \frac{1}{v(S_A)}$. Thus, $v(S_A) = v(M)$.

8. Concluding remarks

In this paper, we defined the concept of value of a linear transformation relative to a self-dual cone. While we have extended some classical results to this general setting, many interesting problems/issues arise for further study and exploration, such as the dependence of value on the chosen point e in K° and the concept of value of a linear transformation L from one inner product space (or one self-dual cone) to another. We end this paper with an open problem.

- We noted in Proposition 1, item (6), that the value of a **P**-transformation is positive. In the classical setting, the converse is known to hold for **Z**-matrices as every positive stable **Z**-matrix is a **P**-matrix, see [1]. Whether such a result holds in the general situation is an open problem. We state this as a **Conjecture**: *If the value of a* **Z***-transformation on a Euclidean Jordan algebra is positive (that is, the transformation is positive stable), then it is a* **P***-transformation.* We remark that the answer is 'yes' for a Lyapunov-like transformation, see [10].

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